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Main tools
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The three main model spaces
The model spaces M_k^n

Metric spaces of bounded curvature
CAT(k) spaces
Some common properties

Fixed point
Fixed point Theorems in CAT(k) spaces

Teoremas de punto fijo en espacios métricos de curvatura acotada

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Geodesic spaces

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- (M, d) a metric space.

We say $\gamma : [a, b] \to M$ is a path in M if it is a continuos mappings.

Geodesic spaces

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- (M, d) a metric space.

We say $\gamma : [a, b] \to M$ is a path in M if it is a continuos mappings.

We call length of a path γ, $L(\gamma)$, to the supremum of the sums

$$\sum(Y) = \sum_{i=1}^{N} d(\gamma(y_{i-1}), \gamma(y_i)),$$

where Y is any partition of $[a, b]$.

Geodesic spaces

Geodesic

A geodesic path between x e $y \in M$ is a path $c : [a, b] \to M$ such that

- $c(a) = x$, $c(b) = y$
- $d(x, y) = L(c)$, i.e.,

The geodesics are path which minimize the distance between its ends.

Geodesic segment

The image α of a geodesic c is said to be a *geodesic segment* which joins x and y.

Geodesic spaces

Convexity

A subset C of M is said to be convex (D-convex) if each pair of points $x, y \in C$ (such that $d(x, y) < D$) are joined by a geodesic which image is in C.

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Convexity

A subset C of M is said to be convex (D-convex) if each pair of points $x, y \in C$ (such that $d(x, y) < D$) are joined by a geodesic which image is in C.

Geodesic spaces

A metric space M is said to be geodesic if there exists at least one geodesic joining any two points of the space. M will be said uniquely geodesic if each of these geodesics is unique (up to parametrization), i.e., each geodesic segment between each pair of points is unique.

Triangles in geodesic spaces

The main tool to develop the theory of metric spaces of bounded curvature.

Triangles in geodesic spaces

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The main tool to develop the theory of metric spaces of bounded curvature.

Geodesic triangle

A geodesic triangle $\triangle(p, q, r)$ in a metric space M consists of:

- Three points in M (the vertices of $\triangle$) and
- Three geodesic segments which join each pair of vertices.

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Model spaces

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- The Euclidean space (curvature 0)
- The Spherical space (curvature 1)
- The Hyperbolic space (curvature -1)

The Spherical space

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n-dimensional Sphere

The n-dimensional sphere $\mathbb{S}^n$ is the set of points $\{x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid (x|x) = 1\}$, where $(\cdot|\cdot)$ denote the Euclidean scalar product.

The Spherical space

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n-dimensional Sphere

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Definition of the Spherical metric

Let $d : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{R}$ be the function that assigns to each pair of points A and B in the sphere the unique real number $d(A, B) \in [0, \pi]$ such that $\mathbf{\cos(d(A, B)) = (A|B)}$.

- This new function, the Spherical distance, is a metric.

Spherical space

$(\mathbb{S}^n, d)$ is called <u>Spherical space</u> and is a geodesic metric space.

The Spherical space

Spherical triangle

Spherical triangle $\triangle$ in $\mathbb{S}^n$:

- Three different points p,q, and r in $\mathbb{S}^n$ (vertices)
- Three Spherical segments joining them pairwise.

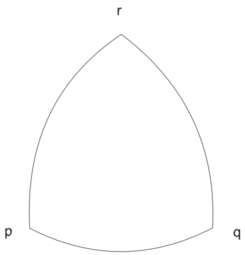

Figura: Spherical triangle

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- **Characteristics:**
  - If A and B are two points in $(\mathbb{S}^n, d)$ such that $d(A, B) < \pi$, then there exists a unique geodesic joining A to B.
  - Any open (resp. closed) ball of radius $r \leq \pi/2$ (resp. $r < \pi/2$) in $(\mathbb{S}^n, d)$ is convex.

The Hyperbolic space

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- $E^{n,1}$: vector space $\mathbb{R}^{n+1}$ endowed with the symmetric bilinear form that associates to vector u and v the real number

$$\langle u|v\rangle = -u_{n+1}v_{n+1} + \sum_{i=1}^{n} u_i v_i .$$

The upper sheet of the real hyperboloid

The upper sheet of the real hyperboloid, denoted by $\mathbb{H}^n$, is the set of points
$\{u = (u_1, \ldots, u_{n+1}) \in E^{n,1} | \langle u|u\rangle = -1 \text{ and } u_{n+1} > 0\}$.

The Hyperbolic space

- Hyperbolic metric.
  - unique non-negative number $d(A,B) \geq 0$ such that $\cosh \mathbf{d}(\mathbf{A},\mathbf{B}) = -\langle \mathbf{A}|\mathbf{B}\rangle$.
- $(\mathbb{H}^n, d)$ will be called the hyperbolic space.

Proposition

The Hyperbolic space $(\mathbb{H}^n, d)$ is a geodesic metric space.

The Hyperbolic space

Hyperbolic triangle

Hyperbolic triangle $\triangle$ in $\mathbb{H}^n$:

- Three different points p,q, and r in $\mathbb{H}^n$ (vertices)
- Three Hyperbolic segments joining them pairwise.

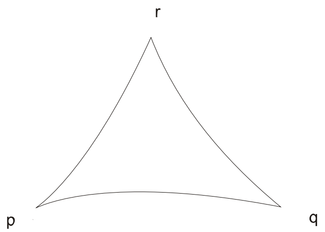

Figura: Hyperbolic triangle

The Hyperbolic space

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- **Characteristics:**
  - $\mathbb{H}^n$ is a uniquely geodesic metric space .
  - All balls in $\mathbb{H}^n$ are convex.

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The model spaces M^n_k

Let k be a real number.

Model spaces M^n_k

(1) If $k = 0$, M^n_0 is the Euclidean space $\mathbb{E}^n$;

(2) If $k > 0$, M^n_k is obtained from the Spherical space $\mathbb{S}^n$ by multiplying the distance function by $1/\sqrt{k}$;

(3) If $k < 0$, M^n_k is obtained from the Hyperbolic space $\mathbb{H}^n$ by multiplying the distance function by $1/\sqrt{-k}$.

- $\mathbb{E}^n = M^n_0$,
- $\mathbb{S}^n = M^n_1$,
- $\mathbb{H}^n = M^n_{-1}$.

Model spaces M_k^n

Properties

- M_k^n is a geodesic metric space.

(1) If $k \leq 0$
- M_k^n is uniquely geodesic.
- All balls in M_k^n are convex.

(2) If $k > 0$
- M_k^n is $\pi/\sqrt{k}$-uniquely geodesic.
- Closed balls in M_k^n of radius $r < \pi/(2\sqrt{k})$ are convex.

(3) If D_k denote the diameter of M_k^n:
$D_k = \pi/\sqrt{k}$ if $k > 0$.
$D_k = \infty$ if $k \leq 0$.

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How to compare?

Triangle of comparison in M_k^2

A comparison triangle in M_k^2 of a geodesic triangle $\triangle$ in (M, d) is a triangle in M_k^2 with vertices $\bar{p}, \bar{q}, \bar{r}$ such that $d(p, q) = d(\bar{p}, \bar{q})$, $d(q, r) = d(\bar{q}, \bar{r})$ y $d(p, r) = d(\bar{p}, \bar{r})$.

- This triangle always exits (if $k > 0$ we have to assume that $d(p, q) + d(q, r) + d(r, p) < 2D_k$).
- It is unique up to an isometry in M_k^2.
- We will denote it as $\overline{\triangle}(p, q, r)$ or $\triangle(\bar{p}, \bar{q}, \bar{r})$.

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CAT(k) inequality. Comparison axiom

CAT(k) spaces

CAT(k) inequality. Comparison axiom

- (M, d) metric space.
- k real number.
- $\triangle$ geodesic triangle in M which perimeter is less than $2D_k$.
- $\overline{\triangle} \subseteq M_k^2$ a comparison triangle for $\triangle$.

CAT(k) spaces

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CAT(k) inequality. Comparison axiom

- (M, d) metric space.
- k real number.
- $\triangle$ geodesic triangle in M which perimeter is less than $2D_k$.
- $\overline{\triangle} \subseteq M^2_k$ a comparison triangle for $\triangle$.

▶ $\triangle$ satisfy the **CAT(k) inequality** if:

$x, y \in \triangle$
$\bar{x}, \bar{y} \in \overline{\triangle}$

$$d(x, y) \leq d(\bar{x}, \bar{y}).$$

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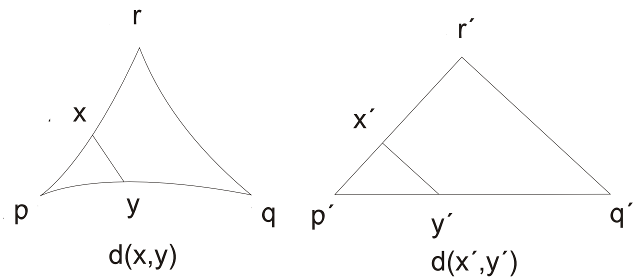

FIGURA: CAT(k) inequality

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CAT(k) SPACE

- M is a CAT(k) space for $k \leq 0$ if:
  - M is a geodesic space.
  - All its geodesic triangles satisfy the CAT(k) inequality.
- M is a CAT(k) space for $k > 0$ if:
  - M is D_k-geodesic.
  - All geodesic triangles in M of perimeter less than $2D_k$ satisfy the CAT(k) inequality.

CAT(k) spaces

Metric spaces of curvature bounded above

(M, d) is said to be of curvature $\leq k$ (or M is of curvature bounded above by k) if it is locally a CAT(k) space , i.e., if $\forall x \in M$, $\exists r_x > 0$ / $B(x, r_x)$ is a CAT(k).

CAT(k) SPACES

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METRIC SPACES OF CURVATURE BOUNDED ABOVE

(M, d) is said to be of curvature $\leq k$ (or M is of curvature bounded above by k) if it is locally a CAT(k) space , i.e., if $\forall x \in M$, $\exists r_x > 0$ / $B(x, r_x)$ is a CAT(k).

THEOREM

$CAT(k) \Rightarrow CAT(k')\ \forall k' \geq k.$

CAT(k) spaces

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Inherited properties

In a CAT(k) space,

- there exists just one geodesic segment between each pair of points (each pair of points with $d(x, y) < D_k$ when $k > 0$).
- the balls with radio $r < D_k/2$ are convex.

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Let a Hadamard space be a complete CAT(0) space.

Common properties

Hadamard and uniformly convex Banach spaces

Common properties

Hadamard and uniformly convex Banach spaces

- Closed and convex subsets are uniquely proximal,
- Decreasing sequences of bounded closed and convex subsets have nonempty intersection, and
- The spaces have normal structure (in fact, uniform normal structure).

$$\begin{cases} \text{Normal structure} & : rad(K) < diam(K) \\ \text{Uniform normal structure} & : rad(K) \leq c\,diam(K), c < 1. \end{cases}$$

Hadamard and Hilbert spaces

Common properties

Hadamard and uniformly convex Banach spaces

- Closed and convex subsets are uniquely proximal,
- Decreasing sequences of bounded closed and convex subsets have nonempty intersection, and
- The spaces have normal structure (in fact, uniform normal structure).

$$\begin{cases} \text{Normal structure} & : rad(K) < diam(K) \\ \text{Uniform normal structure} & : rad(K) \leq c\,diam(K), c < 1. \end{cases}$$

Hadamard and Hilbert spaces

- Orthogonal projection of points onto closed and convex subsets are nonexpansive.

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Mappings

Definition (Nonexpansive mapping)

A mapping $T : X \to X$ is said to be nonexpansive if $d(T(x), T(y)) \leq d(x, y)$ for all $x, y \in X$.

Definition (Uniformly L-lipschitzian mapping)

A mapping $T : X \to X$ is said to be uniformly L-lipschitzian if there exists a constant L such that $d(T^n x, T^n y) \leq L d(x, y)$ for all $x, y \in X$ and $n \in \mathbb{N}$.

CAT(0) and CAT(1)

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Theorem (W.A. Kirk (2002))

- M a Hadamard space.
- K a nonempty bounded closed and convex subset of M.
- $f : K \to K$ a nonexpansive mapping.
- ▶ f has a fixed point in K.

Theorem (W.A. Kirk(2002); Espínola,F-L(2009))

- M a complete CAT(1) space such that $diam(M) \leq \pi$
- K a nonempty closed and convex subset of M such that $\mathbf{rad}_X(K) < \pi/2$.
- $T : K \to K$ a nonexpansive mapping
- ▶ T has at least one fixed point in K.

The Lifšic Characteristic

$c - regular$ balls

Balls in X are said to be $c - regular$ if

$\forall s < c \; \exists \; \mu, \alpha \in (0,1)$ such that

- if $x, y \in X$ and $r > 0$ with $d(x,y) \geq (1-\mu)r$,
- $\exists \; z \in X$ such that

$$B(x;(1+\mu)r) \bigcap B(y;s(1+\mu)r) \subset B(z;\alpha r).$$

Definition

The Lifšic characteristic $\kappa(X)$ of X is defined as:

$$\kappa(X) = \sup\{c \geq 1 : \text{ balls in } X \text{ are c-regular}\}.$$

Fixed point in *uniformly L-lipschitzian mappings*

Theorem (E.A. Lifšic (1975))

Let (X, d) be a bounded complete metric space. Then every uniformly L-lipschitzian mapping $T : X \to X$ with $L < \kappa(X)$ has a fixed point.

- Previous conjeture of "S. Dhompongsa, W.A. Kirk, Brailey Sims - Fixed points of uniformly lipschitzian mappings":
  - Lifšic characteristic of a CAT(k) with constant curvature k for $k < 0$ is a continuous decreasing function which takes values in $(\sqrt{2}, 2)$.

Fixed point in *uniformly L-lipschitzian mappings*

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Proposition (Espínola,F-L(2009))

If $k < 0$, $\kappa(M_k^2) = \dfrac{\operatorname{arccosh}(\cosh^2 \sqrt{-k})}{\sqrt{-k}}$.

Proposition (Espínola,F-L(2009))

Let $k < 0$. *If* (X, d) *is a complete CAT*(k) *space, then* $\kappa(X) \geq \kappa(M_k^2)$.

Theorem (Espínola,F-L(2009))

Let $k < 0$. *If* (X, d) *is a bounded complete CAT*(k), *then every uniformly L-lipschitzian mapping* $T : X \to X$ *with* $L < \kappa(M_k^2)$ *has a fixed point.*

THANKS FOR YOUR ATTENTION.

- For more details :

R. Espínola and A. Fernández–León, *CAT(k)–spaces, weak convergence and fixed points*, J. Math. Anal. Appl. (1) **353** (2009), 410–427.