Miembros del grupo de Investigación

Juán Manuel Delgado Sánchez
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Tesis Doctorales dirigidas

- Begoña Marchena González (1999): *Subconjuntos del rango de una medida vectorial*
- J. M. Delgado Sánchez (2002): *Conjuntos uniformemente sumantes de operadores*
- E. Serrano Aguilar (2005): *Conjuntos equicompactos de operadores definidos en espacios de Banach*

Líneas de investigación

- Rango de una medida vectorial
- Teoría de operadores en espacios de Banach: operadores p-sumantes, compactos, p-compactos, etc.
- Propiedad de aproximación de orden p

p-Compact Operators

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Department of Mathematics
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Outline

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1. Motivation
2. *p*-Compact sets and *p*-compact operators
3. Relationship with *p*-summing operators

D. P. Sinha, A. K. Karn, " *Compact operators whose adjoints factor through subspaces of* ℓ_p", Studia Math. 150 (2002), no. 1, 17–33.

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Theorem [Grothendieck]

Let X be a Banach space. $K \subset X$ is relatively compact iff there exists $(x_n) \in c_0(X)$ such that

$$A \subset \overline{co}(x_n)$$

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Definition [D. P. Sinha and A. K. Karn, 2002]

Let $p \geq 1$ and $1/p + 1/p' = 1$. A set $K \subset X$ is relatively p-compact if there exists $(x_n) \in \ell_p(X)$ such that

$$K \subset p\text{-co}\,(x_n) := \left\{ \sum_n a_n x_n : \ (a_n) \in B_{\ell_{p'}} \right\}.$$

Definition [D. P. Sinha and A. K. Karn, 2002]

An operator $T \in \mathcal{L}(X, Y)$ is p-compact if $T(B_X)$ is relatively p-compact, i.e., there exists $(y_n) \in \ell_p(Y)$ such that $T(B_X) \subset p\text{-co}\,(y_n)$.

$$\mathcal{K}_p(X, Y) = \{T \in \mathcal{L}(X, Y) : \ T \text{ is } p\text{-compact}\}$$

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Definition

A Banach space X has the approximation property if the identity map I_X can be approximated uniformly on every compact subset of X by finite rank operators.

- p-compact operators defined on same classes of Banach spaces: Hilbert spaces, function spaces,etc.
- Relationship with other operator ideals
- The p-approximation property of Sinha y Karn

Outline

1. Motivation
2. p-Compact sets and p-compact operators
3. Relationship with p-summing operators

Definition

If $\mathcal{L}$ denotes the class of all bounded operators between Banach spaces, we recall that a subclass $\mathcal{A}$ is called an operator ideal if the components $\mathcal{A}(X, Y) = \mathcal{A} \cap \mathcal{L}(X, Y)$ satisfy (for all Banach spaces X and Y):

- $\mathcal{A}(X, Y)$ is a vector subspace of $\mathcal{L}(X, Y)$
- $\mathcal{F}(X, Y) \subset \mathcal{A}(X, Y)$
- $S \circ T \circ R$ belongs to $\mathcal{A}(X, W)$, whenever $S \in \mathcal{L}(Z, W)$, $R \in \mathcal{L}(X, Y)$ and $T \in \mathcal{A}(Y, Z)$

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Definition

If $\mathcal{A}$ is an operator ideal, $\mathcal{A}^d$ is the dual operator ideal defined by

$$\mathcal{A}^d(X, Y) = \{T \in \mathcal{L}(X, Y) : T^* \in \mathcal{A}(Y^*, X^*)\}$$

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$$\mathcal{K}_p(X, Y) = \{T \in \mathcal{L}(X, Y) : \; T \text{ is } p\text{-compact}\}$$

- If $1 \leq p \leq q \leq \infty$, $\mathcal{K}_p(X, Y) \subset \mathcal{K}_q(X, Y)$.
- $\mathcal{K}_p$ is an operator ideal.

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Proposition [D. P. Sinha and A. K. Karn, 2002]

Let $p \geq 1$. Then:

1. $\mathcal{K}_p(X, Y) \subset \Pi_p^d(X, Y)$.
2. $\mathcal{N}_p(X, Y) \subset \mathcal{K}_p^d(X, Y) \subset \Pi_p(X, Y)$

Definition

An operator $T \in \mathcal{L}(X, Y)$ is *p-nuclear* if it admits a representation of the form $T = \sum_n x_n^* \otimes y_n$, where $(y_n) \in \ell_{p'}^w(Y)$ and $(x_n^*) \in \ell_p(X^*)$.

Definition

An operator $T \in \mathcal{L}(X, Y)$ is *p-summing* if it maps p-weakly summable sequences to absolutely p-summable sequences.

Definition

An operator $T \in \mathcal{L}(X, Y)$ is said to be quasi p-nuclear ($T \in \mathcal{QN}_p(X, Y)$) if there exists $(x_n^*) \in \ell_p(X^*)$ such that

$$\|Tx\| \leq \left(\sum_n |\langle x_n^*, x\rangle|^p\right)^{1/p}, \quad \forall x \in X.$$

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Proposition [J. M. Delgado, C. Piñeiro, E. Serrano]

1. $T(B_X) \subset p\text{-co}\,(y_n) \Leftrightarrow \|T^*y^*\| \leq \left(\sum_n |\langle y_n, y^*\rangle|^p\right)^{1/p}, \forall y^* \in Y^*.$

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Corollary

1. *If* $T \in \mathcal{K}_p(X, Y)$ *then* $T^* \in \mathcal{QN}_p(Y^*, X^*)$. $[\mathcal{K}_p \subset \mathcal{QN}_p^d]$

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2. $T^*(B_{Y^*}) \subset p\text{-co}\,(x_n^*) \Leftrightarrow \|Tx\| \leq (\sum_n |\langle x_n^*, x\rangle|^p)^{1/p}, \forall x \in X.$

Corollary

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2. $T \in \mathcal{QN}_p(X, Y)$ *iff* $T \in \mathcal{K}_p(Y^*, X^*)$. $[\mathcal{QN}_p = \mathcal{K}_p^d]$

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Theorem [Reinov (2001)]

Let $p \in [1, +\infty]$, $T \in \mathcal{L}(X, Y)$. If X^* enjoys the approximation property, then the p-nuclearity of the conjugate operator T^* implies T belongs to the space $\mathcal{N}^p(X, Y)$.

- We recall that $T \in \mathcal{N}^p(X, Y)$ if there exist sequences $(x_n^*) \in \ell_{p'}^w(X^*)$ and $(y_n) \in \ell_p(Y)$ such that T admits the representation $T = \sum_n x_n^* \otimes y_n$. Note that
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$$\mathcal{N}^p(X, Y) \subseteq \mathcal{K}_p(X, Y).$$
- The norm in this ideal will be denoted by ν^p and is defined by
$$\nu^p(T) = \inf \|(y_n)\|_p \cdot \varepsilon_{p'}(x_n^*)$$
where the infimun is taken over all possible representations in the above form of the operator T.

- If $A \subset X$ is bounded, consider the operator $U : \ell_1(A) \to X$ defined by $U(\psi_a) = \sum_{a \in A} \psi_a \cdot a$ for all $(\psi_a) \in \ell_1(A)$.

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Proposition

Let $p \in [1, +\infty]$ and $A \subset X$ bounded. The following statements are equivalent:

1. A is relatively p-compact.
2. U is p-compact.
3. U^* is p-nuclear.
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Proof. (1)⇔(2)

$$A \subset U(B_{\ell_1(A)}) \subset \overline{aco}(A).$$

Corollary 1

Let $p \in [1, +\infty]$ and $A \subset X$ bounded. A is relatively p-compact in X iff is relatively p-compact as a subset of X^{**}.

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Corollary 2

If $p \in [1, +\infty]$, then $\mathcal{QN}_p^d = \mathcal{K}_p$.

Outline

If $T \in \mathcal{K}_p(X, Y)$, we consider the natural norm

$$\kappa_p(T) = \inf \left(\sum_n \|y_n\|^p\right)^{1/p},$$

where the infimun runs over all sequences $(y_n) \in \ell_p(Y)$ satisfying

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Proposition

If X and Y are Hilbert spaces, then $\mathcal{K}_2(X, Y)$ and $\mathcal{H}S(X, Y)$ are isometric

Theorem 1

Let $T \in \mathcal{L}(X, Y)$ and $p \in [1, +\infty)$. The following statements are equivalent:

1. T is p-summing.
2. T^* maps relatively compact subsets of Y^* to relatively p-compact subsets of X^*.

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Proof. (1)⇒(2) If (y_n^*) is a null sequence in Y^*, we define $S\colon y \in Y \longrightarrow (\langle y, y_n^* \rangle) \in c_0$. Then S is ∞-nuclear, $S \circ T$ is p-nuclear and

$$\nu_p(S \circ T) \leq \nu_\infty(S)\pi_p(T) \leq \pi_p(T) \sup_n \|y_n^*\|$$

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Therefore $(S \circ T)^*$ belongs to $\mathcal{N}^p(\ell_1, X^*)$ and $\nu^p((S \circ T)^*) \leq \nu_p(S \circ T)$.It is easy to check that $\mathcal{K}_p(\ell_1, X^*)$ and $\mathcal{N}^p(\ell_1, X^*)$ are isometric. Then

$$\kappa_p((S \circ T)^*) \leq \nu_p(S \circ T) \leq \pi_p(T) \sup_n \|y_n^*\| .$$

This proves that the linear map

$$\begin{array}{rcl} U\colon c_0(Y^*) & \longrightarrow & \mathcal{K}_p(\ell_1, X^*) \\ (y_n^*) & \longmapsto & \sum_n e_n^* \otimes T^* y_n^* \end{array}$$

is well defined and $\|U\| \leq \pi_p(T)$. Notice that, in particular, we have proved that the set $\{T^* y_n^* : n \in \mathbb{N}\}$ is relatively p-compact.

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(2)⇒(1)

Proposition[R. Ryan]

$T\colon X \longrightarrow Y$ is p-summing iff there is a constant $C > 0$ such that for every finite dimensional subspace E of X and every finite codimensional subspace F of Y, the finite dimensional operator

$$q_F \circ T \circ i_E : E \longrightarrow X \longrightarrow Y \longrightarrow Y/F$$

satisfies $\pi_p(q_F \circ T \circ i_E) \leq C$. Furthermore, we have $\pi_p(T) = \inf C$, where the infimun is taken over all such pairs, E, F.

Theorem 2

Let $T \in \mathcal{L}(X, Y)$ and $p \in [1, +\infty)$. The following statements are equivalent:

1. T^* is p-summing.
2. T maps relatively compact subsets of X to relatively p-compact subsets of Y.

Theorem 2

Let $T \in \mathcal{L}(X, Y)$ and $p \in [1, +\infty)$. The following statements are equivalent:

1. T^* is p-summing.
2. T maps relatively compact subsets of X to relatively p-compact subsets of Y.

Proposition

Let $p \in [1, +\infty)$. If $S : X \to Y$ is compact and $T : Y \to Z$ has p-summing adjoint, then $T \circ S$ is p-compact and

$$\kappa_p(T \circ S) \leq \pi_p(T^*) \cdot \|S\|.$$

Definition [D. P. Sinha and A. K. Karn, 2002]

Let $p \geq 1$ and $1/p + 1/p' = 1$. A set $K \subset X$ is *relatively weakly p-compact* if there exists $(x_n) \in \ell_p^w(X)$ such that

$$K \subset p\text{-co}\,(x_n) := \left\{ \sum_n a_n x_n : \ (a_n) \in B_{\ell_{p'}} \right\}.$$

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Definition [J.M.F Castillo et al]

$A \subset X$ is said to be relatively weakly p-compact if each bounded sequence in A admits a weakly p-converging subsequence.

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A sequence (x_n) is said to be weakly p-convergent to x if the sequence $(x_n - x)$ is weakly p-summable.

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As the unit ball of ℓ_p is weakly p'-compact, it follows that the first definition is stronger.

Definition [D. P. Sinha and A. K. Karn, 2002]

An operator $T \in \mathcal{L}(X, Y)$ is *weakly p-compact* if $T(B_X)$ is relatively weakly p-compact, i.e., there exists $(y_n) \in \ell_p^w(Y)$ such that $T(B_X) \subset p\text{-co}\,(y_n)$.

$$\mathcal{W}_p(X, Y) = \{T \in \mathcal{L}(X, Y) : \ T \text{ is weakly } p\text{-compact}\}$$

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As usual, we denote by $\mathcal{V}(X, Y)$ the set of all completely continuous operators from X into Y. That is, they map weakly compact subsets of X to compact subsets of Y.

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Definition[D. P. Sinha and A. K. Karn, 2002]

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Definition [D. P. Sinha and A. K. Karn, 2002]

An operator $T \in \mathcal{L}(X, Y)$ is *weakly p-compact* if $T(B_X)$ is relatively weakly p-compact, i.e., there exists $(y_n) \in \ell_p^w(Y)$ such that $T(B_X) \subset p\text{-co}\,(y_n)$.

$$\mathcal{W}_p(X, Y) = \{T \in \mathcal{L}(X, Y) \colon \; T \text{ is weakly } p\text{-compact}\}$$

As usual, we denote by $\mathcal{V}(X, Y)$ the set of all completely continuous operators from X into Y. That is, they map weakly compact subsets of X to compact subsets of Y.

Definition[D. P. Sinha and A. K. Karn, 2002]

Let $p \geq 1$. $T \in \mathcal{V}_p(X, Y)$ if it map weakly p-compact subsets of X to p-compacts subsets of Y.

It is obvious that

$$\Pi_p(X, Y) \subset \mathcal{V}_p(X, Y).$$

Proposition

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Proposition

If Y is a Banach space isomorphic to a Hilbert space, then $\Pi_2(X, Y) = \mathcal{V}_2(X, Y)$ for every Banach space X.

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- p-compactness in classical Banach spaces

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Examples

1. $A \subset \ell_1$ is relatively 1-compact iff A is order bounded.
2. If $A \subset \ell_p$ is relatively p-compact, then A is order bounded
3. In general, the above condition is not sufficient

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- Operators mapping weakly compact sets to weakly p-compact sets