Flux limited diffusion equations: the "relativistic" heat equation

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Abstract

We consider the class of parabolic equations of the form

$$u_t = \operatorname{div} \mathbf{a}(u, Du) \qquad \text{in } (0,T) \times \mathbb{R}^N \tag{1}$$

with initial condition $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, $u_0 \geq 0$ where the flux function $\mathbf{a}(z,p)$, $(z,p) \in \mathbb{R} \times \mathbb{R}^N$ is the sub-differential with respect to the variable p of a function $f(z,p)$ which is convex in p and has linear growth as $|p| \to \infty$. In other words, the flux saturates for large gradients. This type of models were introduced by J.R. Wilson in the context of radiation hydrodynamics in order to ensure a finite speed of propagation radiation's energy density u. An important example in this class is the so-called "relativistic" heat equation (RHE), derived by Y. Brenier using Monge-Kantorovich's mass transport theory,

$$u_t = \nu \operatorname{div} \left(\frac{u \nabla u}{\sqrt{u^2 + \frac{\nu^2}{c^2} |\nabla u|^2}} \right) \qquad \text{in } (0,T) \times \mathbb{R}^N \tag{2}$$

where $\nu > 0$ is a constant representing a kinematic viscosity and c is the speed of light. Our purpose is to give a concept of entropy solution for the class of Flux Limited Diffusion equations (1) (under some assumptions on $\mathbf{a}(z,p)$ covering the case of (2)) proving the existence and uniqueness of such solutions. Then we prove that the support of solutions of the RHE (2) moves with the speed of light c and that there are discontinuity fronts moving with such speed. We also study the two asymptotic limits of (2) as $c \to \infty$ and as $\nu \to +\infty$ showing that solutions of (2) converge to solutions of the heat equation $u_t = \nu \Delta u$ as $c \to \infty$ and to solutions of $u_t = c \operatorname{div}\left(u \frac{\nabla u}{|\nabla u|} \right)$ as $\nu \to \infty$. The features exhibited by the solutions of (2) and its asymptotic limit as $\nu \to \infty$ show that the concept of entropy solution is well adapted to study this class of equations.

References

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