Representation theorems of Banach lattices and spaces of integrable functions.

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Motivation

Vector measures defined on σ-algebras have been already used for representing order continuous or σ-Fatou Banach lattices with a weak order unit as spaces of integrable functions.

The use of p-th powers of the function spaces that appear, allows to introduce p-convexity as a relevant property for obtaining more specialized representation theorems; the case of finite measure (that implies the existence of weak order unit in the space) has been already studied.

E order continuous p-convex Banach lattice with weak unit, $(1 \leq p < infty) \cong L^p(m)$

E p-convex Banach lattice σ-Fatou with weak unit in E_a $(1 \leq p < infty) \cong L^p_w(m)$

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In a similar way, vector measures defined on δ-rings have been already used for representing order continuous Banach lattices as spaces of integrable functions where the requirement of the existence of weak order unit is not needed.

In this talk, we introduce the p-th powers of such spaces to use again vector measures on δ-rings in order to prove a general representation theorem and we present concrete representations of this spaces as a Banach lattices with weak unit when further conditions are imposed to the measure.

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Banach lattices

- Our framework are Banach lattices, i.e. E is a partially ordered Banach space over the reals compatible with the algebraic structure, where each pair of elements has an infimum and a supremum (lattice structure), and the norm is a lattice norm [$\|x\| \leq \|y\|$ whenever $|x| \leq |y|$, where the absolute value $|x|$ of $x \in E$ is defined by $|x| = \sup\{x, -x\}$].
- An element $e > 0$ of a Banach lattice E is said to be a weak unit of E if $\inf\{e, x\} = 0$ implies $x = 0$.
- The norm in E is σ-order continuous if $\|x_n\| \downarrow 0$ whenever the sequence (x_n) decreases to zero in E.
- Let $E_a = \{x \in E : |x| \geq e_n \downarrow 0 \Rightarrow \|e_n\| \downarrow 0\}$ denote the largest closed ideal in E to which the restriction of the norm in E is σ-order continuous.
- A Banach lattice E is order continuous if every order bounded increasing sequence in E converges in the norm topology of X.

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- A Banach lattice E has the σ-Fatou property if for every increasing sequence $(x_n) \geq 0$ in E which is norm bounded, the element $x := \sup x_n$ exists in E and $\|x_n\| \uparrow \|x\|$.
- Let $0 < p < \infty$. The Banach lattice E is called p-convex if there exists a constant $M > 0$ such that

$$\left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \right\|_E \leq M \left(\sum_{i=1}^{n} \|x_i\|_E^p \right)^{\frac{1}{p}}$$

for all $n \in \mathbb{N}$ and every choice of vectors $\{x_i\}_{i=1}^{n}$ in E. The smallest possible value of M is called the p-convexity constant of E.

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- A Banach function space with respect to a σ-finite measure space (Ω, Σ, μ) is a Banach space E of classes of real functions which are integrable with respect to μ over sets with finite measure, satisfying

    1. If f is a measurable function, $g \in E$ and $|f| \leq |g|$ μ-a.e., then $f \in E$ and $\|f\| \leq \|g\|$,
    2. For every $A \in \Sigma$ with $\mu(A) < \infty$ the characteristic function χ_A of A belongs to E.

    where functions which are equal μ-a.e. are identified.

    Note that E is a Banach lattice with the μ-a.e. order ($f \geq 0$ if $f \geq 0 \; \mu - a.e.$) and convergence in norm of a sequence implies μ-a.e. convergence for some subsequence.

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Vector measures in δ-rings

- A δ-ring is a collection $\mathcal{R}$ of subsets of a set Ω such that
  1. if $A, B \in \mathcal{R}$, then $A \setminus B,\ A \cup B \in \mathcal{R}$
  2. $\cap A_n \in \mathcal{R}$ for all sequence (A_n) of sets in $\mathcal{R}$.
- Associated to a δ-ring there is the σ-algebra

$$\mathcal{R}^{loc} = \{A \subset \Omega : A \cap B \in \mathcal{R}, \text{ for all } B \in \mathcal{R}\}.$$

We always have $\mathcal{R} \subset \sigma(\mathcal{R}) \subset \mathcal{R}^{loc}$, so if $\mathcal{R}$ is a σ- algebra Σ, then the last inclusions are now equalities.

- A set function m defined over a δ-ring $\mathcal{R}$ and with values in a Banach space X is a vector measure if for every sequence (An) of disjoint sets in $\mathcal{R}$ such that $\cup A_n \in \mathcal{R}$, the series $\sum_n m(A_n)$ is convergent in X to $m(\cup_n A_n)$.

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Contrary to what happens with vector measures defined on σ-algebras, a vector measure defined on a δ-ring may be unbounded.

- The variation of a real measure $\mu : \mathcal{R}(\Sigma) \to \mathbb{R}$ is the measure $|\mu| : \mathcal{R}^{loc}(\Sigma) \to [0, \infty]$ defined by

$$|\mu|(A) = \sup\{\sum_1^n |\mu(A_i)| : (A_i) \text{ is a partition in } \mathcal{R} \cap 2^A\}.$$

- The semivariation of a vector measure $m : \mathcal{R}(\Sigma) \to X$ is the set function defined on $\mathcal{R}^{loc}(\Sigma)$ by

$$\|m\|(A) = \sup\{|x^* m|(A) : x^* \in B_{X^*}\},$$

where $|x^* m|$ is the variation of the measure $x^* m : \mathcal{R}(\Sigma) \to \mathbb{R}$. [We denote by B_X the unit ball of X and by X^* the topological dual of X].

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- A set $A \in \mathcal{R}^{loc}(\Sigma)$ is m-null if $\|m\|(A) = 0$. A property holds m-a.e. if it holds except on a m-null set.
- An important result of Rybakov states that if m is a vector measure defined over a σ-algebra, then there exists x_0^* in B_{X^*} such that the positive, finite measure $\mu_0 = |x_o^* m|$ and m have the same null sets (that is what we mean by a control measure for m).
- In a similar way, Brooks and Dinculeanu show that every vector measure m defined on a δ-ring has a positive but possibly unbounded measure μ_0 such that m and μ_0 have the same null sets (that is what we mean by a local control measure for m).
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The spaces $L^1_w(m)$ and $L^1(m)$ for m defined on a δ-ring

Let $\mathcal{R}$ (Σ) be a δ-ring, X a Banach space and $m : \mathcal{R}$ (Σ) $\to X$ a vector measure. Let $f : \Omega \to \mathbb{R}$ be a measurable real function on $\mathcal{R}^{loc}$ (Σ) ($f \in \mathcal{M}$).

We denote by $L^1_w(m)$ the space of functions which are integrable with respect to x^*m for all $x^* \in X^*$ (scalarly integrable functions with respect to m). Functions which are equal m-a.e. are identified. The space $L^1_w(m)$ is a Banach space with the norm

$$\|f\|_m = \sup_{x^* \in B_{X^*}} \int |f| \, d\, |x^* m|,$$

containing the $\mathcal{R}$-simple functions (an $\mathcal{R}$-simple function is a simple function with support in $\mathcal{R}$) and in which convergence in norm of a sequence implies m-a.e. convergence of some subsequence.

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Moreover, the space $L^1_w(m)$ is a Banach lattice for the m-a.e order (i.e. $f \geq 0$ if $f \geq 0$ m-a.e.) with the σ-Fatou property. Moreover, $L^1_w(m)$ is an ideal of measurable functions over the measure space $(\Omega, \mathcal{R}^{loc}, |\mu_0|)$ that is, if $f \in \mathcal{M}$ and $g \in L^1_w(m)$ with $|f| \leq |g|$ m-a.e., then $f \in L^1_w(m)$.

Remark that in the case of σ-algebras, $L^1_w(m)$ is a Banach function space with respect to $((\Omega, \Sigma, |\mu_0|))$

A function $f \in L^1_w(m)$ is integrable with respect to m if for each $A \in \mathcal{R}^{loc}(\Sigma)$ there exists an element of X, denoted by $\int_A f\, dm \in X$, such that

$$x^*(\int_A f\, dm) = \int_A f\, dx^*m, \text{ for every } x^* \in X^*.$$

Moreover, the space $L^1_w(m)$ is a Banach lattice for the m-a.e order (i.e. $f \geq 0$ if $f \geq 0$ m-a.e.) with the σ-Fatou property. Moreover, $L^1_w(m)$ is an ideal of measurable functions over the measure space $(\Omega, \mathcal{R}^{loc}, |\mu_0|)$ that is, if $f \in \mathcal{M}$ and $g \in L^1_w(m)$ with $|f| \leq |g|$ m-a.e., then $f \in L^1_w(m)$.

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- The measure $m : \mathcal{R} \to X$ is strongly additive if $(m(A_n))$ converges to zero whenever (A_n) is a disjoint sequence in $\mathcal{R}$.
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Theorem

(Delgado O.) Let $\mathcal{R}$ be a δ-ring of subsets of Ω, X a Banach space and $m : \mathcal{R} \to X$ a vector measure. The followings conditions are equivalent:

a) The measure m is strongly additive.

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The spaces $L^p_w(m)$ and $L^p(m)$ for m defined on a δ-ring, $1 < p < \infty$

Give $1 < p < \infty$, let $\mathcal{R}$ (Σ) be a δ-ring, X a Banach space and $m : \mathcal{R}\ (\Sigma) \to X$ a vector measure. Let $f : \Omega \to \mathbb{R}$ be a measurable real function on $\mathcal{R}^{loc}$ (Σ).

The function f is scalarly p-integrable with respect to m if $|f|^p$ is scalarly integrable with respect to m.

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$$\|f\|_p := \||f|^p\|_m^{1/p}, \quad f \in L^p_w(m)$$

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Theorem

(Kakutani) Any order continuous Banach lattice E can be decomposed into an unconditional direct sum of a (generally uncountable) family of mutually disjoint ideals E_α, each E_α having a weak unit x_α. More precisely, every $y \in E$ has a unique representation of the form $y = \sum_\alpha y_\alpha$ with $y_\alpha \in E_\alpha$, only countable many $y_\alpha \neq 0$ and the series converging unconditionally.

Theorem

(Curbera G.P.) Let E be an order continuous Banach lattice with weak unit. There exists a vector measure m defined on a σ-algebra, with values in E, such that the space E is order isometric to $L^1(m)$.

Theorem

(Fernández A., Mayoral F., Naranjo F., Sáez C., Sánchez Pérez E.A.) Let $1 < p < \infty$. If E is a p-convex Banach lattice with a weak unit and order continuous norm, then there exists a vector measure m defined on a σ-algebra and with values in E, such that $L^p(m)$ and E are order isomorphic.

Theorem

(Kakutani) Any order continuous Banach lattice E can be decomposed into an unconditional direct sum of a (generally uncountable) family of mutually disjoint ideals E_α, each E_α having a weak unit x_α. More precisely, every $y \in E$ has a unique representation of the form $y = \sum_\alpha y_\alpha$ with $y_\alpha \in E_\alpha$, only countable many $y_\alpha \neq 0$ and the series converging unconditionally.

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Let $\mathcal{R}$ be a δ-ring of subsets of Ω, X a Banach space and $m : \mathcal{R} \rightarrow X$ a vector measure.
Since $L^1(m)$ (with m with no further properties) is an order continuous Banach lattice, it can be represented as an unconditional direct sum of a family of disjoints ideals, each one of them having a weak unit. Moreover, each of these ideals is the space L^1 of some vector measure defined on a σ-algebra. The next results gives a concrete representation of such a decomposition.

Theorem

(Delgado O.) The space $L^1(m)$ can be decomposed into an unconditional direct sum of a family of disjoints ideals, each one of them order isomorphic and isometric to a space $L^1(m_A)$, where m_A is the vector measure restricted to a σ-algebra of the type $A \cap \mathcal{R}$ for some $A \in \mathcal{R}$.

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Theorem

(Delgado O.)

a) If m is strongly additive, then $L^1(m)$ coincides with the space $L^1(\hat{m})$ where $\hat{m}$: $\mathcal{R}^{loc} \to X$ is a vector measure which extends m.

b) The vector measure m is strongly additive if and only if $\chi_\Omega \in L^1(m)$.

c) If m is strongly additive, then $L^1(m)$ is a Banach function space with respect to the measure space $(\Omega, \mathcal{R}^{loc}, |x_0^* m|)$, where $|x_0^* m|$ is a bounded control measure for m, for a certain $x_0^* m \in B_{X^*}$.

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If m is now a σ-finite vector measure which is not strongly additive, we cannot ensure that $L^1(m)$ is a Banach function space with respect to some measure space $(\Omega, \mathcal{R}^{loc}, \lambda)$, with λ a local control measure for m. However, $L^1(m)$ is an order continuous Banach lattice having a weak unit and so there exist a measure space (S, Σ, μ) such that $L^1(m)$ is order isometric to Y (Kakutani). Even more, $L^1(m)$ is order isometric to the space $L^1(\hat{m})$ for the vector measure defined by $\hat{m}(A) = \chi_A$. The measure $\hat{m}$ is rather uncertain, like Y and the measure space (S, Σ, μ). The following result gives a representation of $L^1(m)$ as an space L^1 of a more concrete vector measure defined on the σ-algebra $\mathcal{R}^{loc}$ and with values in X.

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(Calabuig J.M., Delgado O, Juan M.A.) *The space $L^p(m)$ can be decomposed into an unconditional direct sum of a family of disjoints ideals, each one of them order isometric to a space $L^p(m_A)$, where m_A is the vector measure restricted to a σ-algebra of the type $A \cap \mathcal{R}$ for some $A \in \mathcal{R}$.*

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