Strictly monotonic preferences on continuum of goods commodity spaces

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Motivation

This work is motivated by the lack of **examples** in the literature of strictly monotonic preferences in economies with a continuum of goods.

For example, how would we picture a strictly monotonic preference relation in the space of bounded function on the real interval [0,1]?

Are there strictly monotonic utility functions in this space?

Are there continuous ones?

Abstract

In consumption set of finite-dimensional commodity spaces, many standard examples of preference relations are strictly monotonic. It is an easy task to find examples of strictly monotonic preferences defined on subsets of infinite-dimensional commodity spaces like l^∞, where a countable number of commodities are considered.

However, it is not easy for consumption sets of continuum of goods commodity spaces.

In this note we investigate the roots of this difficulty.

The Aim

Let K denote de set of commodities. Typically, the commodity space is a subspace of $F(K)$, the space of all the functions defined on K. The preference relations are defined on the consumption set X, a subset of the commodity space.

We show that strictly monotonic preferences on X always exist.

In particular, this shows that on $B\left(K\right)$, the space of bounded functions defined on K, strictly monotonic preference relations always exist.

Our existence proof is an example inspired by the lexicographic order.
Thus, our preference has no utility representation and is discontinuous.

Can we find an example that is continuous?

Even better, can we find a continuous utility function on $B(K)$ which is strictly monotonic?

We will answer these questions **negatively;**

i) no strictly monotonic preference on $B(K)$ can be continuous in a linear topology if K is uncountable;
ii) no such preference has a utility representation.

Thus, our results completely explain the lack of examples in the literature.

The consumption set

We denote by K the set of elementary commodities.
A consumption plan is a real function defined on K that specifies an amount $f(k) \in \mathbb{R}$ of each commodity $k \in K$.

Our consumption set, or the set of alternatives, will be any set of functions X.

We use the **standard** partial order on X. Thus, if $f, g \in X$ we have that $f \geq g$ if $f(k) \geq g(k)$ for every $k \in K$ and we write $f > g$ if $f \geq g$ and $f \neq g$.

If $f \geq g$ we define the interval $[g, f] = \{h \in B; g \leq h \leq f\}$.

Notations and definitions

A preference relation $\succeq$ is a complete and transitive relation on X. Thus, for every $f, g \in X$ either $f \succeq g$ or $g \succeq f$. Moreover, if $f \succeq g \succeq h$ then $f \succeq h$.

For $f, g \in X$ we write $f \succ g$ if $f \succeq g$ but $g \not\succeq f$.

The preference relation on X is strictly monotonic if $f, g \in X$ and $f > g$ implies that $f \succ g$.

A preference relation has a utility representation (in short, is representable) if there is a function $U = U_{\succeq} : X \to \mathbb{R}$ such that $f \succeq g$ if and only if $U(f) \geq U(g)$.

This Talk

- Non-representability of strictly monotonic preferences.
- Existence of monotonic preferences.
- Non-continuity of strictly monotonic preferences.
- Final remarks.

representation

If K is countable it is quite easy to find strictly monotonic preferences on the space of bounded functions on K.

For it, let $a_n > 0$ be such that $\sum_{n=1}^{\infty} a_n < \infty$. The preference relation given by the function $U(x) = \sum_{n=1}^{\infty} a_n x_n$ is strictly monotone, continuous[1] and, obviously, representable by a utility function. However, the next theorem shows that we cannot go much further.

[1] In the weak* topology and in the norm topology a fortiori.

Zermelo's Theorem

Let K be a set and $\leq$ an order on K. We say that $\leq$ **is a well-order** in K if for every non-empty subset $A \subset K$ there exists $\min A$.

The following theorem will be quite useful.

Zermelo's Theorem : For every set K there is a well-ordering of K.

Non-representation

We denote by χ_A characteristic function of the set A. That is, $\chi_A(x) = 1$ if $x \in A$ and is 0 otherwise.

Theorem

Suppose K is uncountable and suppose that the consumption set X contains the characteristic functions χ_A, with $K \supset A$.

Then every strictly monotonic preference relation on X is non-representable.

Proof of Theorem 1

Let $\leq$ be an order of K. Suppose $\succeq$ is a strictly monotonic preference relation on X.

Suppose $U : X \to \mathbb{R}$ represents $\succeq$.
Define for $t \in K$ the functions
$x_t = \chi_{\{k \in K; k < t\}}$ and $y_t = \chi_{\{k \in K; k \leq t\}}$.
Since $y_t > x_t$ we have that $U(y_t) > U(x_t)$.
Now if $t < s$ we have that $x_s > y_t$ and therefore $U(x_s) > U(y_t)$.

In particular we conclude that the set of intervals $\{I_t; t \in K\}$ where $I_t = (U(x_t), U(y_t))$ is pairwise disjoint and uncountable.

An impossibility.

Remark

The same proof would work if instead of supposing that X contains the interval $[0, \chi_K]$ we suppose that X contains an interval $[u, v]$ where **u and v** are such that $v(k) - u(k) > 0$ for every $k \in K$.

The theorem above will have little meaning if there are no strictly monotonic preference relations on X.

We fill this gap in our next result.

Results: Existence

Let K be any set of commodities.

Theorem:

There exists a strictly monotonic preference relation on the set of functions on K.

This result implies that for any set of alternatives X contained in the set of functions on K, there exists a strictly monotonic preference relation on X.

Proof of Theorem 2

Let $\leq_K$ be a well-ordering of K. Let $x \neq y \in X$. Let $k^* = \min\{k; x(k) \neq y(k)\}$. If $x(k^*) > y(k^*)$ we define $x \succ y$. Otherwise we define $y \succ x$.

It is a simple task to verify that the so defined $\succeq$ is complete, transitive and strictly monotonic.

Remark 2

The preference relation defined above is a generalization of the lexicographic ordering.
It is discontinuous and non-representable.

Is it possible to find a strictly monotonic preference on X which is continuous in a suitable topology?
For example, if τ is such that (X, τ) is a Hausdorff topological vector space, can we find such a preference?

Our next result shows that this is not possible.

Continuous Preferences

The preference relation on X is continuous if $\succeq$ is a closed subset of $X \times X$.

The well known classical results on representation of continuous preferences are:

Given a continuous preference relation on the topological space (X, τ), we can always find a continuous representation of the preference relation whenever

(X, τ) is second countable (Debreu (1964)) or

whenever (X, τ) is separable and connected (Eilemberg (1941))

Continuous Preferences

In an infinite dimensional space the above result may not be useful as, in general, we lack the separability of the space.

Monteiro (1987) shows that a continuous preference relation on a path connected topological space (X, τ), has a continuous utility representation if and only if it is countably bounded, i.e., there is some countable subset F of X such that for all x in X there exist y and z in F with $y \succeq x \succeq z$.

An easy corollary states that any continuous preference on (X, τ) which has a best and a worst point has a continuous representation.

Continuity

Theorem:
Suppose (X, τ) is a topological vector space containing the interval $[0, \chi_K]$.
If $\succeq$ is a continuous preference relation on X then $\succeq$ restricted to $[0, \chi_K]$ has a utility representation.

Proof:
Let $\succeq' := \succeq \cap ([0, \chi_K] \times [0, \chi_K])$. That is, $\succeq'$is $\succeq$ restricted to $[0, \chi_K]$. It is therefore continuous, has a **most preferred** point, namely $\chi_K \equiv 1$ and has a **worst** point, 0. Thus, since τ is a linear topology, and $[0, \chi_K]$ is a path connected space, **the preference** has a utility representation by Corollary 2, page 150 of Monteiro (JME, 1987).

Non-Existence of Continuous Preferences

We have proved that in the space of functions on K there exist strictly monotonic preferences.

We have also proved that if the consumption set X is rich enough (the set of commodities is a continuum and X contains the characteristic functions) then the strictly monotonic preferences on X are never representable.

However, any strictly monotonic preference has both a worst and a best point in the interval $[0, 1]$. Therefore if the preference is continuous, the restriction of this preference to the interval will be representable.

A contradiction.

Final Comments 1

In non-separable metric spaces, there is always a non-representable continuous preference relation (M. Estévez and C. Hervés-Beloso. JME, 1995, page 306).

Our results complements this nicely: It is not possible to strengthen the result by requiring strict monotonicity.[2]

[2] In spaces like B of course.

Example

To make a counterpoint to the above results, let us consider the following example.

Example

There is a continuous strictly monotonic utility function on the space $l^1_+([0,1])$. For this, define $U(f) = |f|_1$.

This does not contradict our theorem since any $f \in l^1_+([0,1])$ is such that $\{t; f(t) \neq 0\}$ is countable.
In this consumption space, no agent ever consumes an uncountable set of commodities.

Conclusion

Having considered a general scenario where monotonicity is meaningful, we have proved that strictly monotonic preferences always exist. However, quite surprisingly, strict monotonicity is incompatible with continuity and utility representation if there is a continuum of goods.

It is important to highlight that our incompatibility results do not apply, for example, to the case of the Banach spaces of the class of integrable functions.
In fact, the argument that leads to the proof of our theorem requires that the characteristic functions of the intervals $[0, x)$ and $[0, x]$ are different. However, this is not the case if we consider classes of integrable functions.