Incompressibility in relativistic continuous media

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(Received 18 February 1981; accepted for publication 24 April 1981)

A formalism of continuous media in general relativity is given and the concept of shock waves is defined using manifolds with boundary and the transversality theory of submanifolds. A definition of relativistic incompressibility is proposed and one gets that the shock waves are longitudinal with the speed of light.

PACS numbers: 04.20. − q, 04.40. + c

1. INTRODUCTION

In this paper we propose to study the concept of incompressibility in General Relativity. This topic has been studied for perfect fluids (Eddington,[1] Lichnerowicz,[2] B. Coll,[3] Olivert[4]), but it has not been treated with whole generality.

Our aim is to extend the concept of dynamic volume given in[3,4] and require that its spatial projection does not vary along the stream lines. Then we shall examine the relativistic waves in continuous media.

When one deals with extended bodies in General Relativity, it is usual to avoid possible definitions of such bodies. Nevertheless, in the scientific literature, one finds some suggestions in order to structure them; for instance, Cattaneo[5] with his "ideal state," Choquet-Bruhat and Lamoureux-Brousse[6] with the "frame state," Beiglböck,[7] and so on.

In this paper, we start with the idea of scheme given by Lichnerowicz.[8] We introduce the concept of local scheme which, intersecting with the Landau manifold of an observer,[9] allows a possible definition of a relativistic continuous medium.

According to the notations used in this paper, let us consider the space–time as a connected pseudo-Riemannian manifold X, of Hausdorff type and four dimensions, in which a metric tensor field of hyperbolic signature (3,1) has been defined. The coordinate x^4 of a point, belonging to a coordinate neighborhood of a given local chart F, coincides with the coordinate time in the system described by F. It is assumed that the measure units have been chosen such that the constant light velocity in the vacuum has the value 1.

We notice that the indices represented by latin letters take values 1 to 4 and the Greeck indices are restricted the the values 1, 2, 3.

Finally, we represent by T_pX the tangent space of X at p, and by ϕ_{*m} the derived linear function at $m\in X$ of a differentiable function ϕ defined in X. We write $M\pitchfork N$ to say that the two submanifolds M and N are transverse.

2. LOCAL DOMAINS AND SIMULTANEITY IN THE SPACE-TIME

Let D be a domain of the space–time X. Being D an open manifold, D is a locally compact and connected manifold with boundary V. This manifold of dimension 4 is called a local domain.

The study of the local domain in the Minkowski space allows us to give, in Theorem 2.1, a structure for certain sets of simultaneous points, which are subsets of a domain D.

A local domain $\mathbf{V}$ of Minkowski space $\mathbf{X}$ is called a transversally spatial domain if its boundary $\partial\mathbf{V}$ cuts transversally[10] with the whole spacelike hyperplane which contains a point of $\mathbf{IntV}$ (interior of $\mathbf{V}$).

Lemma 2.1: For each $\mathbf{p}$ of an open set $\mathbf{D}$ of Minkowski space, there is a transversally spatial and convex local domain $\mathbf{V}\subset\mathbf{D}$ such that $\mathbf{p}\in\mathbf{IntV}$.

Proof: It is easy to prove that a closed ball of center $\mathbf{p}$ is a local domain satisfying the adequate conditions.

Lemma 2.2: Let $\mathbf{D}$ be an open set and π a spacelike hyperplane of Minkowski space $\mathbf{X}$. Then there is a local domain $\mathbf{V}\subset\mathbf{D}$ such that $\mathbf{M}=\mathbf{V}\cap\pi$ is a compact and convex manifold with boundary of 3 dimensions immersed in $\mathbf{R}^3$.

Proof: As the dimensions of $\mathbf{V}$ and $\mathbf{X}$ are the same,

$$T_x\mathbf{V}+T_x\pi=T_x\mathbf{X},\quad \forall x\epsilon\pi\cap\mathbf{V}.$$

That is,

$$\mathbf{V}\pitchfork\pi. \tag{1.1}$$

Therefore, as $\mathbf{V}$ is transversally spatial, π spacelike, and $\mathbf{p}\in\mathbf{IntV}\cap\pi$, we get

$$\partial\mathbf{V}\pitchfork\pi. \tag{1.2}$$

From (2.1) and (2.2), one derives[10] that $\mathbf{M}=\mathbf{V}\cap\pi$ is a manifold with boundary of dimension 3.

Finally, as $\mathbf{V}$ is compact and π is closed, $\mathbf{M}$ is compact into $\pi\simeq\mathbf{R}^3$ and it is connected, inasmuch as $\mathbf{V}$ and π are convex.

Theorem 2.1: Let D be a domain and C an observer (a differentiable timelike curve) of the space–time X. For each $p\in C$ such that $L_p\cap D\neq\phi$ (L_p being the Landau manifold of p), there is a local domain $V\subset D$ so that $M_p=L_p\cap V$ is a compact connected Riemannian manifold with boundary of dimension 3.

Proof: Let $p\in C$ be such that $L_p\cap D\neq\phi$. As the space–time is a linear connected manifold, at each point $p\in X$ there exist a neighborhood B_p diffeomorphic to a neighborhood of $0\in T_pX=\mathbf{X}$ by the map $\exp_p$. Then $\mathbf{D}_p=\exp_p^{-1}(B_p\cap D)$ is an open set of Minkowski space which satisfies

$$\mathbf{E}_p\cap\mathbf{D}_p=\exp_p^{-1}(L_p\cap D)\neq\phi,$$

where $\mathbf{E}_p$ is the physical space of C at p. $\mathbf{E}_p$ is a spacelike hyperplane.

Then we can apply Lemma 2.2. There is a local domain $\mathbf{V}\subset\mathbf{D}_p$ so that $\mathbf{M}_p=\mathbf{V}\cap\mathbf{E}_p$ is a compact connected Riemannian manifold with boundary of 3 dimensions.

Let us consider the local domain $V=\exp_p\mathbf{V}\subset D$. Then $M_p=V\cap L_p$ is a compact connected manifold with boundary of 3 dimensions, as it is the image of $\mathbf{M}_p$ by $\exp_p$.

Therefore, we can restrict M_p to the positive definite metric of the Landau manifold: $\bar{\gamma}^0 = i^*\bar{g}$, where i is the natural injection of L_p in X.

3. ALMOST-THERMODYNAMIC MATERIAL SCHEMES

Following Lichernowicz,[8] a scheme is a domain D of the space–time in which one has defined a tensor field of second order which is called the energy-momentum tensor $\bar{T}$. We say that D is normal is $\bar{T}$ is symmetric and has a timelike eigenvector; that is, there exist a scalar field r and a unitary timelike vector field $\bar{u}$ so that

$$\bar{T}(\bar{u}) = -r\bar{u}. \tag{3.1}$$

Let us now solve the following question: As a physical body with a material distribution has a density, is it possible to endow a material density to a point set of a scheme D (i. e., the M_p of Theorem 2.1)? In this case, M_p would generalize the concept of continuous medium; we take it as a compact connected manifold with boundary of 3 dimensions immersed in $\mathbf{R}^3$, on which a scalar field $\sigma > 0$ is defined as the density of medium.

Theorem 3.1: Let D be a normal scheme with energy–momentum tensor $\bar{T}$ such that $r > 0$; and C an observer of space–time. For each $p\epsilon C$ such that $L_p \cap D \neq \phi$, there is a local scheme V (local domain of a scheme) of D so that $M_p = L_p \cap V$ is a compact connected Riemannian manifold endowed with a scalar field $\sigma > 0$.

Proof: Due to Theorem 2.1, it only remains to show the existence of the field σ. But $r = \bar{T}(\bar{u},\bar{u}) > 0$, by hypothesis. Then, if i is the injection from M_p in D,

$$\sigma = i^*r = r \circ i > 0$$

is a positive scalar field on M_p.

The schemes which satisfy the hypotheses of Theorem 3.1 with unique unitary timelike eigenvector $\bar{u}$ are called material schemes and the sets M_p are relativistic continuous media (diffeomorphic, on the other hand, to a continuous medium, according to the construction made in Theorem 2.1). The scalar field r is the so-called proper mass–energy density of the scheme. The tensor field $\bar{\tau} = i^*\bar{T}$ is a symmetric tensor field of second order on M_p which generalizes the classic stress tensor. The stream lines of the scheme are integral curves of the field $\bar{u}$. This field, which by definition of material scheme is unique, is a so-called 4-velocity of the scheme.

The relativistic stress tensor $\bar{t}$ is defined as the spatial projection (by means of Eckart tensor $\bar{\gamma} = \bar{g} + \bar{u} \otimes \bar{u}$) of the energy–momentum tensor. The spatial components of $\bar{t}$ in the local inertial proper system of point $p \in D$ (event of a stream line or observer) coincides at p with the components of $\bar{\tau}$ in the chart induced in M_p.

From the definition of $\bar{t}$, one derives that the energy–momentum tensor takes the expression

$$\bar{T} = \rho(1+\epsilon)\bar{u}\otimes\bar{u} + \bar{t}, \tag{3.2}$$

where we have adopted the Taub hypothesis[11] over the decomposition of the proper mass–energy density in two terms: ρ is the mass–energy density and ϵ is the specific internal energy. This introduction of the thermodynamic variable ϵ leads us to say that an almost-thermodynamic material scheme is such that its energy–momentum tensor takes the expression (3.2).

Let us assume that there is matter conservation in an almost-thermodynamic material scheme, a condition which can be expressed by the continuity equation

$$\nabla_i(\rho u^i) = 0 \Leftrightarrow u^i\nabla_i\rho + d^{\,i}_{\,i} = 0, \tag{3.3}$$

where $\bar{d} = \frac{1}{2}L_{\bar{u}}\bar{\gamma}$ is the deformation tensor of the scheme and $L_{\bar{u}}$ is the Lie derivative. The tensor $\bar{d}$ has the components

$$d_{ij} = 1/2[\gamma_i^k\nabla_k u_j + \gamma_j^k\nabla_k u_i]. \tag{3.4}$$

The time and space components of the conservation equations $\nabla_i T^i_j = 0$ can be written as

$$\rho u^i\nabla_i\epsilon + t^{ij}d_{ij} = 0, \tag{3.5}$$

$$\rho f^r_j u^k\nabla_k u^j + \gamma^r_i\gamma^k_j\nabla_k t^{ij} = 0, \tag{3.6}$$

where $\bar{f} = (1+\epsilon)\bar{\gamma} + (1/\rho)\bar{t}$ is the tensor index[12] of the scheme (below we shall give its physical significance).

4. INCOMPRESSIBILITY OF AN ALMOST-THERMODYNAMIC MATERIAL SCHEME

Let us recall that in perfect fluids one has characterized the incompressibility for isoentropic processes by

$$\nabla_{\bar{u}}k = 0, \tag{4.1}$$

k being the dynamic volume defined by

$$k = f/\rho \tag{4.2}$$

and f the fluid index[2] which, at once, is given as a function of the pressure, the specific internal energy, and the material density according to the expression

$$f = 1 + \epsilon + p/\rho. \tag{4.3}$$

The generalization of these concepts is suggested by the definition of the tensor index, which is given by Maugin[12] for elastic media, and we extend it to almost-thermodynamic material schemes:

$$\bar{f} = (1+\epsilon)\bar{\gamma} + (1/\rho)\bar{t}. \tag{4.4}$$

From (4.4), we propose the following.

Definition 4.1: We call dynamic volume of an almost-thermodynamic material scheme to the 2-tensor

$$\bar{k} = v\bar{f},$$

where v is the specific volume.

From this definition and (4.4), one derives

$$\bar{k} = v\bar{\gamma} + v\bar{e} \tag{4.5}$$

and $\bar{e} = \epsilon\bar{\gamma} + v\bar{t}$ is called the tensor enthalpy, according to the analogy with classical thermodynamics.

Formula (4.5) hints that the dynamic volume is the sum of a term which stands for the specific volume and another term, a function of the enthalpy, which is called enthalpic volume. This enthalpic volume reduces to zero in classical physics ($c \to \infty$), as one can prove if we correct the above mentioned formulas in order to make them homogeneous.

Definition 4.2: We say that an almost-thermodynamic material scheme D is incompressible if the spatial change of dynamic volume along the stream lines is null, i. e.,

$$\gamma_i^r \gamma_j^l \nabla_{\bar{u}} k^{ij} = 0. \tag{4.6}$$

Let us see that this idea of incompressibility generalizes the one which is given by (4.1) for perfect fluids. For them, the relativistic stress tensor is given by $\bar{t} = p\bar{\gamma}$; therefore, Eq. (4.4) can be written as $\bar{f} = f\bar{\gamma}$, and therefore

$$\bar{k} = k\bar{\gamma}. \tag{4.7}$$

Transporting (4.7) in the definition of incompressibility (4.6), one gets

$$\gamma^{rl} \nabla_{\bar{u}} k = 0, \tag{4.8}$$

hence (4.1) is obviously satisfied.

Theorem 4.1: The condition of incompressibility (4.6) leads to the expression

$$\gamma_i^r \gamma_j^l u^k \nabla_k t^{ij} + (2t^{rl} + \rho(1+\epsilon)\gamma^{rl})\gamma_i^k \nabla_k u^i$$
$$= \gamma^{rl} t^{ki} d_{ki}.$$

Proof: If we develop the covariant derivative of the dynamic volume using $v = 1/\rho$, we obtain

$$\nabla_{\bar{u}} k^{ij} = 1/\rho^2 \{\rho\gamma^{ij}\nabla_{\bar{u}}\epsilon + \rho(1+\epsilon)(u^i\nabla_{\bar{u}}u^j + u^j\nabla_{\bar{u}}u^i)$$
$$+ \nabla_{\bar{u}} t^{ij} - ([2/\rho]t^{ij} + (1+\epsilon)\gamma^{ij})\nabla_{\bar{u}}\rho\}. \tag{4.9}$$

From this expression, by virtue of (4.6), one derives

$$\rho\gamma^{rl} u^k \nabla_k \epsilon + \gamma_i^r \gamma_j^l u^k \nabla_k t^{ij}$$
$$- ([2/\rho]t^{rl} + (1+\epsilon)\gamma^{rl}) u^k \nabla_k \rho = 0. \tag{4.10}$$

Using Eqs. (3.3) and (3.5), we eliminate the derivatives of ϵ and ρ, and the theorem is proved.

5. RELATIVISTIC WAVES IN THE INCOMPRESSIBLE ALMOST-THERMODYNAMIC SCHEMES

In order to study the relativistic waves for incompressible almost-thermodynamic schemes, we introduce some properties of the timelike characteristic hypersurfaces.

Theorem 5.1: Let W be a timelike hypersurface of a material scheme D, and p a point of D such that $L_p \cap D \neq \phi$. Then $H_p = W \cap L_p$ is a regular submanifold of dimension two.

Proof: It is easy to see that the tangent vectors to L_p are spacelike. Then, as W is timelike, one has that, in each $q \epsilon W \cap L_p$, there is a timelike vector tangent to W which will not be tangent to L_p. This, together with the fact that W and L_p are submanifolds of three dimensions and D of four dimensions, leads to the transversality condition

$$T_q L_p + T_q W = T_q D$$

and it tells us that H_p is a regular submanifold of dimension two.

Definition: We call shock waves of a timelike characteristic hypersurface W, of an observer C, the submanifolds

$$H_p = L_p \cap W, \quad p \in C.$$

If B_p is the domain of $\exp_p^{-1}$, then $\mathbf{W}_p = \exp_p^{-1}(W \cap B_p)$ is a hypersurface into the Minkowski space $T_p X$ associated with W at p. Thereby, we can consider the set

$$\mathbf{H}_p = \mathbf{W}_p \cap \mathbf{E}_p = \exp_p^{-1} H_p, \tag{5.1}$$

which is a submanifold of dimension two in $T_p X$, and it will be called a "spacelike shock wave" of the observer C.

Once the shock waves are defined, let us introduce their speed. This leads us to the well-known expression given by Lichnerowicz[2,8] and Pham Mau Quan.[13] Hereby we proceed to derive from the definitions some properties which attempt to fill a vacuum existing in this respect in the scientific literature.

Let p be a point of timelike hypersurface W and C the stream line of the scheme which contains p. In order to associate a speed to the shock waves $H_p, p \in C$, let us take a vector $\bar{w}$ tangent to W at p orthogonal to H_p. Let us see that $\bar{w}$ is timelike. As

$$T_p H_p = \mathbf{E}_p \cap T_p W, \tag{5.2}$$

then $\{\bar{h}_1, \bar{h}_2, \bar{w}\}$ will be a basis of $T_p W$ if $\{\bar{h}_1, \bar{h}_2\}$ is a basis of $T_p H_p$, because $\bar{w}$ is orthogonal to H_p. Therefore, $\bar{w}$ is timelike, inasmuch as W. If we consider that $\bar{w}$ is unitary, it is called a four-velocity of H_p.

By virtue of (5.2), there is a unitary vector $\bar{\lambda} \in \mathbf{E}_p$ orthogonal to H_p. This vector gives us the direction of the observed propagation. Hereafter we shall express $T_p H_p$ by $\mathbf{H}_\lambda$.

Theorem 5.2: The vectors $\bar{l}$ (orthogonal to W at p), $\bar{w}$, $\bar{\lambda}$, and $\bar{u}$ (four-velocity of the observer at p) are in the same two-plane; that is, $\bar{l}$ and $\bar{w}$ can be written as linear combinations of $\bar{\lambda}$ and $\bar{u}$.

Proof: Let us define the vector

$$\bar{L} = \bar{l} - a\bar{u} \quad \text{with} \quad a = -\bar{g}(\bar{u}, \bar{l}). \tag{5.3}$$

This vector belongs to the physical space $\mathbf{E}_p$, inasmuch as

$$\bar{g}(\bar{u}, \bar{L}) = 0. \tag{5.4}$$

But by virtue of (5.2), all $\bar{v} \in \mathbf{H}_\lambda$ is in $\mathbf{E}_p$ and in $T_p W$; hence it will be orthogonal to $\bar{L}$. Thus, $\bar{L}$ has the direction of $\bar{\lambda}$. This implies

$$\bar{l} = \|\bar{L}\|\bar{\lambda} + a\bar{u}. \tag{5.5}$$

With $b = -\bar{g}(\bar{w}, \bar{u})$, $\bar{w} - b\bar{u}$ belongs to the physical space $\mathbf{E}_p$. Moreover, as $\bar{w}$ is orthogonal to H_p, it will be orthogonal to all vectors of $\mathbf{H}_\lambda$, and so

$$\bar{g}(\bar{v}, \bar{w} - b\bar{u}) = 0, \quad \forall \bar{v} \in \mathbf{H}_\lambda.$$

That is, if $\bar{w} - b\bar{u} \neq 0$, it will have the direction of $\bar{\lambda}$ and there will be a $c \in R$ such that $c\bar{\lambda} = \bar{w} - b\bar{u}$, and hence

$$\bar{w} = b\bar{u} + c\bar{\lambda}. \tag{5.6}$$

If $\bar{w} - b\bar{u} = 0$, it is enough to take $c = 0$ so that (5.6) is satisfied.

The expressions (5.5) and (5.6) prove the theorem.

The speed $\mathbf{U}$ of propagation of the shock wave is the relative in the Minkowski space between the moving point of four-velocity $\bar{\mathbf{w}}$ and the observer of four-velocity $\bar{\mathbf{u}}$, where

$$\bar{\mathbf{u}} = (\exp_p^{-1})_* \bar{u}, \quad \bar{\mathbf{w}} = (\exp_p^{-1})_* \bar{w}. \tag{5.7}$$

As $\exp_p$ conserves the metric at p,

$$b = -w^i u_i = -\mathbf{w}^i \mathbf{u}_i = 1/(1 - \mathbf{U}^2)^{1/2}. \tag{5.8}$$

From (5.6) and (5.8), we get

$$c^2 = \mathbf{U}^2/(1 - \mathbf{U}^2). \tag{5.9}$$

On the other hand, as $\bar{w}$ and $\bar{l}$ are orthogonal, the expressions (5.5) and (5.6) lead to

$$a/\|\bar{L}\| = c/b. \tag{5.10}$$

Substituting (5.8) and (5.9) into (5.10), we arrive at

$$\mathbf{U} = a/\|\bar{L}\|. \tag{5.11}$$

[We have taken the positive root in (5.9). One would arrive at the same final result taking the negative root.]

Now, as W is a characteristic hypersurface, we may apply the Hadamard discontinuities, so that if ϕ is one of the tensor components, of any type, with discontinuities at the first derivatives, one satisfies[14]

$$[\phi,_i] = \delta\phi l_i. \tag{5.12}$$

As the Christoffel symbols are continuous, one gets

$$[\nabla_i\phi] = \delta\phi l_i. \tag{5.13}$$

Furthermore, using (5.5), we obtain

$$[\gamma^i_j\nabla_i\phi] = \|\bar{L}\|\lambda_j\delta\phi, \tag{5.14}$$

and keeping in mind (5.3) and (5.11), we derive

$$[u^i\nabla_i\phi] = -\mathbf{U}\|\bar{L}\|\delta\phi. \tag{5.15}$$

It is easy to verify that the components $\delta\phi$ yield to the lowering and raising of indices, because $\bar{g}$ is continuous in X.

If the W is a characteristic hypersurface by the discontinuities of the first derivatives of the four-velocity, by virtue of (5.12) it leads us for each component to

$$[\partial_j u^i] = \delta u^i l_j. \tag{5.16}$$

With theses discontinuities, we define the vector $\delta\bar{u} = (\delta u^i)$, which will be called "infinitesimal discontinuity of the four-velocity $\bar{u}$".

It is easy to prove that

$$u_i\delta u^i = 0, \tag{5.17}$$

and hence $\delta\bar{u}$ is a vector of the physical space.

With the decomposition made by Maugin,[12]

$$\delta\bar{u} = \delta\bar{u}_\perp + \bar{\lambda}\delta u_\|, \tag{5.18}$$

where $\delta u_\| = \lambda_i\delta u^i$, $\delta u^i_\perp = S^i_j\delta u^j$, $\bar{S} = \bar{\gamma} - \bar{\lambda}\otimes\bar{\lambda}$, we come to formulate the following

Theorem 5.3: Let $\bar{P}$ be a spacelike symmetric tensor of second order ($P_{ij}u^i = 0$). The discontinuity system of $\delta\bar{u}$,

$$P_{ri}\delta u^i = 0, \tag{5.19}$$

may be written in the form

$$P_{\perp ri}\delta u^i_\perp + p_r\delta u_\| = 0, \tag{5.20}$$

$$p_i\delta u^i_\perp + P_\|\delta u_\| = 0, \tag{5.21}$$

$\bar{P}_\perp$, $\bar{p}$, $P_\|$ are defined as

$$P_{\perp ri} = S^k_r S^l_i P_{kl}, \tag{5.22}$$

$$P_\| = P_{kl}\lambda^k\lambda^l, \tag{5.23}$$

$$p_i = \lambda^k P_{kl}S^l_i. \tag{5.24}$$

Proof: The use of (5.22) leads to

$$\bar{P} = \bar{P}_\perp + \bar{p}\otimes\bar{\lambda} + \lambda\otimes\bar{p} + P_\|\bar{\lambda}\otimes\bar{\lambda} \tag{5.25}$$

and from (5.24) to

$$p_i = P_{ik}\lambda^k - P_\|\lambda_i. \tag{5.26}$$

Placing (5.25) and (5.18) into (5.19), one gets

$$P_{ri}\delta u^i_\perp + p_r\delta u_\| + \lambda_r p_i\delta u^i + P_\|\lambda_r\delta u_\| = 0. \tag{5.27}$$

The substitution of (5.26) into (5.27) leads to (5.20). Finally, by means of the contraction of (5.19), one obtains (5.21).

Let us study the relativistic waves in the incompressible almost-thermodynamic scheme. For it, we analyze the equations by virtue of which the Cauchy problem is established.

As we are not going to study the gravitational waves, we assume that the second derivatives of g_{ij} are continuous, and thus the use of either the Einstein field equations or material waves will not be necessary, so that $\mathbf{U}\neq 0$.

Infinitesimal discontinuities can exist at the derivatives of the four-velocity components, of the relativistic stress tensor, of the material mass density, and of the specific internal energy; i. e.,

$$\delta u^i,\ \delta t_{ij},\ \delta\rho,\ \delta\epsilon. \tag{5.28}$$

Nevertheless, from the conservation equations (3.5) and (3.6), the continuity equation (3.3), and the equation given by Theorem 4.1, one has that not all these discontinuites are independent; but they satisfy the following.

Theorem 5.4: The study of discontinuities (5.28) reduces to the system

$$P_{ri}u^i = 0,$$

with

$$P_{ri} = \rho\mathbf{U}^2 f_{ri} + \lambda_r\lambda_k t^k_i - (2t^k_r\lambda_k + \rho(1+\epsilon)\lambda_r)\lambda_i.$$

Proof: Calculating discontinuities in the formula of Theorem 4.1, keeping in mind (5.14), (5.15), and (3.4), one has

$$-\gamma^r_i\gamma^l_j\mathbf{U}\delta t^{ij} = \gamma^{rl}t^k_i\lambda_k - (2t^{rl} + \rho(1+\epsilon)\gamma^{rl})\lambda_i\delta u^i, \tag{5.29}$$

and from (3.6) we obtain

$$-\rho f^r_i\mathbf{U}\delta u^i + \gamma^r_i\lambda_j\delta t^{ij} = 0. \tag{5.30}$$

Equations (5.29) and (5.30) lead us to the statement of the theorem.

Let us restrict the principal shock waves. We use the following definition given by Maugin[12]: We say that a shock wave is principal if $\bar{\lambda}$ (propagation direction) is the eigenvector of the relativistic stress tensor $\bar{t}$:

$$t_{ij}\lambda^j = t_\|\lambda_i. \tag{5.31}$$

Theorem 5.5: In a principal shock wave, one satisfies

$$P_{\perp ri}\delta u^i_\perp = 0, \quad P_\|\delta u_\| = 0, \tag{5.32}$$

where

$$P_{\perp ri} = \rho\mathbf{U}^2 S^s_r S^j_i f_{sj}, \tag{5.33}$$

$$P_\| = (\mathbf{U}^2 - 1)(\rho(1+\epsilon) + t_\|). \tag{5.34}$$

Proof: Due to (5.31), $\bar{P}$ is symmetric and takes the form

$$\bar{P} = \rho\mathbf{U}^2\bar{f} - (\rho(1+\epsilon) + t_\|)\bar{\lambda}\otimes\bar{\lambda}. \tag{5.35}$$

Then, we may apply Theorem 5.3 to Theorem 5.4, and hence one obtains straightforwardly the expressions (5.32), because of

$$p_j = S^i_j P_{ri}\lambda^r = 0$$

by virtue of (5.35). Further, (5.33) and (5.34) are derived from the definition given in Theorem 5.3.

Theorem 5.6: In an incompressible almost-thermodynamic scheme the principal shock waves are longitudinal and their speeds are $\mathbf{U} = 1$.

Proof: Due to the uniqueness of the four-velocity of a material scheme

$$\rho(1+\epsilon)+t_\alpha \neq 0, \quad \alpha=1,2,3. \tag{5.36}$$

From (5.32) and (5.33) one derives, by virtue of $\mathbf{U}\neq 0$, $\delta \bar{u}_\perp \in \mathbf{H}_\lambda$,

$$\rho f_{ri} \delta u_\perp^i = 0. \tag{5.37}$$

But from (5.36), the system (5.37) leads to $\delta\bar{u}_\perp = 0$; that is, to the longitudinal shock waves

$$\delta\bar{u} = \delta u_{\|}\bar{\lambda}, \quad \delta u_{\|} \neq 0.$$

This condition carries us with (5.32) to

$$P_{\|} = 0. \tag{5.38}$$

Finally, the expression (5.34) of $P_{\|}$, with (5.36) and (5.38), implies the unity value for the speed $\mathbf{U}$ of shock waves.

6. DISCUSSION

In this paper we have studied what we can understand as a relativistic continuous medium in General Relativity. For it, we have made use of the theory of manifolds with boundary and use them to satisfy some topologic properties in order to verify the integrability conditions in these manifolds. The integration is essential to definition in these continuous media; for instance, their mass, center-of-mass, and so on.

Nevertheless, our exposition must be considered as a first order approximation to the reality, because the boundaries of the natural extended bodies are not always differentiable at all points. It is suggestive to extend our definition to the manifolds with corners; but these mathematical structures have not been deeply studied so very few properties are known to be applied to this topic.

Once the material schemes have been defined, we have restricted the discussion to the so-called almost-thermodynamic material schemes, in order to avoid introducing in this paper the concepts of enthropy and temperature.

We have obtained an incompressibility condition for the almost-thermodynamic schemes, following the line labelled by Lichnerowicz.[2] With it, the transversal shock waves disappear (at most, they are material: $\mathbf{U}=0$), and the longitudinal shock waves speed takes the value 1.

Therefore, if the almost-thermodynamic scheme is a perfect fluid, the incompressibility condition reduces to the known one in B. Coll[3] and J. Olivert[4] for isoenthropic processes.

Likewise, it is easy to prove that the incompressibility condition, in the hypoelastic media,[12] leads us to consistent results:

$$\rho(\lambda+2\mu)+2t_{\|} = \rho(1+\epsilon), \tag{6.1}$$

$$\rho\mu + t_\alpha = 0, \; \alpha = 2,3, \tag{6.2}$$

where λ, μ are the Lamé coefficients. With it, one gets the same results for the speeds for shock waves as this paper.

The expressions (6.1) and (6.2) are obtained establishing the equations for the Hadamard discontinuity described in Sec. 5, by replacing the space component of conservation equation (3.6) by the relativistic Hooke law given by Choquet-Bruhat and Lamoureux-Brouse,[6] equivalent to the one proposed by Maugin[12] when the continuity equation is satisfied.

ACKNOWLEDGMENTS

We would like to thank Professor A. M. Naveira and Dr. A. Montesinos for suggestions related to the mathematical content of this work, and especially Professor J. Bernabeu for his care in reading the manuscript and suggestions about its physical content.

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