Some constructions of non-separable $\mathcal{L}_\infty$ spaces

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The aim of this talk is to present some recent constructions of *"non-trivial"* non-separable $\mathcal{L}_\infty$-spaces, and discuss about future perspectives.

Definition

Recall that a Banach space space X is called $\mathcal{L}_{\infty,\lambda}$ (for $\lambda > 1$) if for every finite dimensional subspace F of X there is some subspace G of X containing F and such that $d(G, \ell_\infty^{\dim G}) \leq \lambda$.

Typical examples of $\mathcal{L}_\infty$ spaces are c_0 and $C(K)$. Not so well known example is the *Gurarij* space $\mathfrak{G}$, characterized isometrically by the following properties:

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$\mathfrak{G}$ is separable and for every $\varepsilon > 0$, every pair $F \subseteq G$ of f.d. spaces and isometry $T : F \to \mathfrak{G}$ there is $U : G \to \mathfrak{G}$ such that $U \upharpoonright F = T$ and $(1-\varepsilon)\|x\| \leq \|U(x)\| \leq (1+\varepsilon)\|x\|$.

Note that the Gurarij space is universal (almost-isometrically) for separable Banach spaces.

Each of the examples mentioned above contain copies of c_0, but this is not always the case.

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By “non-trivial” we mean:

- In the separable structure:Not containing c_0,having the RNP, having the RNP and being Asplund space.
- In the non-separable structure:Not having nice renorming (as for example, a renorming with the Mazur intersection property (MIP)), or not having uncountable biorthogonal sequences.

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Theorem (1980)

There is a separable Asplund $\mathcal{L}_\infty$ space with the RNP and reflexively saturated.

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There is a separable $\mathcal{L}_\infty$ space with the RNP and the Schur property.

Both examples are the result of a parametrized construction of a direct limit of a direct (indeed linear) system of ℓ_∞^n's and isomorphism between them. The key is to take into account the natural projections between $\ell_\infty^m \subseteq \ell_\infty^n$.

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Kunen and Shelah spaces

Kunen space is a $C(K)$ (consequently, a $\mathcal{L}_{\infty,1+}$-space) with the following properties:

1. $C(K)$ is non-separable and Asplund (i.e. K is non-metrizable and scattered).
2. $(C(K), w)^n$ is hereditarily Lindelöf (HL) for every integer n. Consequently, $C(K)$ cannot be renormed to have the MIP. This space is built with the extra help of the Continuum Hypothesis.

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Theorem (LA-Todorcevic 2010)

1. *There are $X \subseteq Y$ non-separable such that:*
   1. *X is Asplund and c_0-sat., Y is Gurarij and $Y/\mathfrak{G} \equiv X$.*
   2. *Both $(X, w)^n$ and $(Y, w)^n$ are HL for every integer n.*
   3. *Both X and Y have no supported sets.*
2. *A pair X and Y related as above such that X have uncountable fundamental ε-biorthogonal sequences for every $\varepsilon > 0$ but no uncountable biorthogonal sequences.*
3. *A non-separable space X with uncountable ε-Schauder basic sequences for every $\varepsilon > 0$ but no uncountable monotone basic sequences.*

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4 All of the spaces above have few operators.

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For every non-separable Banach space X of density $\aleph_1$ there is a $Y \supseteq X$ such that Y/X has both the Schur and the RNP.

Ingredients of the proof:

1. Follow the Bourgain-Pisier construction.
2. Now step-up the B-P construction to $\aleph_1$ by using the following combinatorial property of $\aleph_1$:

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Inductive family

There is a family $\mathcal{F}$ consisting on finite subsets of ω_1 such that:

1. $\mathcal{F}$ is cofinal (i.e. for every $s \subseteq \omega_1$ finite there is $t \in \mathcal{F}$ such that $s \subseteq t$).
2. For every $s \in \mathcal{F}$ there is a total ordering $\preceq_s$ on $\mathcal{F} \restriction s := \{t \in \mathcal{F} \,:\, t \subseteq s\}$ such that
   1. $\preceq_s$ extends the inclusion.
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Hint: use the existence of such families in $\aleph_1$ to step-up now the Bourgain-Delbaen construction.

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Is there a non-separable Asplund $\mathcal{L}_\infty$-space with the RNP and without nice renorming?

Hint: Use our approach to build non-trivial $\mathcal{L}_\infty$ spaces together with the Bourgain-Delbaen construction. Note that such space (if exists) would be the first example of an Asplund space without smooth bump functions (Based on a work of Deville-Godefroy and Zizler)

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