The $p$-approximation property in Banach spaces

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Outline

1. Introduction
2. Density of finite rank operators and the $p$-approximation property
3. A trace characterization of the $p$-approximation property
4. Open problems
Introduction

Density of finite rank operators and the $p$-approximation property

A trace characterization of the $p$-approximation property

Open problems
Notation

- $X, Y$ Banach spaces, $B_X = \{ x \in X : \|x\| \leq 1 \}$
- $\mathcal{L}(Y, X)$ is the space of bounded operators from $Y$ into $X$
- $\mathcal{F}(Y, X) = \{ T \in \mathcal{L}(Y, X) : T$ has finite rank$\}$
- $\mathcal{K}(Y, X) = \{ T \in \mathcal{L}(Y, X) : T$ is compact$\}$
- $p \in [1, \infty) \Rightarrow p' = \frac{p}{p - 1} \quad (\Leftrightarrow \frac{1}{p} + \frac{1}{p'} = 1)$
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- \(p \in [1, \infty) \Rightarrow p' = \frac{p}{p - 1} \quad (\Leftrightarrow \frac{1}{p} + \frac{1}{p'} = 1)\)
- \(\ell_p(X) = \{(x_n) \subset X : \sum_n \|x_n\|^p < \infty\}\)
  \(\|(x_n)\|_p = (\sum_n \|x_n\|^p)^{1/p}\)
- \(\ell^w_p(X) = \{(x_n) \subset X : \sum_n |\langle x^*, x_n \rangle|^p < \infty, \text{ for all } x^* \in X^*\}\)
  \(\|(x_n)\|_p^w = \sup_{x^* \in B_{X^*}} (\sum_n |\langle x^*, x_n \rangle|^p)^{1/p}\)
Notation

- **X, Y** Banach spaces,  \( B_X = \{ x \in X : \| x \| \leq 1 \} \)
- **\( \mathcal{L}(Y, X) \)** is the space of bounded operators from **Y** into **X**
- **\( \mathcal{F}(Y, X) = \{ T \in \mathcal{L}(Y, X) : T \text{ has finite rank} \} \)**
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- **\( \ell_p(X) = \{ (x_n) \subset X : \sum_n \| x_n \|^p < \infty \} \)**
  \[ \| (x_n) \|_p = (\sum_n \| x_n \|^p)^{1/p} \]
- **\( \ell_w^p(X) = \{ (x_n) \subset X : \sum_n |\langle x^*, x_n \rangle|^p < \infty, \text{ for all } x^* \in X^* \} \)**
  \[ \| (x_n) \|_{w^p} = \sup_{x^* \in B_{X^*}} (\sum_n |\langle x^*, x_n \rangle|^p)^{1/p} \]
- **\( \mathcal{N}_p(Y, X) = \{ T \in \mathcal{L}(Y, X) : T \text{ is } p\text{-nuclear} \} \)**
- **\( T : Y \rightarrow X \text{ } p\text{-nuclear} \iff \exists (y_n^*) \in \ell_p(Y^*) \exists (x_n) \in \ell_w^{p'}(X) : T = \sum_n y_n^* \otimes x_n \)**
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A trace characterization of the $p$-approximation property
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- $\ell^w_p(X) = \{(x_n) \subset X : \sum_n |\langle x^*, x_n \rangle|^p < \infty, \text{ for all } x^* \in X^*\}$
  \[\|(x_n)\|_p^w = \sup_{x^* \in B_{X^*}} (\sum_n |\langle x^*, x_n \rangle|^p)^{1/p}\]
- $\mathcal{N}_p(Y, X) = \{T \in \mathcal{L}(Y, X) : T$ is $p$-nuclear$\}$
- $T : Y \rightarrow X$ $p$-nuclear $\iff$ \(\exists (y_n^*) \in \ell_p(Y^*)\)
  \(\exists (x_n) \in \ell^w_{p'}(X) : T(\cdot) = \sum_n \langle y_n^*, \cdot \rangle x_n\)
The approximation property

\[(T_n) \subset \mathcal{F}(Y, X) \quad \Rightarrow \quad T \in \mathcal{K}(Y, X)\]
The approximation property

\[ (T_n) \subset \mathcal{F}(Y, X) \]

\[ T_n \xrightarrow{\| \cdot \|} T \]

\[ \Rightarrow T \in \mathcal{K}(Y, X) \iff \mathcal{F}(Y, X) \subset \mathcal{K}(Y, X) \]
The approximation property

\[
(T_n) \subset \mathcal{F}(Y, X) \quad \text{implies} \quad T \in \mathcal{K}(Y, X) \iff \overline{\mathcal{F}(Y, X)}_{\|\cdot\|} \subset \mathcal{K}(Y, X)
\]

- \(X\) has Schauder basis

\Rightarrow \text{For every Banach } Y, \overline{\mathcal{F}(Y, X)}_{\|\cdot\|} = \mathcal{K}(Y, X)
The approximation property

**Theorem [Grothendieck, 1955]**

Let $X$ be a Banach space. The following statements are equivalent:

1. For every Banach space $Y$, $\mathcal{F}(Y, X)\|\cdot\| = \mathcal{K}(Y, X)$.
2. The identity map $I_X$ belongs to $\mathcal{F}(X, X)^{\tau_c}$.
The approximation property

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$X$ a Banach space. The following statements are equivalent:

1. For every Banach space $Y$, $\mathcal{F}(Y, X)^{\|\cdot\|} = \mathcal{K}(Y, X)$.

2. The identity map $I_X$ belongs to $\mathcal{F}(X, X)^{\tau_c}$.

Definition

A Banach space $X$ has the approximation property (AP) if the identity map $I_X$ can be approximated by finite rank operators uniformly on every compact subset of $X$ ($\equiv I_X \in \mathcal{F}(X, X)^{\tau_c}$).
The approximation property

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A Banach space $X$ has the *approximation property* (AP) if the identity map $I_X$ can be approximated by finite rank operators uniformly on every compact subset of $X$ ($\equiv I_X \in \mathcal{F}(X, X)^{\tau_c}$).

- All the classical Banach spaces of sequences and functions have the AP.
The approximation property

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**Definition**

A Banach space $X$ has the *approximation property* (AP) if the identity map $I_X$ can be approximated by finite rank operators uniformly on every compact subset of $X$ ($\equiv I_X \in \mathcal{F}(X, X)^{\tau_c}$).

- All the classical Banach spaces of sequences and functions has the AP.
- Enflo (1973): $\mathcal{L}(\ell_2, \ell_2)$ does not have the AP.
Approximation property in terms of tensor products

\[ Y^* \otimes_{\pi} X \]

\[ \sum_n y_n^* \otimes x_n \quad \sum_n \| y_n^* \| \| x_n \| < \infty \]

\[ J_1 \rightarrow \mathcal{N}_1(Y, X) \]

\[ \sum_n \langle y_n^*, \cdot \rangle x_n \]
Theorem [Grothendieck, 1955]

Let $X$ be a Banach space. The following statements are equivalent:

1. $X$ has the AP.
2. For every Banach space $Y$, $\mathcal{F}(Y, X) \| \cdot \| = \mathcal{K}(Y, X)$.
3. For every Banach space $Y$, $Y^* \hat{\otimes}_\pi X \cong \mathcal{N}_1(Y, X)$.
Chevet–Sapĥar’s tensor norm: \( p \in [1, \infty) \)

\[
g_p(u) = \inf \left\{ \| (y_n^*)\|_p \| (x_n)\|_{\ell^w_p(X)} : u = \sum_{n=1}^{m} y_n^* \otimes x_n \in Y^* \otimes X \right\}
\]
Chevet–Saphar’s tensor norm: \( p \in [1, \infty) \)

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\]

\[
J_p \quad Y^* \hat{\otimes}_{g_p} X \quad \rightarrow \quad N_p(Y, X) \quad \sum_n y_n^* \otimes x_n \quad \mapsto \quad \sum_n \langle y_n^*, \cdot \rangle x_n
\]
Approximation property of order $p$ via tensor products

- Chevet–Saphar’s tensor norm: $p \in [1, \infty)$

$$g_p(u) = \inf \left\{ \| (y_n^*) \|_p \| (x_n) \|_{\ell^w_p(X)} : u = \sum_{n=1}^{m} y_n^* \otimes x_n \in Y^* \otimes X \right\}$$

$$Y^* \hat{\otimes}_{g_p} X \xrightarrow{J_p} \mathcal{N}_p(Y, X)$$

$$\sum_n y_n^* \otimes x_n \mapsto \sum_n \langle y_n^*, \cdot \rangle x_n$$

- Saphar (1970’s): $p \in [1, \infty]$

$X$ has the approximation property of order $p$ ($\text{AP}_p$) if, for every Banach space $Y$, $Y^* \hat{\otimes}_{g_p} X \simeq \mathcal{N}_p(Y, X)$. 

$\text{AP}_1 = \text{AP}$
Approximation property of order $p$ via tensor products

- Chevet–Saphar’s tensor norm: $p \in [1, \infty)$
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g_p(u) = \inf \left\{ \|y_n^*\|_p \|x_n\|_{\ell_w^p(X)} : u = \sum_{n=1}^{m} y_n^* \otimes x_n \in Y^* \otimes X \right\}
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\[J_p \quad Y^* \hat{\otimes}_{g_p} X \quad \rightarrow \quad \mathcal{N}_p(Y, X)\]

\[\sum_n y_n^* \otimes x_n \quad \rightarrow \quad \sum_n \langle y_n^*, \cdot \rangle x_n\]

- Saphar (1970’s): $p \in [1, \infty]$
  $X$ has the \textit{approximation property of order $p$ (AP$_p$)} if, for every Banach space $Y$, $Y^* \hat{\otimes}_{g_p} X \simeq \mathcal{N}_p(Y, X)$.

- Reinov (1980’s): $p \in (0, 1]$
  $X$ has \textit{approximation property of order $p$ (AP$_p$)} if, for every Banach space $Y$, the restriction of $J_1$ to $H_p$ is injective, where

  \[H_p = \{ u = \sum_n y_n^* \otimes x_n : \sum_n (\|y_n^*\| \|x_n\|)^p < \infty \} \subset Y^* \hat{\otimes}_\pi X.\]
Approximation property of order $p$ via tensor products

- Chevet–Saphar’s tensor norm: $p \in [1, \infty)$

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g_p(u) = \inf \left\{ \|y_n^*\|_p \|x_n\|_{\ell^w_p(X)} : u = \sum_{n=1}^m y_n^* \otimes x_n \in Y^* \otimes X \right\}
\]

\[
J_p : Y^* \hat{\otimes}_{g_p} X \rightarrow \mathcal{N}_p(Y, X)
\]

\[
\sum_n y_n^* \otimes x_n \mapsto \sum_n \langle y_n^* , \cdot \rangle x_n
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- Saphar (1970’s): $p \in [1, \infty]$
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  \]

- AP$_1$ = AP
Approximation property of order $p$ via tensor products

**Theorem [Grothendieck, 1955]**

Let $X$ be a Banach space. The following statements are equivalent:

1. $X$ has the AP (i.e., $l_X$ can be approximated by finite rank operators uniformly on compact subsets $K \subset X$).
2. For every Banach space $Y$, $Y^* \hat{\otimes} \pi X \simeq \mathcal{N}_1(Y, X)$. 

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Theorem [Grothendieck, 1955]

$X$ a Banach space.

The following statements are equivalent:

1. $X$ has the AP ($\equiv l_X$ can be approximated by finite rank operators uniformly on compact subsets $K \subset X$).
2. For every Banach space $Y$, $Y^* \hat{\otimes}_\pi X \simeq \mathcal{N}_1(Y, X)$.

A set $K \subset X$ is relatively compact if and only if there exists $(x_n) \in c_0(X)$ such that $K \subset \overline{\text{aco}} (x_n) := \{ \sum_n a_n x_n : (a_n) \in B_{\ell_1} \}$. 
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**Approximation property of order $p$ via tensor products**

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**Theorem [Grothendieck, 1955]**

Let $X$ be a Banach space.

- The following statements are equivalent:
  1. $X$ has the AP ($l_X$ can be approximated by finite rank operators uniformly on compact subsets $K \subset X$).
  2. For every Banach space $Y$, $Y^* \hat{\otimes}_\pi X \simeq J_1(Y, X)$.

- A set $K \subset X$ is relatively compact if and only if there exists $(x_n) \in c_0(X)$ such that $K \subset \overline{\operatorname{aco}}\{x_n\} := \{\sum_n a_n x_n : (a_n) \in B_{\ell_1}\}$.

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**Theorem [Bourgain and Reinov, 1984-85]**

A Banach space $X$ has the AP$_p$ ($p \in (0, 1)$) if and only if $l_X$ can be approximated by finite rank operators uniformly on subsets $K \subset X$ for which there exists $(x_n) \in \ell_q(X)$ such that $K \subset \{\sum_n a_n x_n : (a_n) \in B_{\ell_1}\}$ ($p^{-1} - q^{-1} = 1$).
Definition [Sinha and Karn, 2002]

Let $p \geq 1$.

- $K \subset X$ is relatively $p$-compact if there exists $(x_n) \in \ell_p(X)$ such that $K \subset p\text{-co}(x_n) := \left\{ \sum_n a_nx_n : (a_n) \in B_{\ell_p} \right\}$.
Definition [Sinha and Karn, 2002]

Let $p \geq 1$.

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- A Banach space $X$ has the $p$-approximation property ($p$-AP) if the identity map $I_X$ can be approximated by finite rank operators uniformly on every $p$-compact subset of $X$. 
Definition [Sinha and Karn, 2002]

Let $p \geq 1$.

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- A Banach space $X$ has the $p$-approximation property ($p$-AP) if the identity map $I_X$ can be approximated by finite rank operators uniformly on every $p$-compact subset of $X$.

- $\infty$-AP=AP.
Definition [Sinha and Karn, 2002]

Let $p \geq 1$.

- $K \subset X$ is \textit{relatively $p$-compact} if there exists $(x_n) \in \ell_p(X)$ such that $K \subset p\text{-co}\ (x_n) \coloneqq \left\{ \sum_n a_n x_n : (a_n) \in B_{\ell_p'} \right\}$.

- A Banach space $X$ has the \textit{$p$-approximation property} ($p$-AP) if the identity map $l_X$ can be approximated by finite rank operators uniformly on every $p$-compact subset of $X$.

- $\infty$-AP=AP.

- All Banach spaces have the $p$-AP for all $p \in [1, 2]$.

- For every $p > 2$, there exist Banach spaces without the $p$-AP.

- A necessary condition in terms of the trace is obtained for Banach spaces having the $p$-AP.
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Let $p \geq 1$ and $1/p + 1/p' = 1$.

**Definition [Sinha and Karn, 2002]**

- $K \subset X$ is *relatively $p$-compact* if there exists $(x_n) \in \ell_p(X)$ such that $K \subset p\text{-co}(x_n) := \left\{ \sum_n a_n x_n : (a_n) \in B_{\ell_p'} \right\}$.
- $T \in \mathcal{L}(X, Y)$ is *$p$-compact* if $T(B_X)$ is relatively $p$-compact.

\[
\mathcal{K}_p(X, Y) = \{ T \in \mathcal{L}(X, Y) : T \text{ is } p\text{-compact} \}.
\]
The ideal $\mathcal{K}_p$ of $p$-compact operators

Let $p \geq 1$ and $1/p + 1/p' = 1$.

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- $K \subset X$ is *relatively $p$-compact* if there exists $(x_n) \in \ell_p(X)$ such that $K \subset p\text{-co} (x_n) := \left\{ \sum_n a_n x_n : (a_n) \in B_{\ell_p} \right\}$.
- $T \in \mathcal{L}(X, Y)$ is *$p$-compact* if $T(B_X)$ is relatively $p$-compact.

\[ \mathcal{K}_p(X, Y) = \{ T \in \mathcal{L}(X, Y) : T \text{ is } p\text{-compact} \} \]

- If $1 \leq p \leq q \leq \infty$, $\mathcal{K}_p(X, Y) \subset \mathcal{K}_q(X, Y)$.
- $\mathcal{K}_p$ is an operator ideal.
Theorem [Oja, Piñeiro, Serrano and Delgado, 2009]

Let $X$ be a Banach space, $p \in [1, +\infty]$. The following statements are equivalent:

1. $X$ has the $p$-AP.
2. For every Banach $Y$, $\mathcal{F}(Y, X)$ is $\| \cdot \|$-dense in $\mathcal{K}_p(Y, X)$.
3. For every Banach $Y$, $\mathcal{F}(Y, X)$ is $\tau_c$-dense in $\mathcal{K}_p(Y, X)$. 
$p$-approximation property and $p$-compact operators

- $\Pi_p^d(Y, X) = \{ T \in \mathcal{L}(Y, X) : T^* \text{ is } p\text{-summing} \}$

- $T \in \Pi_p^d(Y, X) \iff T$ maps relatively compact sets in $Y$ to relatively $p$-compact sets in $X$.

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**$p$-approximation property and $p$-compact operators**

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4. For every Banach $Y$, $\mathcal{F}(Y, X)$ is $\tau_c$-dense in $\Pi^d_p(Y, X)$.
1. $X$ has the $p$-AP.
2. For every Banach $Y$, $\mathcal{F}(Y, X)$ is $\tau_c$-dense in $\mathcal{K}_p(Y, X)$.

$3 \Rightarrow 1$
X has the $p$-AP.

For every Banach $Y$, $\mathcal{F}(Y, X)$ is $\tau_c$-dense in $\mathcal{K}_p(Y, X)$.

3$\Rightarrow$1

$\varepsilon > 0$

$K = p\text{-co}(x_n), (x_n) \in \ell_p(X)$

$R \in \mathcal{F}(X, X)$ satisfying $\sup_{x \in K} \|Rx - x\| < \varepsilon$
$p$-approximation property and $p$-compact operators

1. $X$ has the $p$-AP.
2. For every Banach $Y$, $\mathcal{F}(Y, X)$ is $\tau_c$-dense in $\mathcal{K}_p(Y, X)$.

3$\Rightarrow$1

$\varepsilon > 0$

$K = \text{p-co}(x_n), (x_n) \in \ell_p(X)$

$\ell'_p \xrightarrow{\phi_x} X$

$\phi_x(e_n) = x_n$

$R \in \mathcal{F}(X, X)$ satisfying $\sup_{x \in K} \|Rx - x\| < \varepsilon$
$p$-approximation property and $p$-compact operators

1. $X$ has the $p$-AP.
2. For every Banach $Y$, $\mathcal{F}(Y, X)$ is $\tau_c$-dense in $\mathcal{K}_p(Y, X)$.

$3 \Rightarrow 1$

$\varepsilon > 0$

$K = p\text{-co}(x_n), (x_n) \in \ell_p(X) \Rightarrow \exists (\alpha_n) \downarrow 0: (z_n) := (\alpha_n^{-1} x_n) \in \ell_p(X)$

$\ell_p' \xrightarrow{\phi_x} X$

$\phi_x(e_n) = x_n$

$R \in \mathcal{F}(X, X)$ satisfying $\sup_{x \in K} \|Rx - x\| < \varepsilon$
1. $X$ has the $p$-AP.

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$K = p\text{-co}(x_n), (x_n) \in \ell_p(X) \Rightarrow \exists (\alpha_n) \downarrow 0: (z_n) := (\alpha_n^{-1}x_n) \in \ell_p(X)$

$\ell_p' \xrightarrow{\phi_x} X$

$D_\alpha \downarrow \phi_z /$

$\ell_p'$

$\phi_x(e_n) = x_n$

$\phi_z(e_n) = z_n$

$D_\alpha(\beta_n) = (\alpha_n\beta_n)$

$R \in \mathcal{F}(X,X)$ satisfying $\sup_{x \in K} \|Rx - x\| < \varepsilon$
$p$-approximation property and $p$-compact operators

1. $X$ has the $p$-AP.

2. For every Banach $Y$, $\mathcal{F}(Y, X)$ is $\tau_c$-dense in $\mathcal{K}_p(Y, X)$.

3 $\Rightarrow$ 1

$\varepsilon > 0$

$K = p\text{-co}(x_n), (x_n) \in \ell_p(X) \Rightarrow \exists (\alpha_n) \downarrow 0: (z_n) := (\alpha_n^{-1} x_n) \in \ell_p(X)$

\[
\begin{align*}
\ell_p' & \xrightarrow{\phi_x} \quad X \\
D_\alpha & \downarrow \\
\ell_p' & \xrightarrow{Q} \quad Y := \ell_p'/\ker \phi_z
\end{align*}
\]

$\phi_x(e_n) = x_n$

$\phi_z(e_n) = z_n$

$D_\alpha(\beta_n) = (\alpha_n \beta_n)$

$\hat{\phi}_z[(\beta_n)] = \phi_z(\beta_n)$

$R \in \mathcal{F}(X, X)$ satisfying $\sup_{x \in K} \|Rx - x\| < \varepsilon$
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**$p$-approximation property and $p$-compact operators**

1. $X$ has the $p$-AP.

2. For every Banach $Y$, $\mathcal{F}(Y, X)$ is $\tau_c$-dense in $K_p(Y, X)$.

### 3$\Rightarrow$1

- $\varepsilon > 0$
- $K = p$-co$(x_n), (x_n) \in \ell_p(X) \Rightarrow \exists (\alpha_n) \downarrow 0: (z_n) := (\alpha_n^{-1} x_n) \in \ell_p(X)$

\[ \begin{align*}
\ell_p' & \xrightarrow{\phi_x} X \\
D_\alpha & \downarrow \phi_z \\
\ell_p' & \xrightarrow{Q} Y := \ell_p'/\text{Ker} \phi_z
\end{align*} \]

\[ \begin{align*}
\phi_x(e_n) &= x_n \\
\phi_z(e_n) &= z_n \\
D_\alpha(\beta_n) &= (\alpha_n/\beta_n) \\
\hat{\phi}_z[(\beta_n)] &= \phi_z(\beta_n)
\end{align*} \]

$H := Q \circ D_\alpha(B_{\ell_p'})$ compact in $Y$

\[ R \in \mathcal{F}(X, X) \text{ satisfying } \sup_{x \in K} \|Rx - x\| < \varepsilon \]
\section*{Introduction}

Density of finite rank operators and the \( p \)-approximation property

A trace characterization of the \( p \)-approximation property

Open problems

\section*{\( p \)-approximation property and \( p \)-compact operators}

1. \( X \) has the \( p \)-AP.

3. For every Banach \( Y \), \( \mathcal{F}(Y, X) \) is \( \tau_c \)-dense in \( \mathcal{K}_p(Y, X) \).

3 \( \Rightarrow \) 1

\( \varepsilon > 0 \)

\( K = p\text{-co}(x_n), (x_n) \in \ell_p(X) \Rightarrow \exists (\alpha_n) \downarrow 0: (z_n) := (\alpha_n^{-1} x_n) \in \ell_p(X) \)

\[
\begin{align*}
\ell_{p'} & \xrightarrow{\phi_x} X \\
\downarrow D_\alpha & \xrightarrow{\phi_z} \uparrow \hat{\phi}_z \\
\ell_{p'} & \xrightarrow{Q} Y := \ell_{p'}/\ker \phi_z \\
S = \sum_{k=1}^N & \psi_k \otimes u_k \in \mathcal{F}(Y, X): \sup_{h \in H} \|Sh - \hat{\phi}_z h\| < \varepsilon
\end{align*}
\]

\( H := Q \circ D_\alpha(B_{\ell_{p'}}) \)

compact in \( Y \)

\( \phi_x(e_n) = x_n \)

\( \phi_z(e_n) = z_n \)

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\( R \in \mathcal{F}(X, X) \) satisfying \( \sup_{x \in K} \|Rx - x\| < \varepsilon \)
\( p \)-approximation property and \( p \)-compact operators

1. \( X \) has the \( p \)-AP.
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\[ \varepsilon > 0 \]
\[ K = p\text{-co}(x_n), (x_n) \in \ell_p(X) \Rightarrow \exists (\alpha_n) \downarrow 0: (z_n) := (\alpha_n^{-1} x_n) \in \ell_p(X) \]

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\[ \hat{\phi}_z \text{ injective} \Rightarrow \exists u_k^* \in X^*: \sup_{h \in H} \left| \left\langle \phi_z^* u_k^* - \psi_k, h \right\rangle \right| < \varepsilon, \; k = 1, \ldots, N \]

\[ R \in \mathcal{F}(X, X) \text{ satisfying } \sup_{x \in K} \| Rx - x \| < \varepsilon \]
$p$-approximation property and $p$-compact operators

1. $X$ has the $p$-AP.

3. For every Banach $Y$, $\mathcal{F}(Y, X)$ is $\tau_c$-dense in $\mathcal{K}_p(Y, X)$.

3$\Rightarrow$1

$\varepsilon > 0$

$K = p$-co$(x_n), (x_n) \in \ell_p(X) \Rightarrow \exists (\alpha_n) \downarrow 0: (z_n) := (\alpha_n^{-1} x_n) \in \ell_p(X)$

\[
\begin{align*}
\ell_p' & \xrightarrow{\phi_x} X & \phi_x(e_n) = x_n \\
\ell_p' & \xrightarrow{\phi_z} Y := \ell_p'/\ker \phi_z & \phi_z(e_n) = z_n \\
D_\alpha \downarrow & \phi_z & D_\alpha(\beta_n) = (\alpha_n\beta_n) \\
\uparrow & \phi_z & \hat{\phi_z}[(\beta_n)] = \phi_z(\beta_n)
\end{align*}
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$\hat{\phi_z}$ injective $\Rightarrow \exists u_k^* \in X^*: \sup_{h \in H} \left| \langle \phi_z^* u_k^* - \psi_k, h \rangle \right| < \varepsilon, \ k = 1, \ldots, N$

$R = \sum_{k=1}^{N} u_k^* \otimes u_k$

$R \in \mathcal{F}(X, X)$ satisfying $\sup_{x \in K} \|Rx - x\| < \varepsilon$
Theorem [Grothendieck, 1955]

Let $X$ be a Banach space. The following statements are equivalent:

1. $X^*$ has the $p$-AP, i.e., for every Banach $Y$, $\|F(Y, X^*)\| = K(Y, X^*)$.

2. For every Banach $Y$, $\|\mathcal{F}(X, Y)\| = K(X, Y)$. 

---

$p$-approximation property for dual Banach spaces
Theorem [Grothendieck, 1955]

$X$ a Banach space. The following statements are equivalent:

1. $X^*$ has the AP, i.e., for every Banach $Y$, $\mathcal{F}(Y, X^*)\|\cdot\| = \mathcal{K}(Y, X^*)$.
2. For every Banach $Y$, $\mathcal{F}(X, Y)\|\cdot\| = \mathcal{K}(X, Y)$.

$T \in \mathcal{QN}_p(X, Y)$ (quasi $p$-nuclear) $\iff$ There exists $(x_n^*) \in \ell_p(X^*)$ such that $\|Tx\| \leq (\sum_n |\langle x_n^*, x \rangle|^p)^{1/p}, \forall x \in X$. 

$T \in \mathcal{K}_p(X, Y) \iff T^* \in \mathcal{QN}_p(Y^*, X^*)$. 

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The $p$-approximation property for dual Banach spaces
Theorem [Grothendieck, 1955]

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2. For every Banach Y, \( \mathcal{F}(X, Y) \| \cdot \| = \mathcal{K}(X, Y) \).

- \( T \in \mathcal{QN}_p(X, Y) \) (T quasi \( p \)-nuclear) \( \iff \) There exists \( (x^*_n) \in \ell_p(X^*) \) such that \( \|Tx\| \leq (\sum_n |\langle x^*_n, x \rangle|^p)^{1/p}, \forall x \in X \).

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**p-approximation property for dual Banach spaces**

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Let $X$ be a Banach space. The following statements are equivalent:

1. $X^*$ has the $p$-AP, i.e., for every Banach $Y$, $\overline{\mathcal{F}(Y, X^*)}_{\|\cdot\|} = \mathcal{K}(Y, X^*)$.

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Relation to Saphar’s approximation property $\text{AP}_p$

Saphar (1970-72):
- For every $p \neq 2$, there exist Banach spaces without $\text{AP}_p$. 
Relation to Saphar’s approximation property $\text{AP}_p$

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- For every $p \neq 2$, there exist Banach spaces without $\text{AP}_p$.
- $X^{**}$ has the $\text{AP}_{p'} \Rightarrow \mathcal{F}(X, Y)$ is $\pi_p$-dense in $\mathcal{QN}_p(X, Y)$. 

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J. M. Delgado
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\( X \) a Banach space. The following statements are equivalent:

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- For every \( p \neq 2 \), there exist Banach spaces without \( \text{AP}_p \).
- \( X^{**} \) has the \( \text{AP}_{p'} \) \( \Rightarrow \) For every Banach \( Y \), \( \mathcal{F}(X, Y) \) is \( \pi_p \)-dense in \( \mathcal{QN}_p(X, Y) \).

Corollary

- \( X^{**} \) has the \( \text{AP}_{p'} \) \( \Rightarrow \) \( X^* \) has the \( p \)-AP, \( p \in (1, \infty) \).
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Bourgain and Reinov (1985):

- $H^{**}, H^{****}, \ldots$ have the $\text{AP}_p$, $p \in (1, \infty)$

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- $X^{**}$ has the $\text{AP}_{p'}$ $\Rightarrow$ $X^*$ has the $p$-AP, $p \in (1, \infty)$
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Theorem [Oja, Piñeiro, Serrano and Delgado, 2009]

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- $X^{**}$ has the $\text{AP}_{p'}$ $\Rightarrow$ $X^*$ has the $p$-AP, $p \in (1, \infty)$
- $H^{*}, H^{***}, \ldots$ have the $p$-AP, $p \in [1, \infty)$
Outline

1. Introduction
2. Density of finite rank operators and the $p$-approximation property
3. A trace characterization of the $p$-approximation property
4. Open problems
The trace functional

- $\mathcal{N}_1(X, Y)$ is a quotient of $X^* \hat{\otimes}_\pi Y \Rightarrow \mathcal{N}_1(X, Y)^* \hookrightarrow \mathcal{L}(Y, X^{**})$. 
The trace functional

- \( \mathcal{N}_1(X, Y) \) is a quotient of \( X^* \hat{\otimes}_\pi Y \Rightarrow \mathcal{N}_1(X, Y)^* \hookrightarrow \mathcal{L}(Y, X^{**}). \)

- \( T = \sum_{n=1}^{m} x_n^* \otimes x_n^{**} \in \mathcal{F}(X, X^{**}) \)

\[
\text{trace}(T) := \sum_{n=1}^{m} \langle x_n^{**}, x_n^* \rangle \leq \sum_{n=1}^{m} \|x_n^{**}\| \|x_n^*\|
\]
The trace functional

- $\mathcal{N}_1(X, Y)$ is a quotient of $X^* \widehat{\otimes}_\pi Y \Rightarrow \mathcal{N}_1(X, Y)^* \hookrightarrow \mathcal{L}(Y, X^{**})$.
- $T = \sum_{n=1}^m x_n^* \otimes x_n^{**} \in \mathcal{F}(X, X^{**})$ \quad \begin{align*}
\text{trace}(T) &:= \sum_{n=1}^m \langle x_n^{**}, x_n^* \rangle \\
|\text{trace}(T)| &\leq \sum_{n=1}^m \|x_n^{**}\| \|x_n^*\|
\end{align*}
- Finite-nuclear norm of $T \in \mathcal{F}(X, X^{**})$:
  $\nu_0(T) := \inf\{\sum_{n=1}^m \|x_n^{**}\| \|x_n^*\| : T = \sum_{n=1}^m x_n^* \otimes x_n^{**}\}$
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Theorem [Grothendieck, 1955]
$X$ a Banach space. The following statements are equivalent:

1. $X$ has the AP.
2. If $(x_n) \subset X$ and $(x_n^*) \subset X^*$ are such that $\sum_{n} \|x_n^*\| \|x_n\| < \infty$ and $\sum_{n} \langle x_n^*, x \rangle x_n = 0$ for all $x \in X$, then $\sum_{n} \langle x_n^*, x \rangle x_n = 0$.

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- $\mathcal{N}_1(X, Y)$ is a quotient of $X^* \hat{\otimes}_\pi Y \Rightarrow \mathcal{N}_1(X, Y)^* \hookrightarrow \mathcal{L}(Y, X^{**})$.
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- Finite-nuclear norm of $T \in \mathcal{F}(X, X^{**})$: $\nu_0(T) := \inf\{\sum_{n=1}^{m} \|x_n^{**}\| \|x_n^*\| : T = \sum_{n=1}^{m} x_n^* \otimes x_n^{**}\}$

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- The linear map $T \in \mathcal{N}_1(X, X^{**}) \leftrightarrow \text{trace}(T) \in \mathbb{R}$ is continuous if and only if $X^*$ has the AP.
The trace functional

- $\mathcal{N}_1(X, Y)$ is a quotient of $X^* \hat{\otimes}_\pi Y \Rightarrow \mathcal{N}_1(X, Y)^* \hookrightarrow \mathcal{L}(Y, X^{**})$.
- $T = \sum_{n=1}^{m} x_n^* \otimes x_n^{**} \in \mathcal{F}(X, X^{**})$ s.t. \[
\begin{align*}
\text{trace}(T) &= \sum_{n=1}^{m} \langle x_n^{**}, x_n^* \rangle \\
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- Finite-nuclear norm of $T \in \mathcal{F}(X, X^{**})$:
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Theorem [Grothendieck, 1955]

$X$ a Banach space. The following statements are equivalent:

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- The linear map $T \in \mathcal{N}_1(X, X^{**}) \mapsto \text{trace}(T) \in \mathbb{R}$ is continuous if and only if $X^*$ has the AP.
- If $X^*$ has the AP, then $\mathcal{N}_1(X, Y)^* \simeq \mathcal{L}(Y, X^{**})$. 

J. M. Delgado
A trace characterization of the \( p \)-AP

**Proposition [Sinha and Karn, 2002]**

If \( X \) has the \( p \)-AP then the following holds:

For every \( (x_n) \in \ell_p(X) \) and every \( (x_n^*) \in \ell_1(X^*) \) such that \( \sum_n \langle x_n^*, x \rangle x_n = 0 \) for all \( x \in X \), we have \( \sum_n \langle x_n^*, x_n \rangle = 0 \).
A trace characterization of the p-AP

Proposition [Sinha and Karn, 2002]

If $X$ has the $p$-AP then the following holds:
For every $(x_n) \in \ell_p(X)$ and every $(x_n^*) \in \ell_1(X^*)$ such that
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Proposition [Oja, Piñeiro, Serrano and Delgado, 2009]

The following statements are equivalent:

1. $X$ has the $p$-AP.
2. For every relatively $p$-compact sequence $(x_n) \subset X$ and every $(x_n^*) \in \ell_1(X^*)$ such that $\sum_n \langle x_n^*, x \rangle x_n = 0$ for all $x \in X$, we have $\sum_n \langle x_n^*, x_n \rangle = 0$. 

Corollary

$X^{**}$ has the $p$-AP $\Rightarrow$ $X$ has the $p$-AP.
A trace characterization of the $p$-AP

**Proposition [Sinha and Karn, 2002]**

If $X$ has the $p$-AP then the following holds:

For every $(x_n) \in \ell_p(X)$ and every $(x_n^*) \in \ell_1(X^*)$ such that

$$\sum_n \langle x_n^*, x \rangle x_n = 0$$

for all $x \in X$, we have

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**Corollary**

$X^{**}$ has the $p$-AP $\Rightarrow$ $X$ has the $p$-AP
About the subspace structure of $\mathcal{N}_1(X, Y)^*$

\[
\mathcal{N}_p(X, X^{**}) := \left\{ T = \sum_n x_n^* \otimes x_n^{**} : (x_n^{**}) \subset X^{**} \text{ relatively } p\text{-compact} \right\}
\]

$\Pi_p(Y, X^{**}) \hookrightarrow N_1(X, Y)^*$,
About the subspace structure of $\mathcal{N}_1(X, Y)^*$

\[ \mathcal{N}_p(X, X^{**}) := \left\{ T = \sum_n x_n^* \otimes x_n^{**} : (x_n^{**}) \subset X^{**} \text{ relatively } p\text{-compact} \right\} \]

- $\mathcal{N}_p(X, X^{**}) \subset \mathcal{N}_1(X, X^{**})$
- If $X^{**}$ has the $p$-AP, the map $T \in \mathcal{N}_p(X, X^{**}) \mapsto \text{trace}(T) \in \mathbb{R}$ is well-defined, linear and continuous.
About the subspace structure of $\mathcal{N}_1(X, Y)^*$

$$\mathcal{N}_1(X, X^{**}) := \left\{ T = \sum_n x_n^* \otimes x_n^{**} : \begin{array}{c} (x_n^{**}) \subset X^{**} \text{ relatively } p\text{-compact} \\ (x_n^*) \in \ell_1(X^*) \end{array} \right\}$$

- $\mathcal{N}_1(X, X^{**}) \subset \mathcal{N}_1(X, X^{**})$
- If $X^{**}$ has the $p\text{-AP}$, the map $T \in \mathcal{N}_1(X, X^{**}) \mapsto \text{trace}(T) \in \mathbb{R}$ is well-defined, linear and continuous.

**Corollary**

$p \in [1, \infty)$.

- If $X^{**}$ has the $p\text{-AP}$, then $\Pi^d_p(Y, X^{**}) \subset \mathcal{N}_1(X, Y)^*$.
- If $Y^*$ has the $p\text{-AP}$, then $\Pi_p(Y, X^{**}) \subset \mathcal{N}_1(X, Y)^*$.
About the subspace structure of $\mathcal{N}_1(X, Y)^*$

$\mathcal{N}_{(p)}(X, X^{**}) := \left\{ T = \sum_n x_n^* \otimes x_n^{**} : (x_n^{**}) \subset X^{**} \text{ relatively } p\text{-compact} \right\}$

- $\mathcal{N}_{(p)}(X, X^{**}) \subset \mathcal{N}_1(X, X^{**})$
- If $X^{**}$ has the $p$-AP, the map $T \in \mathcal{N}_{(p)}(X, X^{**}) \mapsto \text{trace}(T) \in \mathbb{R}$ is well-defined, linear and continuous.

Corollary

$p \in [1, \infty)$.

- If $X^{**}$ has the $p$-AP, then $\Pi^d_p(Y, X^{**}) \subset \mathcal{N}_1(X, Y)^*$.
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About the subspace structure of $\mathcal{N}_1(X, Y)^*$

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\mathcal{N}_p(X, X^{**}) := \left\{ T = \sum_n x_n^* \otimes x_n^{**} : \quad \begin{aligned}
(x_n^{**}) &\subset X^{**} \text{ relatively } p\text{-compact} \\
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\end{aligned} \right\}
\]

- $\mathcal{N}_p(X, X^{**}) \subset \mathcal{N}_1(X, X^{**})$
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**Corollary**

$p \in [1, \infty)$.

- If $X^{**}$ has the $p$-AP, then $\Pi_p^d(Y, X^{**}) \subset \mathcal{N}_1(X, Y)^*$.
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\langle S, u \rangle = \quad \forall u \in \mathcal{N}_1(X, Y)
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Density of finite rank operators and the $p$-approximation property

A trace characterization of the $p$-approximation property

Open problems

About the subspace structure of $\mathcal{N}_1(X, Y)^*$

\[ \mathcal{N}_{(p)}(X, X^{**}) := \left\{ T = \sum_n x_n^* \otimes x_n^{**} : (x_n^{**}) \subset X^{**} \text{ relatively } p\text{-compact} \right\} \]

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- If $X^{**}$ has the $p$-AP, the map $T \in \mathcal{N}_{(p)}(X, X^{**}) \mapsto \text{trace}(T) \in \mathbb{R}$ is well-defined, linear and continuous.

Corollary

$p \in [1, \infty)$.

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$$\Pi_p(Y, X^{**}), \Pi^d_p(Y, X^{**}) \hookrightarrow \mathcal{N}_1(X, Y)^* \quad (p \in [1, 2]).$$
Outline

1. Introduction
2. Density of finite rank operators and the $p$-approximation property
3. A trace characterization of the $p$-approximation property
4. Open problems
The Banach ideal $\mathcal{K}_p$

- $T \in \mathcal{K}_p(X, Y)$ when $T(B_X)$ is relatively $p$-compact, i.e., if there exists $(y_n)_n \in \ell_p(Y)$ such that

$$T(B_X) \subset p\text{-co} (y_n)_n := \left\{ \sum_n a_n y_n : (a_n)_n \in B_{\ell_p'} \right\}$$  \hspace{1cm} (1)
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- $[\mathcal{K}_p, \kappa_p]$ is a Banach operator ideal.
Open problems

\[ \mathcal{F}(Y, X)^{K_p} = \mathcal{K}_p(Y, X), \text{ for all } Y \iff \text{? ?} \]
Open problems

- \( \mathcal{F}(Y, X)^{\kappa_p} = \mathcal{K}_p(Y, X) \), for all \( Y \)
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- A general definition including most of the mentioned approximation properties of order \( p \).