Perturbation Techniques for Nonexpansive Mappings with Applications

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- INTRODUCTION: NOTIONS AND RESULTS.
- MAIN RESULT.
- APPLICATIONS TO CONVEX AND OPTIMIZATION PROBLEM.

Problem

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X real Banach space
$C \in X$ nonempty closed convex subset
$T : C \longrightarrow C$ nonexpansive mapping
$Fix(T) = \{x \in C : x = Tx\} \neq \emptyset$

$$\text{find } x \in Fix(T)$$

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The aim is to define an algorithm which generates

$$\{x_n\} \text{ converging to } x \in Fix(T).$$

Nonexpansive Type Mappings

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  $T : C \to X$ is *nonexpansive* if

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  where $\lambda \in (0, 1)$ and T is nonexpansive.

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  where $\lambda \in (0, 1)$ and T is nonexpansive.
- H Hilbert space,
  $T : C \to H$ is *firmly nonexpansive* if
  $$\|Tx - Ty\|^2 \leq (x - y, Tx - Ty)\,, \forall x, y \in C.$$

Metric Projection

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- H Hilbert space,
  $C \subset H$ closed convex.
  The *metric projection* onto C:

$$P_C(x) = \{y \in C : d(x, y) = d(x, C)\}.$$

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Theorem. H Hilbert space,
$D \subset C \subset H$ closed convex,
$P : C \to D$ retraction ($P(x) = x \; \forall x \in D$).
Equivalent:

(a) P is the metric projection from C onto D.

(b) $\|Px - Py\|^2 \leq \langle x - y, Px - Py \rangle, \forall x, y \in C$.

(c) $\langle x - Px, y - Px \rangle \leq 0, \forall x \in C$ and $\forall y \in D$.

PROJECTION

$$\Downarrow$$

FIRMLY NONEXPANSIVE

$$\Downarrow$$

AVERAGED

$$\Downarrow$$

NONEXPANSIVE

Sunny Nonexpansive Retraction

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  $D \subset C \subset X$ closed convex subsets,
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  Q is sunny if $\forall x \in C$ and $\forall t \in [0,1]$

$$Q(tx + (1-t)Q(x)) = Q(x).$$

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$$Q(tx + (1-t)Q(x)) = Q(x).$$

Theorem. X smooth Banach space. Equivalent:

(a) Q is sunny and nonexpansive.

(b) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle, \forall x, y \in C.$

(c) $\langle x - Qx, J(y - Qx)\rangle \leq 0, \forall x \in C, y \in D.$

There is at most one sunny nonexpansive retraction from C onto D.

Duality Mapping

Duality Mapping

- A *gauge* is a continuous strictly increasing function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$\phi(0) = 0 \text{ and } \lim_{t \to \infty} \phi(t) = \infty.$$

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- X Banach space.
  The *duality mapping* is the mapping

$$J_\phi : X \longrightarrow 2^{X^*}$$

$$J_\phi(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x^*\|\,\|x\|, \phi(\|x\|) = \|x^*\|\}.$$

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- If $\phi(t) = t$, J_ϕ is the *normalized duality map*,

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

Subdifferential

Subdifferential

- $f : X \to (-\infty, \infty]$ is *subdifferentiable* at $x \in X$ if there exists $x^* \in X^*$, *subgradient* of f at x, such that

$$f(y) - f(x) \geq (x^*, y - x), \forall y \in X.$$

Subdifferential

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$$f(y) - f(x) \geq (x^*, y - x), \forall y \in X.$$

- The subdifferential of f is the mapping

$$\partial f : X \to 2^{X^*}$$

$$\partial f(x) = \{x^* \in X^* : (x^*, y - x) \leq f(y) - f(x), \forall y \in X\}.$$

- f proper convex lsc function $\Rightarrow f$ subdifferentiable on Int D(f).

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- f proper convex continuous function, Gâteaux differentiable at $x \in IntD(f) \Leftrightarrow$ has a unique subgradient

$$\partial f(x) = \nabla f(x).$$

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- f has a minimum value at $x \Leftrightarrow 0 \in \partial f(x)$.

Duality Mapping

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If ϕ is a gauge and $\Phi(t) = \int_0^t \phi(s)ds$,

$$J_\phi(x) = \partial\Phi(\|x\|).$$

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$$J_\phi(x) = \partial\Phi(\|x\|).$$

Subdifferential inequality:

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, j_\phi(x+y)\rangle$$

where $\quad j_\phi(x+y) \in J_\phi(x+y).$

Duality Mapping

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where $\quad j_\phi(x+y) \in J_\phi(x+y).$

For the normalized duality map J:

$$\Phi(t) = t^2/2$$

$J(x) = \partial f(x)$ where $f(x) = \frac{1}{2}\|x\|^2.$

Duality Mapping

If ϕ is a gauge and $\Phi(t) = \int_0^t \phi(s)ds$,

$$J_\phi(x) = \partial\Phi(\|x\|).$$

Subdifferential inequality:

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, j_\phi(x + y)\rangle$$

where $\quad j_\phi(x + y) \in J_\phi(x + y).$

For the normalized duality map J:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle.$$

where $\quad j(x + y) \in J(x + y).$

- X is smooth $\Leftrightarrow J_\phi$ is single-valued .

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- X is uniformly smooth $\Leftrightarrow J_\phi$ is single-valued and norm-to-norm uniformly continuous on bounded sets of X.

- X is smooth $\Leftrightarrow$ J_ϕ is single-valued .
- X is uniformly smooth $\Leftrightarrow$ J_ϕ is single-valued and norm-to-norm uniformly continuous on bounded sets of X.

- J_ϕ is *weakly continuous* if single-valued and weak-to-weak* sequentially continuous

$$x_n \rightharpoonup x \;\Rightarrow\; J_\phi(x_n) \rightharpoonup^* J_\phi(x).$$

Fixed Point Algorithms

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- Algorithms that generate $\{x_n\}$,

$$x_n \text{ converging to } x \in Fix(T)$$

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$$\boxed{x_{n+1} = Tx_n, \quad n \geq 0}$$

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$C \subset X$ closed convex,
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- T contraction: for some $\alpha \in (0, 1)$,

$$\|T(x) - T(y)\| \leq \alpha \|x - y\|, \; \forall x, y \in X,$$

Banach's Contraction Principle, 1922:
There exists unique fixed point x of T and

$$x_n \to x \in Fix(T).$$

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Mann's iteration

$$\boxed{x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0}$$

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Halpern's iteration

$$\boxed{x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \quad n \geq 0}$$

where $u \in C$ arbitrary and $\{\alpha_n\} \subset [0, 1]$

Mann’s Iteration

Mann's Iteration

Theorem (Reich, 1979)
X uniformly convex with Fréchet differentiable norm
T nonexpansive self-mapping on C with $F(T) \neq \emptyset$,

(i) $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = +\infty$.

Then $x_n \rightharpoonup x \in F(T)$.

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$$\boxed{x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_n x_n, \quad n \geq 0}$$

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Theorem (Xu, 2006)
X uniformly convex with Fréchet differentiable norm
T nonexpansive self-mapping on C with $F(T) \neq \emptyset$,
$\{T_n\}$ sequence of nonexpansive self-mappings on C,

(i) $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = +\infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n D_\rho(T_n, T) < \infty, \quad \forall \rho > 0$

$$D_\rho(T_n, T) = \sup\{\|T_n x - Tx\| : \|x\| \leq \rho\}.$$

Then $x_n \rightharpoonup x \in F(T)$.

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$$\boxed{x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \geq 0}$$

Theorem (Halpern, Lions, Wittmann, Xu, 1967-2006).
X either *uniformly smooth*
or *reflexive* with a *weakly continuous duality map* J_ϕ,

(H1) $\lim_{n\to\infty} \alpha_n = 0$,

(H2) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(H3) either $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or
$\lim_{n\to\infty} \frac{|\alpha_{n+1}-\alpha_n|}{\alpha_{n+1}} = 0$.

Then $x_n \to x \in F(T)$.

Halpern's Iteration - Averaged

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$$\boxed{x_{n+1} = \alpha_n u + (1-\alpha_n)(\lambda I + (1-\lambda)T)x_n, \quad n \geq 0}$$

for $\lambda \in (0,1)$.

Halpern's Iteration - Averaged

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Theorem (Suzuki, Chidume, 2006).
X uniformly smooth Banach space.

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$$\boxed{x_{n+1} = \alpha_{n+1} u + (1 - \alpha_{n+1}) S_{n+1} x_n, \quad n \geq 0}$$

$u, x_0 \in C$ arbitrary (but fixed)
$\alpha_n \in [0, 1]$
$S_n = (1 - \lambda) I + \lambda T_n$
$\lambda \in (0, 1)$
$T_n : C \to C$ nonexpansive
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- Under some conditions on X and $\{\alpha_n\}$

$$x_n \to x \in Fix(T)$$

where x is a specific fixed point.

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$$Q(tx + (1-t)Q(x)) = Q(x).$$

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There is at most one sunny nonexpansive retraction from C onto D.

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Theorem (Reich, Xu).
If X is either *uniformly smooth*
or has a *weakly continuous duality map* J_ϕ

$$Q(u) = \lim_{t \to 0^+} z_t = z \in Fix(T)$$

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Theorem (Reich, Xu).
If X is either *uniformly smooth*
or has a *weakly continuous duality map* J_ϕ

$$Q(u) = \lim_{t \to 0^+} z_t = z \in Fix(T)$$

Moreover, $Q : C \to Fix(T)$ is the unique
sunny nonexpansive retraction from C to $Fix(T)$

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(H2) $\sum_{n=0}^{\infty} \alpha_n = \infty$

(i) $\lim_{n\to\infty} \|T_n y_n - T y_n\| = 0,\ \forall \{y_n\}$ bounded

(ii) $\sum_{n=0}^{\infty} \|T_n f - T f\| < \infty,\ \forall f \in Fix(T)$

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$$\boxed{x_{n+1} = \alpha_{n+1} u + (1-\alpha_{n+1})\big(\lambda I + (1-\lambda)T_{n+1}\big)x_n}$$

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$$\boxed{x_{n+1} = \alpha_{n+1} u + (1 - \alpha_{n+1})\big(\lambda I + (1-\lambda) T_{n+1}\big) x_n}$$

Then

$$x_n \to Q(u)$$

$Q : C \to Fix(T)$ unique sunny nonexpansive retraction

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(iii) $\sum_{n=0}^{\infty} D_\rho(T_n, T) < \infty \; \forall \rho > 0$

$$D_\rho(T_n, T) = \sup\{\|T_n x - Tx\| : \|x\| \leq 0\}$$

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$$D_\rho(T_n, T) = \sup\{\|T_n x - Tx\| : \|x\| \leq 0\}$$

$$(iii) \Rightarrow \begin{cases} (i)\,\text{lím}_{n\to\infty}\|T_n y_n - Ty_n\| = 0, & \{y_n\} bounded \\ (ii)\sum_{n=0}^{\infty}\|T_n f - Tf\| < \infty, & f \in Fix(T) \end{cases}$$

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$$(iii) \Rightarrow \quad (ii) \sum_{n=0}^{\infty} \|T_n f - Tf\| < \infty, \quad f \in Fix(T)$$

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Then

$$x_n \longrightarrow Q(u)$$

$Q : C \longrightarrow Fix(T)$ unique sunny nonexpansive retraction

Split Feasibility Problem

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H_1, H_2 Hilbert spaces
$C \subset H_1$, $Q \subset H_2$ nonempty convex subsets
$A : H_1 \longrightarrow H_2$ linear bounded operator

$$\text{find } x^* \in C \text{ such that } Ax^* \in Q$$

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$$\text{find } x^* \in C \text{ such that } Ax^* \in Q$$
$$\Updownarrow$$
$$\text{mín}_{x\in C} f(x), \quad f(x) = \frac{1}{2}\|P_Q Ax - Ax\|^2$$

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$$\Updownarrow$$
$$\text{mín}_{x\in C}\, f(x), \quad f(x) = \tfrac{1}{2}\|P_Q Ax - Ax\|^2$$
$$\Updownarrow$$
$$x^* = P_C(I - \gamma A^*(I - P_Q)A)x^*$$

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$$\Updownarrow$$
$$\text{mín}_{x \in C} f(x), \quad f(x) = \frac{1}{2}\|P_Q Ax - Ax\|^2$$
$$\Updownarrow$$
$$x^* = P_C(I - \gamma A^*(I - P_Q)A)x^*$$

If $\gamma \in (0, 2/\delta)$ with δ the spectral radius of A^*A:

- $T = P_C(I - \gamma A^*(I - P_Q)A)$ nonexpansive.

Split Feasibility Problem

To avoid difficulties with the implementation of the projections (Zhao, Yang, 2006)

- $T_n = P_{C_n}(I - \gamma A^*(I - P_{Q_n})A)$,
  where $C_n \in H_1$ and $Q_n \in H_2$ closed convex.

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To avoid difficulties with the implementation of the projections (Zhao, Yang, 2006)

- $T_n = P_{C_n}(I - \gamma A^*(I - P_{Q_n})A)$,
  where $C_n \in H_1$ and $Q_n \in H_2$ closed convex.

T_n is nonexpansive if $\gamma \in (0, 2/\delta)$.

Split Feasibility Problem

Theorem. $Fix(T) \neq \emptyset$

$$\boxed{x_{n+1} = \alpha_{n+1} u + (1 - \alpha_{n+1})\big((1-\lambda)x_n + \lambda T_{n+1} x_n\big)}$$

(H1) $\lim_{n\to\infty} \alpha_n = 0$

(H2) $\sum_{n=0}^{\infty} \alpha_n = \infty$

(iii) $\sum_{n=0}^{\infty} d_\rho(C_n, C) < \infty \ \forall \rho > 0$

$\sum_{n=0}^{\infty} d_\rho(Q_n, Q) < \infty$

Split Feasibility Problem

Theorem. $Fix(T) \neq \emptyset$

$$\boxed{x_{n+1} = \alpha_{n+1} u + (1-\alpha_{n+1})\big((1-\lambda)x_n + \lambda T_{n+1} x_n\big)}$$

(H1) $\lim_{n\to\infty} \alpha_n = 0$

(H2) $\sum_{n=0}^{\infty} \alpha_n = \infty$

(iii) $\sum_{n=0}^{\infty} d_\rho(C_n, C) < \infty \ \forall \rho > 0$

$\sum_{n=0}^{\infty} d_\rho(Q_n, Q) < \infty$

$$d_\rho(C_1, C_2) = \sup_{\|x\| \leq \rho} \|P_{C_1} x - P_{C_2} x\|$$

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Then $x_n \to x^*$ solution of SFP

Zeros of m-accretive operator

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X real Banach space
$A : X \to 2^X$ multivalued *m-accretive* operator

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- $\langle y_1 - y_2, j(x_1 - x_2)\rangle \geq 0$
  $y_i \in Ax_i,\ j(x_1 - x_2) \in J(x_1 - x_2)$
- $R(I + \lambda A) = X, \quad \forall \lambda > 0.$

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- The *resolvent* of A:

$$J_r = (I + rA)^{-1}.$$

  - J_r is single-valued and nonexpansive $\forall r > 0$.
  - $Fix(J_r) = A^{-1}(0), \ \forall r > 0.$

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- $T = J_r$ and $T_n = J_{r_n}$ where $\{r_n\} \in (0, +\infty)$.

$$\bigcap_{n \geq 0}^{\infty} Fix(T_n) = Fix(T).$$

Zeros of m-accretive operator

Theorem 1.
If X is either *uniformly smooth*
or *reflexive* with a *weakly continuous duality map* J_ϕ
$C = \overline{D(A)}$ convex, $A^{-1} \neq \emptyset$

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$$(iii) \Rightarrow \quad (i) \lim_{n\to\infty} \|T_n x_n - T x_n\| = 0, \quad \{x_n\} bounded$$

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Zeros of m-accretive operator

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X, A, $\{\alpha_n\}$, $\{r_n\}$ as in Theorem 1

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$$\sum_{n=0}^{\infty} \|e_n\| < \infty \Rightarrow (ii) \sum_{n=0}^{\infty} \|T_n f - Tf\| < \infty, \; f \in Fix(T)$$

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Minimizer Problem

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- $A : X \longrightarrow 2^{X^*}$ is *monotone* if $\forall x, y \in X$

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H Hilbert

maximal monotone = m-accretive

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$C \subset H$ closed convex
$f : C \to \mathbb{R}$ convex lower semicontinuous

$$\min_{x \in C} f(x)$$

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$$0 \in \partial f(x) + N_C(x)$$

N_C the normal cone over C.

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N_C the normal cone over C.

- $\partial f(x) + N_C(x)$ is maximal monotone.

SPLIT FEASIBILITY PROBLEM

$$\Uparrow$$

FIXED POINT PROBLEM

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$$\Downarrow$$

ZEROS OF M-ACCRETIVE OPERATORS

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"Perturbation Techniques for Nonexpansive Mappings with Applications" G. López, V. Martín and H-K Xu, preprinted.