Generalized Non-Quasianalytic Classes and Applications

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Index

Index

1. Non-Quasianalytic Classes $\mathcal{E}_*(\Omega)$ of Ultradifferentiable Functions
2. Generalized Non-Quasianalytic Classes $\mathcal{E}_{P,*}(\Omega)$
3. Hypoelliptic and elliptic polynomials and the growth of $\mathcal{E}_{P,*}(\Omega)$
4. Fréchet Spaces Invariant Under Differential Operators

Index

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Weight Functions

We follow the point of view of Braun-Meise-Taylor.

Definition

Let $\omega : [0,\infty[\to [0,\infty[$ *be an increasing and continuous function.* ω *is a* **non-quasianalytic weight function** *if it satifies:*

(α) $\exists K \geq 1$ *such that* $\omega(2t) \leq K(1+\omega(t))$ *for all* $t \geq 0$.

(β) $\displaystyle\int_1^\infty \frac{\omega(t)}{1+t^2}dt < \infty$.

(γ) $\displaystyle\lim_{t\to\infty} \frac{\log(1+t)}{\omega(t)} = 0$.

(δ) $\varphi : [0,\infty[\to [0,\infty[$, $\varphi(t) := \omega(e^t)$ *is convex.*

Young Conjugate

Definition

Let $\varphi : [0, \infty[\to [0, \infty[$ *be an increasing and convex function with* $\varphi(0) = 0$ *and* $\lim_{x \to \infty} \frac{x}{\varphi(x)} = 0$*. We define the* **Young conjugate** φ^* *of* φ *by*

$$\varphi^* : [0, \infty[\to [0, \infty[, \ \varphi^*(y) := \sup_{x \geq 0} \{xy - \varphi(x)\}.$$

Ultradifferentiable Functions (Braun-Meise-Taylor)

Let ω be a weight function,

- Let K be a compact subset and $\lambda > 0$, we consider the seminorm

$$p_{K,\lambda}(f) := \sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^N} \left| f^{(\alpha)}(x) \right| \exp\left(-\lambda \varphi^* \left(\frac{|\alpha|}{\lambda} \right) \right).$$

- For an open subset Ω of $\mathbb{R}^N$. We set
Beurling:

$$\mathcal{E}_{(\omega)}(\Omega) := \{ f \in C^\infty(\Omega) : \forall K \subset\subset \Omega, \forall \lambda > 0, p_{K,\lambda}(f) < \infty \}.$$

Roumieu:

$$\mathcal{E}_{\{\omega\}}(\Omega) := \{ f \in C^\infty(\Omega) : \forall K \subset\subset \Omega, \exists \lambda > 0, p_{K,\lambda}(f) < \infty \}.$$

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Example: Gevrey classes

If $\omega(t) = t^\alpha$ and $s := 1/\alpha > 1$, then

$$\mathcal{E}_{\{\omega\}}(\Omega) = G^s(\Omega).$$

$$G^s(\Omega) = \{f \in \mathcal{C}^\infty(\Omega) : \ \forall K \subset\subset \Omega \ \exists C > 0 \text{ satisfying}$$
$$\max_{x \in K} |f^{(\alpha)}(x)| \leq C^{|\alpha|+1}(\alpha!)^s\}.$$

Other Examples

- $\omega(t) = \log^{\beta}(1+t)$, $(\beta > 1)$.
- $\omega(t) = \frac{t}{(\log(e+t))^{-\beta}}$, $\beta > 1$.
- $\omega(t) = \exp(\beta(\log(1+t))^{\alpha})$, $0 < \alpha < 1$.

$$\omega(t) = \ln t \Rightarrow \mathcal{E}_{(\omega)}(\Omega) = \mathcal{C}^{\infty}(\Omega)$$

are not weight functions

$$\omega(t) = t \Rightarrow \mathcal{E}_{\{\omega\}}(\Omega) = \mathcal{A}(\Omega)$$

- Generalized Non-Quasianalytic Classes $\mathcal{E}_{P,*}(\Omega)$

Motivation: The Theorem of H.Komatsu

In 1960, **H.Komatsu**, using tools introduced by L.Hörmander, characterized when a smooth function in an open subset $\Omega \subset \mathbb{R}^N$is a real analytic function in terms of the successive iterates of a elliptic partial differential operator $P(D)$.
In particular, given a elliptic partial differential operator $P(D)$ with order m, a smooth function $f \in \mathcal{C}^\infty(\Omega)$ is real analytic if and only if for each K compact subset in Ω there exists a constant $C > 0$ such that $\forall j \in \mathbb{N}_0$,

$$\|P^j(D)f\|_{2,K} \leq C^{j+1}(j!)^m,$$

where $P^j(D)$ is the j-th iterate of $P(D)$, i.e.,

$$P^j(D) = P(D)\underbrace{\circ \cdots \circ}_{j} P(D).$$

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The Work of E. Newberger and Z.Zielezny

In 1973, **E.Newberger** and **Z.Zielezny** treated this problem in the setting of the Gevrey classes. These authors proved the following result: let $\mathcal{G}^d(\Omega)$ be the Gevrey class of exponent $d > 1$ and let $\mathcal{G}^d_P(\Omega)$ be the class of smooth functions in Ω such that for each K compact subset in Ω there exists a constant $C > 0$ such that $\forall j \in \mathbb{N}_0$,

$$\|P^j(D)f\|_{2,K} \leq C^{j+1}(j!)^d,$$

then

$$\mathcal{G}^d(\Omega) = \mathcal{G}^{md}_P(\Omega)$$

whenever P is elliptic operator with degree m.

The Work of E. Newberger and Z.Zielezny

Moreover, for P and Q hypoelliptic polynomials, it is proved the equivalence between the inequality $|Q(\xi)|^2 \leq C(1+|P(\xi)|^2)^h$, $\forall \xi \in \mathbb{R}^N$ and the inclusion $\mathcal{G}_P^d(\Omega) \subset \mathcal{G}_Q^{dh}(\Omega)$.

This research has been continued by several authors like **P.Bolley**, **J.Camus**, **L.Rodino**, **L.Zanghirati**, **Langenbruch**, **Bouzar** and **Chiali**.

The **problem of the iterates** consists in giving conditions on P in order to guarantee the equality

$$\mathcal{G}^d(\Omega) = \mathcal{G}_P^{md}(\Omega).$$

Aim

Our aim is to introduce generalized non-quasianalytic classes in a more general setting (the sense of Braun-Meise-Taylor) and study some topological properties in order to extend the results of H.Komatsu and E.Newberger-Z.Zielezny and treat the problem of the iterates.

Classes $\mathcal{E}_{P,*}(\Omega)$

Let ω a weight function and let P be a polynomial,

- For each K compact subset and $\lambda > 0$, we consider the seminorm

$$\|f\|_{K,\lambda} := \sup_{j\in\mathbb{N}_0} \|P^j(D)f\|_{2,K} \exp\left(-\lambda\varphi^*(\frac{j}{\lambda})\right)$$

  where $P^j(D)$ is the j-th iterate of $P(D)$, i.e.,

$$P^j(D) = P(D)\underbrace{\circ\cdots\circ}_{j}P(D).$$

- Let Ω be an open subset of $\mathbb{R}^N$. We set

  Beurling Case:

$$\mathcal{E}_{P,(\omega)}(\Omega) := \{f \in C^\infty(\Omega) : \forall K \subset\subset \Omega, \forall\lambda > 0, \|f\|_{K,\lambda} < \infty\}$$

  Roumieu Case:

$$\mathcal{E}_{P,\{\omega\}}(\Omega) := \{f \in C^\infty(\Omega) : \forall K \subset\subset \Omega, \exists\lambda > 0, \|f\|_{K,\lambda} < \infty\}$$

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The Spaces $\mathcal{E}_{P,*}(\Omega)$ Are Not Necessarily Complete

Example: $\mathcal{E}_{P,(\omega)}(\Omega)$ is not complete if we consider $\Omega = \mathbb{R}^2$ and $P(x,y) = x$.
Let $\{\rho_m\}_{m\in\mathbb{N}}$ be a regularizing sequence in $\mathbb{R}$. We take $g(y) \in \mathcal{C}(\mathbb{R}) \setminus \mathcal{C}^\infty(\mathbb{R})$ and we set $f_m(x,y) := (\rho_m * g)(y)$.
$\{f_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{E}_{P,(\omega)}(\mathbb{R}^2)$ which is not convergent because only g can be the corresponding limit. g is not $\mathcal{C}^\infty$-function.

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A Complete Topology for the Beurling Case

Let $\Omega \subset \mathbb{R}^N$ be an open subset and $\{K_n\}$ a compact exhaustion of Ω. We set

$$\|f\|_n := \|f\|_{K_n,n} = \sup_{j\in\mathbb{N}_0} \|P^j(D)f\|_{2,K_n} \exp\left(-n\varphi^*(\frac{j}{n})\right)$$

and we call

$$p_n(f) := \sup_{|\alpha|\le n} \sup_{x\in K_n} |f^{(\alpha)}(x)|$$

the seminorms of $\mathcal{E}(\Omega)$.
Then

$$\{\max(\|\cdot\|_n, p_m)\}_{n,m\in\mathbb{N}}$$

is a fundamental system of seminorms of $\mathcal{E}_{P,(\omega)}(\Omega)$.

A Complete Topology for the Beurling Case

Theorem $\mathcal{E}_{P,(\omega)}(\Omega)$ endowed with the topology defined by $\{\max(\|\cdot\|_n, p_m)\}_{n,m\in\mathbb{N}}$ is a Fréchet space.

A Complete Topology for the Roumieu Case

Let $n \in \mathbb{N}$ and let $K \subset \Omega$ be a compact subset of Ω, we endow $\mathcal{E}_{P,\omega}^{\frac{1}{n}}(K)$ with the topology defined by the fundamental system of

$$\{\max(\|\cdot\|_{K,\frac{1}{n}}, p_m)\}_{m\in\mathbb{N}}.$$

We know $\mathcal{E}_{P,\omega}^{\frac{1}{n}}(K)$ is a Fréchet space.
Now, we endow $\mathcal{E}_{P,\{\omega\}}(\Omega)$ with the topology

$$\mathcal{E}_{P,\{\omega\}}(\Omega) = \underset{K\subset\subset\Omega}{\underset{\longleftarrow}{\operatorname{proj}}}\ \underset{n\in\mathbb{N}}{\underset{\longrightarrow}{\operatorname{ind}}}\ \mathcal{E}_{P,\omega}^{\frac{1}{n}}(K).$$

Note $\underset{n\in\mathbb{N}}{\underset{\longrightarrow}{\operatorname{ind}}}\ \mathcal{E}_{P,\omega}^{\frac{1}{n}}(K)$ is a LF-space.

A Complete Topology for the Roumieu Case

Theorem $\mathcal{E}_{P,\{\omega\}}(\Omega)$ endowed with the previous topology is complete.

Characterization of completeness

Let $\mathcal{E}_{P,*}(\Omega)$ endowed with the topology defined by

$$\|f\|_{K,\lambda} := \sup_{j\in\mathbb{N}_0} \|P^j(D)f\|_{2,K} \exp\left(-\lambda\varphi^*(\frac{j}{\lambda})\right).$$

Theorem $\mathcal{E}_{P,*}(\Omega)$ is complete $\iff$ P is hypoelliptic.

A Payley-Wiener Theorem for $\mathcal{E}_{P,*}(\Omega)$

Theorem. Let P be a hypoelliptic polynomial and ω a weight function. Then, the Fourier-Laplace transform of a function in $\mathcal{D}_{P,(\omega)}(\mathbb{R}^N)$ verifies

$$|\widehat{f}(z)| \leq Ce^{A|z|} \ \forall z \in \mathbb{C}^N$$

for some constants $C, A > 0$ and for every $\lambda > 0$,

$$\left(\int_{\mathbb{R}^N} |\widehat{f}(x)|^2 \exp(\lambda\omega(|P(x))|)dx\right)^{\frac{1}{2}} < \infty.$$

Conversely, every entire function satisfying the above conditions is the Fourier-Laplace transform of a function in $\mathcal{D}_{P,(\omega)}(\mathbb{R}^N)$.

Corollary. Let P be a hypoelliptic polynomial and ω a weight function. Then,

- A $\mathcal{C}^\infty$-function with compact support in Ω belongs to $\mathcal{D}_{P,(\omega)}(\Omega)$

$$\iff \forall \lambda > 0, \ \left(\int_{\mathbb{R}^N} |\widehat{f}(x)|^2 \exp(\lambda\omega(|P(x))|)dx \right)^{\frac{1}{2}} < \infty.$$

- $\mathcal{D}_{P,(\omega)}(\Omega)$ and $\mathcal{E}_{P,(\omega)}(\Omega)$ are nuclear.

Corollary. If we consider the hypoelliptic heat polynomial in two variables, $P(t,x) = it + x^2$, and Gevrey weights $\omega(t) = t^a$ for $a \in]0, \frac{1}{2}]$, then $\mathcal{D}_{P,*}(\Omega)$ is an algebra.

Corollary. Let P be a hypoelliptic polynomial and ω a weight function. Then,

- A $\mathcal{C}^\infty$-function with compact support in Ω belongs to $\mathcal{D}_{P,(\omega)}(\Omega)$

$$\Longleftrightarrow \forall\lambda > 0, \ \left(\int_{\mathbb{R}^N} |\widehat{f}(x)|^2 \exp(\lambda\omega(|P(x))|)dx\right)^{\frac{1}{2}} < \infty.$$

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- Hypoelliptic and elliptic polynomials and the growth of $\mathcal{E}_{P,*}(\Omega)$

Theorem Let $\Omega \subset \mathbb{R}^N$ be an open subset of $\mathbb{R}^N$. For a weight function ω and a polynomial P with degree m, the inclusion

$$\mathcal{E}_{*(t)}(\Omega) \subseteq \mathcal{E}_{P,*(t^{\frac{1}{m}})}(\Omega)$$

holds and the inclusion map is continuous.
Moreover, the inclusion map has dense range.

Hörmander's Theorem

Let P be a hypoelliptic polynomial and let Q be any polynomial. Then, there are constants $h > 0$ and $C > 0$ such that

$$|Q(\xi)|^2 \leq C(1 + |P(\xi)|^2)^h, \; \forall \xi \in \mathbb{R}^N.$$

Moreover, we can take an smaller h which is a rational number.

Extending the Results of E.Newberger-Z.Zielezny

Theorem Let P be a hypoelliptic polynomial, Q any polynomial, Ω an open subset of $\mathbb{R}^N$ and ω a weight function, then

- $\exists m_0$ such that $m \geq m_0$ implies

$$\mathcal{E}_{P,*(t^{\frac{1}{m}})}(\Omega) \subseteq \mathcal{E}_{Q,*(t^{\frac{1}{mh}})}(\Omega)$$

  with inclusion map continuous.

- If $\exists h \geq 1$ such that $\mathcal{E}_{P,*(t)}(\Omega) \subseteq \mathcal{E}_{Q,*(t^{\frac{1}{h}})}(\Omega)$, then

$$|Q(\xi)|^2 \leq C(1 + |P(\xi)|^2)^h.$$

whenever ω verifies a growth condition of Bonet-Meise-Melikhov:

$$\exists H \geq 1 \text{ such that } \forall t \geq 0,\ 2\omega(t) \leq \omega(Ht) + H.$$

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Extending the Theorem of Komatsu

Theorem Let P be a elliptic polynomial, then

$$\mathcal{E}_{P,*(t^{\frac{1}{m}})}(\Omega) \subseteq \mathcal{E}_{*(t)}(\Omega)$$

and the inclusion map is continuous.
As a consequence, if P is elliptic then $\mathcal{E}_{P,*(t^{\frac{1}{m}})}(\Omega) = \mathcal{E}_{*(t)}(\Omega)$ holds algebraically and topologically.

Extending the Theorem of Komatsu

Theorem Let P be a elliptic polynomial, then

$$\mathcal{E}_{P,*(t^{\frac{1}{m}})}(\Omega) \subseteq \mathcal{E}_{*(t)}(\Omega)$$

and the inclusion map is continuous.
As a consequence, if P is elliptic then $\mathcal{E}_{P,*(t^{\frac{1}{m}})}(\Omega) = \mathcal{E}_{*(t)}(\Omega)$ holds algebraically and topologically.

Problem of the iterates

Theorem Let ω be a weight function verifying the property B-M-M. Given P polynomial with degree m,

$$\mathcal{E}_{P,*(t^{\frac{1}{m}})}(\Omega) = \mathcal{E}_{*(t)}(\Omega) \text{ algebraically} \Longrightarrow P \text{ elliptic.}$$

As a consequence,

$$\mathcal{E}_{P,*(t^{\frac{1}{m}})}(\Omega) = \mathcal{E}_{*(t)}(\Omega) \text{ algebraically} \Longleftrightarrow P \text{ elliptic.}$$

In this case the equality $\mathcal{E}_{P,*(t^{\frac{1}{m}})}(\Omega) = \mathcal{E}_{*(t)}(\Omega)$ is also topological.

- Fréchet Spaces Under Differential Operators

Motivation

Theorem. (Langenbruch-Voigt, 2.000) If E is a Fréchet space continuously included in $\mathcal{D}'(\Omega)$ and E is stable for every partial differential operator, i.e., $P(D)E \subset E$ for every partial differential operator, then E is continuously included in $\mathcal{C}^\infty(\Omega)$.

It is enough that the Fréchet space E is stable under a single hypoelliptic differential operator P(D).

Example

¿E Fréchet included in $\mathcal{D}'_{(\omega)} \Longrightarrow E \subset \mathcal{E}_{(\omega)}$?

This fact is not true for non-quasianalytic classes: the space of ultradistributions of Roumieu type, $\mathcal{D}'_{\{\omega\}}$, is a Fréchet space included in $\mathcal{D}'_{(\omega)}$ which is stable under partial differential operators and clearly $\mathcal{D}'_{\{\omega\}} \subsetneq \mathcal{C}^{\infty}, \mathcal{E}_{(\omega)}$.

Differential Operators of Infinite Order

Let $G \in \mathcal{H}(\mathbb{C}^N)$ such that $\log|G(z)| = \mathcal{O}(\omega(|z|))$, i.e, $\ln|G(z)| \leq C(1+\omega(z))$. Then, given $\varphi \in \mathcal{E}_{(\omega)}(\mathbb{R}^N)$

$$T_G(\varphi) := \sum_{\alpha\in\mathbb{N}_0^N} (-i)^{|\alpha|} \frac{G^{(\alpha)}(0)}{\alpha!} \varphi^{(\alpha)}(0)$$

defines an ultradistribution $T_G \in \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$. The operator

$$G(D) : \mathcal{D}'_{(\omega)}(\mathbb{R}^N) \to \mathcal{D}'_{(\omega)}(\mathbb{R}^N), \qquad G(D)\nu := T_G * \nu$$

is called an **ultradifferential operator of class** (ω). We note that, for every $f \in \mathcal{E}_{(\omega)}(\mathbb{R}^N)$,

$$(G(D)f)(x) = \sum_{\alpha\in\mathbb{N}_0^N} (i)^{|\alpha|} \frac{G^{(\alpha)}(0)}{\alpha!} f^{(\alpha)}(x).$$

Definition. An ultradifferential operator $G(D)$ is called

- **(ω)-hypoelliptic** if $G(D)f \in \mathcal{E}_{(\omega)}(\mathbb{R}^N) \Longrightarrow f \in \mathcal{E}_{(\omega)}(\mathbb{R}^N)$.
- **strongly (ω)-hypoelliptic** if there is a constant $C > 0$ such that the entire function G satisfies $C\omega(x) \leq \log|G(x)|\,,\ x \in \mathbb{R}^N$.
- **elliptic** if $G(D)f \in \mathcal{A}(\mathbb{R}^N) \Longrightarrow f \in \mathcal{A}(\mathbb{R}^N)$

Theorem. Let E be a Fréchet space which is continuously included in $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ and such that $G(D)E \subset E$ for some strongly (ω)-hypoelliptic ultradifferential operator $G(D)$ of class (ω). Then $E \subset \mathcal{E}_{(\omega)}(\mathbb{R}^N)$ with continuous inclusion.

(ω,P)-stability

Definition. Let E be a Fréchet space such that $E \subset \mathcal{D}'_{(\omega)}(\mathbb{R}^d)$ with continuous inclusion and let $P(D)$ be a differential operator of degree m. Then E is said to be $(\omega, P(D))-$stable if $P(D)E \subset E$ and, moreover, for every $k \in \mathbb{N}$, the sequence of operators

$$P^j(D)e^{-k\varphi^*(m\frac{j}{k})} : E \longrightarrow E$$

is equicontinuous.

In the limit case $\omega(t) = \log(1+t)$, $(\omega, P(D))-$stable simply means that $P(D)E \subset E$.

Theorem. Let $P(D)$ be an elliptic differential operator of degree m such that its principal part has real coefficients. If the Fréchet space $E \subset \mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ is $(\omega, P(D))-$stable then $E \subset \mathcal{E}_{(\omega)}(\mathbb{R}^N)$ with continuous inclusion.

Theorem. We assume that the hypoelliptic differential operator $P(D)$ of degree m has real coefficients. There is a weight function $\lambda(t) := \omega(t^r)$ where the constant $0 < r < 1$ only depends on P such that if $\lim_{t\to\infty} \frac{\omega(t)}{t^r} = 0$ and the Fréchet space $E \subset \mathcal{D}'_{(\lambda)}(\mathbb{R}^N)$ is $(\omega, P(D))-$stable then $E \subset \mathcal{E}_{(\lambda)}(\mathbb{R}^N)$ with continuous inclusion.

Corollary. Let $\omega(t) = \log^{\beta}(1+t)$, $\beta > 1$, be given and $P(D)$ a hypoelliptic differential operator with real coefficients and degree m. Then $\mathcal{E}_{P,(\omega(t^{\frac{1}{m}}))}(\mathbb{R}^N) = \mathcal{E}_{(\omega(t))}(\mathbb{R}^N)$.

The weight $\omega(t) = \log^{\beta}(1+t)$, $\beta > 1$, **does not satisfy the property B-M-M**.