A Quantitative Stability Analysis in Optimization

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1. Introduction and preliminaries

1.1. Parameterized convex optimization problems

We consider convex optimization problems in the form:

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\begin{align*}
\text{Inf} & \quad f(x) + \langle c^*, x \rangle \\
\text{s. t.} & \quad g_t(x) \leq b_t, \quad t \in T,
\end{align*}
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We consider convex optimization problems in the form:

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subject to

$$g_t(x) \leq b_t, \quad t \in T,$$

and their linear counterpart

$$f \equiv 0, \quad g_t(x) = \langle a_t^*, x \rangle.$$
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- \(T\) is the index set.
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Example (a LSIP problem, with $X = \mathbb{R}^2$)

$$\pi : \inf x_1$$

s. t. $- t^{-2} x_1 - x_2 \leq -2t^{-1}$, $t \in T := ]0, +\infty[.$
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Parameter spaces in this talk

Continuous case: $X = \mathbb{R}^n$, $T$ compact Hausdorff,

$$P(c, b) : \quad \inf f(x) + \langle c, x \rangle$$

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$$ (t,x) \mapsto g_t(x) \text{ continuous on } T \times \mathbb{R}^n.$$
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- Parameter $(c, b) \in \mathbb{R}^n \times C(T, \mathbb{R})$, 
  $$\|(c, b)\| = \max \{\|c\|, \|b\|\}, \quad \|b\| = \max_{t \in T} |b_t|.$$
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\]

- Feasible/Optimal set mapping, \( \mathcal{F} : C (T, \mathbb{R}) \Rightarrow \mathbb{R}^n \), \( S : \mathbb{R}^n \times C (T, \mathbb{R}) \Rightarrow \mathbb{R}^n \),

\[
\mathcal{F} (b) := \{ x \in \mathbb{R}^n : g_t (x) \leq b_t, \ \text{for all} \ t \in T \}.
\]

\[
S (c, b) := \arg \min \{ f (x) + \langle c, x \rangle, \ x \in \mathcal{F} (b) \}.
\]
General case: $X$ arbitrary Banach, $T$ arbitrary
Parameterized convex infinite inequality system

$$\sigma(p) := \{g_t(x) \leq p_t, \ t \in T\},$$
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- $g_t: X \to \mathbb{R} \cup \{\infty\}, \ t \in T$, are proper lsc convex functs.,
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- In the linear case, $g_t(x) = \langle a^*_t, x \rangle - b_t$, with $\{(a^*_t, b_t)\}_{t \in T}$ arbitrarily given set in $X^* \times \mathbb{R}$. 
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- Parameter $p = (p_t)_{t \in T} \in l_{\infty}(T)$, $\|p\| = \sup_{t \in T}|p_t|$.
- Feasible set mapping, $\mathcal{F} : l_{\infty}(T) \rightrightarrows X$,

$$\mathcal{F}(p) := \{x \in X : g_t(x) \leq p_t, \, \text{for all } t \in T\}.$$
### 1.2. Stability criteria and direct antecedents

<table>
<thead>
<tr>
<th>Condition</th>
<th>Description</th>
<th>Source(s)</th>
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<td>( lsc ) and/or ( usc ) of ( F ) (feasible set map.) and ( S ) (optimal set map.)</td>
<td>( X = \mathbb{R}^n )</td>
<td>( X ) arbitrary</td>
</tr>
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<td>( \text{Bro}(84), \text{Fis}(83), \text{GoLoTo} (96,97) )</td>
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<tr>
<td>( \text{GoLo} (98) )</td>
<td>( \text{CaLoPaTo} (99) )</td>
<td>( \text{CaLoPa}(02) )</td>
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</tbody>
</table>

2. SIOPT 18, 717-732
3. SVAN 16, 511-538
4. Submitted
Definition

Let $X, Y$ be metric spaces. $M : X \ni \Rightarrow Y$ is **Lipschitz-like** (Aubin, pseudo-Lipschitz) at $(\Rightarrow around) \ (\bar{x}, \bar{y}) \in \text{gph}M$ if $\exists U$ neigh. of $\bar{x}$, $\exists V$ neigh. of $\bar{y}$, $\exists \kappa \geq 0$ s.t.

$$d \left( y, M \left( x \right) \right) \leq \kappa d \left( x, x' \right) \ \forall x, x' \in U, \ \forall y \in M \left( x' \right) \cap V.$$
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$\text{lip} M(\bar{x}, \bar{y})$, the exact Lipschitzian bound of $M$ at $(\bar{x}, \bar{y})$, is the infimum of such a $\kappa$'s.
Definition

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$$d(y, \mathcal{M}(x)) \leq \kappa d(x, x') \ \forall x, x' \in U, \forall y \in \mathcal{M}(x') \cap V.$$ 

$\text{lip } \mathcal{M}(\bar{x}, \bar{y})$, the exact Lipschitzian bound of $\mathcal{M}$ at $(\bar{x}, \bar{y})$, is the infimum of such a $\kappa$'s.
Definition (see Mordukhovich (2006))

$X, Y$ Banach spaces, $\mathcal{M} : X \rightrightarrows Y$. The **coderivative** of $\mathcal{M}$ at $(\bar{x}, \bar{y}) \in \text{gph}\mathcal{M}$, $D^*\mathcal{M}(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ is given by

$$D^*\mathcal{M}(\bar{x}, \bar{y})(y^*) := \left\{ x^* \in X^* \mid (x^*, -y^*) \in N((\bar{x}, \bar{y}); \text{gph}\mathcal{M}) \right\},$$

where $N(\cdot; \Omega)$ is the basic, or limiting, or Mordukhovich normal cone (the usual normal cone of convex analysis if $\text{gph}\mathcal{M}$ is locally convex around $(\bar{x}, \bar{y})$).
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Illustration in finite dimension:

- $Y = Y^*$
- $(x^*, -y^*)$
- $\mathcal{M}$
- $X = X^*$
Example (Coderivatives of smooth functions in $\mathbb{R}^n$)

The coderivative of a smooth (single-valued) function $F : \mathbb{R}^n \to \mathbb{R}^m$, $D^* F (\bar{x}, \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, is given by

$$D^* F (\bar{x}, \bar{y}) (y) = \nabla F (\bar{x})^* y, \text{ for all } y \in \mathbb{R}^m,$$

where $\nabla F (\bar{x})^*$ represents the adjoint (transpose) of the Jacobian.
Theorem (Mordukhovich (1993))

Let $X$ and $Y$ be finite-dimensional Banach spaces, and $M : X \rightrightarrows Y$ closed-graph mapping. Then $M$ is Lipschitz-like at $(\bar{x}, \bar{y}) \in \text{gph} M$ if and only if

$$D^* M(\bar{x}, \bar{y})(0) = \{0\}.$$ 

Moreover

$$\text{lip } M(\bar{x}, \bar{y}) = \|D^* M(\bar{x}, \bar{y})\|,$$

where

$$\|D^* M(\bar{x}, \bar{y})\| := \sup \left\{ \|x^*\| \mid x^* \in D^* M(\bar{x}, \bar{y})(y^*), \|y^*\| \leq 1 \right\}.$$
The situation is more involved in an infinite dimensional setting;
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See [Mordukhovich (2006), Th. 4.10] for the infinite dimensional counterpart, assuming that $X$ and $Y$ are Asplund spaces;
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Recall that we consider the feasible set mapping

$\mathcal{F} : l_\infty (T) \rightrightarrows X,$

$$\mathcal{F} (p) := \{ x \in X \mid \langle a_t^*, x \rangle \leq b_t + p_t , t \in T \} ;$$
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Observe that if $T$ is infinite, $l_\infty(T)$ is never Asplund.
1.3. Main goals

In terms of the nominal elements $\bar{x}$ and $\bar{p}$, not involving points and parameters in a neighborhood, we:
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1.3. Main goals

In terms of the nominal elements $\bar{x}$ and $\bar{p}$, not involving points and parameters in a neighborhood, we:

- Give a sufficient condition for the Lipschitz-like property of $S$ in continuous CSIP (characterization in continuous LSIP)
- Determine $D^* F (\bar{p}, \bar{x})$ and $\|D^* F (\bar{p}, \bar{x})\|$ in LIP
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- To answer the questions:
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- Give a sufficient condition for the Lipschitz-like property of \( S \) in continuous CSIP (characterization in continuous LSIP)
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- Determine \( \text{lip} \ F (\bar{p}, \bar{x}) \) in LIP.
- To answer the questions:

\[ F \text{ Lipschitz-like at } (\bar{p}, \bar{x}) \iff D^* F (\bar{p}, \bar{x}) (0) = \{0\} \]
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- To answer the questions:

  $F$ Lipschitz-like at $(\bar{p}, \bar{x}) \Leftrightarrow D^* F (\bar{p}, \bar{x}) (0) = \{0\}$?

  $\text{lip } F (\bar{p}, \bar{x}) = \|D^* F (\bar{p}, \bar{x})\|$?
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$$\text{lip} F (\bar{p}, \bar{x}) = \|D^* F (\bar{p}, \bar{x})\|?$$
1.4. Preliminary results

Lemma (extended Farkas lemma, [Dinh,Goberna, López (2006)])

Let $p \in \text{dom} \mathcal{F}$, and let $(x^*, \alpha) \in X^* \times \mathbb{R}$. The following are equivalent:

(i) The inequality $\langle x^*, x \rangle \leq \alpha$ is a consequence of $\mathcal{F}(p)$, i.e.,

$$[\langle a^*_t, x \rangle \leq b_t + p_t \text{ for all } t \in T] \implies [\langle x^*, x \rangle \leq \alpha].$$

(ii) The pair $(x^*, \alpha)$ satisfies the inclusion

$$(x^*, \alpha) \in \text{cl}^* \text{cone} \{ (a^*_t, b_t + p_t) \mid t \in T \} \cup \{ (0,1) \}, \text{ with } 0 \in X^*.$$
Lemma (Dinh, Goberna, López (2008))

Let \( p \in \text{dom} \mathcal{F} \). Then the following properties are equivalent:

(i) \( \mathcal{F} \) is Lipschitz-like at \( (p, x) \) for all \( x \in \mathcal{F}(p) \);
Lemma (Dinh, Goberna, López (2008))

Let \( p \in \text{dom} \mathcal{F} \). Then the following properties are equivalent:

(i) \( \mathcal{F} \) is Lipschitz-like at \((p, x)\) for all \( x \in \mathcal{F}(p) \);

(ii) \((0, 0) \notin \text{cl}^* \mathcal{C}(p)\), where

\[
\mathcal{C}(p) := \text{co}\{(a_t^*, b_t + p_t) \mid t \in T\};
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Lemma (Dinh, Goberna, López (2008))

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\[
\mathcal{C}(p) := \text{co}\{(a_t^*, b_t + p_t) \mid t \in T\};
\]

(iii) \( p \in \text{int}(\text{dom}\mathcal{F}) \);

(iv) \( \mathcal{F} \) satisfies the strong Slater condition (SSC) at \( p \)
(i.e., \( \exists \hat{x} \in X \mid \sup_{t \in T} [\langle a_t^*, \hat{x} \rangle - b_t - p_t] < 0 \)).
2.1 Lipschitz-like property of $S$. A motivating example:
Nominal problem:

$$
\pi : \quad \inf x_1 \\
\text{s.t.} \quad -x_1 + x_2 \leq 0, \quad -x_1 - x_2 \leq 0, \quad -x_1 \leq 0.
$$

Perturbed problems:

$$
\pi_r (\pi'_r) : \quad \inf x_1 \pm (1/r^2)x_2 \\
\text{s.t.} \quad -x_1 + x_2 \leq 0, \quad -x_1 - x_2 \leq 0, \quad -x_1 \leq -1/r.
$$
2.1. Lipschitz-like property of $S$ in LSIP/CSIP

$S ((1,0),(0,0,0))$

$S ((1,1/r^2),(0,0,1/r))$

$S ((1,-1/r^2),(0,0,1/r))$

$S ((1,0),(0,0,0))$

$S ((1,1/r^2),(0,0,1/r))$

$S ((1,-1/r^2),(0,0,1/r))$
A Karush-Kuhn-Tucker (KKT) type condition

- Set of active indices: $T_b(x) := \{ t \in T \mid \langle a_t, x \rangle = b_t \}$
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A Karush-Kuhn-Tucker (KKT) type condition

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- $A_b(x) := \text{cone}(\{a_t, t \in T_b(x)\})$ (conical convex hull)
- KKT optimality condition (see Goberna and López (1998)):

$$x \in \mathcal{F}(b) \text{ and } -c \in A_b(x) \quad \Rightarrow \quad x \in S(c, b)$$

**Definition**

The *Nürnberger condition* (NC) is satisfied at $(\bar{\pi}, \bar{x}) \in \text{gph}(S)$ if

$\text{SSC}$ holds at $\bar{b}$ and there is no $D \subset T_{\bar{b}}(\bar{x})$

with $|D| < n$ such that $-\bar{c} \in \text{cone}(\{a_t, t \in D\})$.
2.1. Lipschitz-like property of $S$ in LSIP/CSIP

Theorem (see [CKLP07, Th. 16])

Let $(c, b), x) \in 2 gph(S).$ The following conditions are equivalent:

(i) $S$ is Lipschitz-like at $(c, b), x);$ 
(ii) NC is satisfied at $(c, b), x) 2 gph(S).$

Exa. $S$ not Lips.-like at $((c, b), x), \text{not NC}$

Exa. $S$ Lips.-like at $((c, b), x, \text{NC}$

$x = (0, 0), b = (0, 0, 0), T_b(x) = \{1, 2, 3\}$

$-c = (1, 0) \in \text{cone}\{a_3\}$

$t = 3$

$t = 1$

$t = 2$

$F((0, 0, 0))$

$T_b(x) = \{1, 2, 3\}, c = (1, 1/2)$

$-c \notin \text{cone}\{a_1\} \cup \text{cone}\{a_2\} \cup \text{cone}\{a_3\}$
Exa. $S$ not Lips.-like at $((c, b), x)$, not NC

Exa. $S$ Lips.-like at $((c, b), x)$, NC

Theorem (see [CKLP07, Th. 16])

Let $((\bar{c}, \bar{b}), \bar{x}) \in \text{gph}(S)$. The following conditions are equivalent:

(i) $S$ is Lipschitz-like at $((\bar{c}, \bar{b}), \bar{x})$;

(ii) NC is satisfied at $((\bar{c}, \bar{b}), \bar{x}) \in \text{gph}(S)$
Lipschitz-like property of $S$ in CSIP

We consider the parameterized convex optimization problem

\[ P(c, b) : \inf f(x) + \langle c, x \rangle \]
\[ \text{s. t. } g_t(x) \leq b_t, \ t \in T. \]
Lipschitz-like property of $S$ in CSIP

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$x \in \mathbb{R}^n, T$ is a compact Hausdorff index set,
Lipschitz-like property of $S$ in CSIP

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- $x \in \mathbb{R}^n, T$ is a compact Hausdorff index set,
- $f : \mathbb{R}^n \to \mathbb{R}$ and $g_t : \mathbb{R}^n \to \mathbb{R}, t \in T,$ are convex functions,
- $(t,x) \mapsto g_t(x)$ is continuous on $T \times \mathbb{R}^n.$
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We consider the parameterized convex optimization problem

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- $x \in \mathbb{R}^n, T$ is a compact Hausdorff index set,
- $f : \mathbb{R}^n \to \mathbb{R}$ and $g_t : \mathbb{R}^n \to \mathbb{R}, t \in T,$ are convex functions,
- $(t,x) \mapsto g_t(x)$ is continuous on $T \times \mathbb{R}^n.$
- Parameter: $(c,b) \in \mathbb{R}^n \times C(T, \mathbb{R})$ (canonical perturbations)
Definition ([CKLP06])

**Extended Nürnberger Condition, ENC** at \(((\bar{c}, \bar{b}), \bar{x}) \in \text{gph}S:\)

\[
\text{SSC} \ (\exists \hat{x} \in \mathbb{R}^n \ | \ g_t(\hat{x}) < \bar{b}_t, \ t \in T) \ \text{and there is no} \ D \subset T_{\bar{b}}(\bar{x}) \\
\text{with} \ |D| < n \ \text{such that} \ (\bar{c} + \partial f(\bar{x})) \cap \text{cone}(\bigcup_{t \in D} (-\partial g_t(\bar{x}))) \neq \emptyset,
\]

where \(T_{\bar{b}}(\bar{x}) := \left\{ t \in T \ | \ g_t(\bar{x}) = \bar{b}_t \right\} \) (set of active indices).

**KKT conditions**: \((\bar{c} + \partial f(\bar{x})) \cap \text{cone}(\bigcup_{t \in D} (-\partial g_t(\bar{x}))) \neq \emptyset\).
Definition ([CKLP06])

Extended Nürnberger Condition, ENC at \( \left( \left( \bar{c}, \bar{b} \right), \bar{x} \right) \in \text{gph}S \):

\[
\text{SSC} \left( \exists \hat{x} \in \mathbb{R}^n \mid g_t(\hat{x}) < \bar{b}_t, \ t \in T \right) \text{ and there is no } D \subset T_{\bar{b}}(\bar{x}) \text{ with } |D| < n \text{ such that } (\bar{c} + \partial f(\bar{x})) \cap \text{cone} \left( \bigcup_{t \in D} (-\partial g_t(\bar{x})) \right) \neq \emptyset,
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where \( T_{\bar{b}}(\bar{x}) := \left\{ t \in T \mid g_t(\bar{x}) = \bar{b}_t \right\} \) (set of active indices).

KKT conditions: \( (\bar{c} + \partial f(\bar{x})) \cap \text{cone} \left( \bigcup_{t \in T_{\bar{b}}(\bar{x})} (-\partial g_t(\bar{x})) \right) \neq \emptyset. \)
2.1. Lipschitz-like property of \( S \) in LSIP/CSIP

\[\text{ENC \ at} \quad (\bar{c}, \bar{b}, \bar{x}) \in \text{gph} S \quad \Rightarrow \quad (\bar{c}, \bar{b}) \in \text{int} \left\{ (c, b) \mid P(c, b) \text{ has a strongly unique min.} \right\} \]

\( S \) is single valued and Lipschitz in a neighb. of \( (\bar{c}, \bar{b}) \)

\( S \) is single valued and continuous in a neighborhood of \( (\bar{c}, \bar{b}) \)

\( S \) is strongly Lipschitz stable

\( \Downarrow \quad \Uparrow \)

All conditions are equivalent in the linear case
On the exact Lipschitzian bound of $S$ in LSIP/CSIP. Some references.


3. Coderivatives and Lipschitz-like in LIP/CIP

3.1. Coderivative analysis in LIP

Let us consider $\mathcal{F} : l_{\infty}(T) \rightrightarrows X,$

$$\mathcal{F}(p) := \{x \in X \mid \langle a^*_t, x \rangle \leq b_t + p_t, \ t \in T\}.$$
3. Coderivatives and Lipschitz-like in LIP/CIP

3.1. Coderivative analysis in LIP

Let us consider \( F : l_\infty (T) \rightrightarrows X, \)

\[ F (p) := \{ x \in X \mid \langle a_t^*, x \rangle \leq b_t + p_t, \ t \in T \}. \]

Without loss of generality the stability analysis will be made at \( p = 0. \)
3. Coderivatives and Lipschitz-like in LIP/CIP

3.1. Coderivative analysis in LIP

Let us consider $\mathcal{F}: l_\infty (T) \rightrightarrows X,$

$$\mathcal{F} (p) := \{ x \in X \mid \langle a_t^*, x \rangle \leq b_t + p_t , \; t \in T \} .$$

Without loss of generality the stability analysis will be made at $p = 0$. $\mathcal{F}$ has a closed and convex graph.

**Definition**

The **coderivative** of $\mathcal{F}$ at $(0, \bar{x})$, $D^* \mathcal{F} (0, \bar{x}) : X^* \rightrightarrows (l_\infty (T))^*$, is given by

$$D^* \mathcal{F} (0, \bar{x}) (x^*) := \{ p^* \in (l_\infty (T))^* \mid (p^*, -x^*) \in N ((0, \bar{x}) ; \text{gph} \mathcal{F}) \} .$$
• \( N ((0, \bar{x}) ; \text{gph} \mathcal{F}) \) is the normal cone (of convex analysis) to \( \text{gph} \mathcal{F} \) at \( (0, \bar{x}) \).
- $N((0, x); \text{gph} \mathcal{F})$ is the normal cone (of convex analysis) to $\text{gph} \mathcal{F}$ at $(0, x)$.
- Specifically, for $(p^*, x^*) \in (l_\infty(T))^* \times X^*$,

\[
(p^*, x^*) \in N((0, x); \text{gph} \mathcal{F}) \iff \left\{ \langle (p^*, x^*), (p, x) - (0, x) \rangle \leq 0, \right. \\
\left. \text{for all } (p, x) \in \text{gph} \mathcal{F}. \right\}
\]
Remark: gph\(F\) may be seen as the feasible set of the linear inequality system \(\{\langle a_t^*, x \rangle \leq b_t + p_t, t \in T \}\) with respect to the variable \((p, x) \in l_\infty(T) \times X\), which may be rewritten as

\[
\{\langle -\delta_t, p \rangle + \langle a_t^*, x \rangle \leq b_t, t \in T \},
\]

where \(\delta_t \in l_\infty(T)^*\) is the classical Dirac measure at \(t\).
Via the extended Farkas Lemma:

**Proposition (computing the normal cone)**

Let \((0, \bar{x}) \in \text{gph} \mathcal{F}\), and let \((p^*, x^*) \in l_\infty(T)^* \times X^*\). Then we have \((p^*, x^*) \in N((0, \bar{x}); \text{gph} \mathcal{F})\) if and only if

\[
(p^*, x^*, \langle x^*, \bar{x} \rangle) \in \text{cl}^* \text{cone} \left[ \{ (-\delta_t, a^*_t, b_t) \mid t \in T \} \cup \{ (0, 0, 1) \} \right],
\]

where \(\delta_t\) denotes the classical **Dirac measure** at \(t \in T\) satisfying \(\langle \delta_t, p \rangle = p_t, \ t \in T\) for \(p = (p_t)_{t \in T} \in l_\infty(T)\).
Via the extended Farkas Lemma:

**Proposition (computing the normal cone)**

Let \((0, \bar{x}) \in \text{gph} F\), and let \((p^*, x^*) \in l_\infty(T)^* \times X^*.\) Then we have \((p^*, x^*) \in N((0, \bar{x}); \text{gph} F)\) if and only if

\[
(p^*, x^*, \langle x^*, \bar{x} \rangle) \in \text{cl}^* \text{cone} \left[ \{(-\delta_t, a_t^*, b_t) \mid t \in T\} \cup \{(0,0,1)\} \right],
\]

where \(\delta_t\) denotes the classical Dirac measure at \(t \in T\) satisfying \(\langle \delta_t, p \rangle = p_t, \ t \in T\) for \(p = (p_t)_{t \in T} \in l_\infty(T)\).

**Theorem (coderivative of the feasible solution map)**

Let \(\bar{x} \in F(0)\). Then \(p^* \in D^* F(0, \bar{x})(x^*)\) if and only if

\[
(p^*, -x^*, -\langle x^*, \bar{x} \rangle) \in \text{cl}^* \text{cone} \left\{(-\delta_t, a_t^*, b_t) \mid t \in T\right\}.
\]
Theorem (computing the coderivative norm)

Let $\bar{x} \in \mathcal{F}(0)$. Assume that SSC is satisfied for $p = 0$ and that $\{a_t^* \mid t \in T\}$ is bounded in $X^*$. One has:

(i) If $\bar{x}$ is a SS point for $p = 0$, then $\|D^* \mathcal{F}(0, \bar{x})\| = 0$.

(ii) If $\bar{x}$ is not a SS point for $p = 0$, then

$$\|D^* \mathcal{F}(0, \bar{x})\| = \max \left\{ \|x^*\|^{-1} \mid (x^*, \langle x^*, \bar{x} \rangle) \in \text{cl}^* C(0) \right\}$$

via the $w^*$-closure of the characteristic set

$$C(0) := \text{co} \left\{ (a_t^*, b_t) \mid t \in T \right\}.$$
3.2. Arbitrary Banach spaces of decision variables

**Theorem (coderivative criterion for Lipschitz-like property)**

Let $\bar{x} \in \mathcal{F}(0)$. Then $\mathcal{F}$ **Lipschitz-like** at $(0, \bar{x})$ if and only if

$$D^* \mathcal{F}(0, \bar{x})(0) = \{0\}.$$
3.2. Arbitrary Banach spaces of decision variables

**Theorem (coderivative criterion for Lipschitz-like property)**

Let \( \bar{x} \in \mathcal{F}(0) \). Then \( \mathcal{F} \) **Lipschitz-like** at \((0, \bar{x})\) if and only if \( D^* \mathcal{F}(0, \bar{x})(0) = \{0\} \).

**Theorem (computing the exact Lipschitzian bound)**

Let \( \bar{x} \in \mathcal{F}(0) \), **SSC** hold at \( b = 0 \) and \( \{a_t^* \mid t \in T\} \) be bounded:

(i) If \( \bar{x} \) is a SS point for \( p = 0 \), then \( \text{lip} \mathcal{F}(0, \bar{x}) = 0 \).

(ii) If \( \bar{x} \) is not a SS point for \( p = 0 \), then

\[
\text{lip} \mathcal{F}(0, \bar{x}) = \|D^* \mathcal{F}(0, \bar{x})\| = \max \left\{ \|x^*\|^{-1} \mid (x^*, \langle x^*, \bar{x} \rangle) \in \text{cl}^* \mathcal{C}(0) \right\}.
\]

The assumption of boundedness can be removed if \( X \) is reflexive.
In the proof the following fact is used:
In the proof the following fact is used:

\[ \text{lip } F (\bar{p}, \bar{x}) = \text{reg } F^{-1} (\bar{x} \mid \bar{p}) = \limsup_{(x,p) \to (\bar{x}, \bar{p})} \frac{d (x, F (p))}{d (p, F^{-1} (x))} \]

(Under the convention \( \frac{0}{0} = 0 \))

**Lemma**

Assume that SSC is satisfied for \( p = 0 \). Then for any \( x \in X \) and \( p \in l_\infty (T) \) we have the extended Ascoli formula

\[ d(x, F (p)) = \max_{(x^*, \alpha) \in \text{cl}^* C(p)} \frac{[\langle x^*, x \rangle - \alpha]_+}{\|x^*\|} \]

If \( X \) is reflexive we can remove \( \text{cl}^* \) in the formula above, but replacing \( \max \) with \( \sup \).

Moreover, for any \( x \in X \) and \( p \in l_\infty (T) \),
Remark: For $X$ arbitrary Banach, $T$ arbitrary, and $\{a_t^* | t \in T\}$ bounded, the set
\[
S := \{x^* | (x^*, \langle x^*, \bar{x} \rangle) \in \text{cl}^* C(0)\}
\]
is non-empty and $w^*$-compact provided that $\bar{x}$ is not a SS point for $p = 0$. Under SSC, $0 \notin S$, and then the $(w^*)$usc mapping $x \mapsto \|x^*\|^{-1}$ attains its maximum on $S$.

Remark: In the continuous case considered in CaDoLoPa05 ($T$ compact Hausdorff, $X = \mathbb{R}^n$ and $t \mapsto (a_t^*, b_t)$ continuous on $T$, under SSC at $p = 0$,
\[
S = \text{co} \{ (a_t^*, b_t) : t \in T_0(\bar{x}) \},
\]
and, hence,
\[
\text{lip} \ F(0, \bar{x}) = \max \left\{ \|x^*\|^{-1} | x^* \in \text{co} \{ a_t^* : t \in T_0(\bar{x}) \} \right\}
\]
\[
= d(0, \text{co} \{ a_t^* : t \in T_0(\bar{x}) \})^{-1}.
\]
Basic bibliography

Linear semi-infinite programming


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Variational analysis

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