A Quantitative Stability Analysis in Optimization

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1. Introduction and preliminaries

1.1. Parameterized convex optimization problems

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Example (a LSIP problem, with $X = \mathbb{R}^2$)

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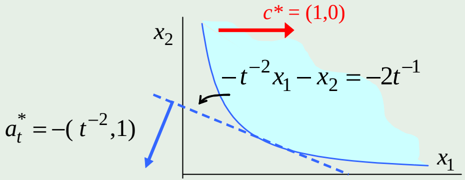

Parameter spaces in this talk

Continuous case: $X = \mathbb{R}^n$, T compact Hausdorff,

$$\begin{array}{lll} P(c,b): & \text{Inf} & f(x) + \langle c, x \rangle \\ & \text{s. t.} & g_t(x) \leq b_t, \ \ t \in T, \end{array}$$

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- Parameter $(c,b) \in \mathbb{R}^n \times C(T,\mathbb{R})$,

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- Feasible/Optimal set mapping, $\mathcal{F} : C(T,\mathbb{R}) \rightrightarrows \mathbb{R}^n$, $\mathcal{S} : \mathbb{R}^n \times C(T,\mathbb{R}) \rightrightarrows \mathbb{R}^n$,

$$\mathcal{F}(b) := \{x \in \mathbb{R}^n : g_t(x) \leq b_t, \text{ for all } t \in T\}.$$

$$\mathcal{S}(c,b) := \arg\min \{f(x) + \langle c, x \rangle, \ x \in \mathcal{F}(b)\}.$$

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1.2. Stability criteria and direct antecedents

	$X = \mathbb{R}^n$	X arbitrary
lsc and/or usc of $\mathcal{F}$ (feasible set map.) and $\mathcal{S}$ (optimal set map.)	Bro(84), Fis(83), GoLoTo (96,97) GoLo (98) CaLoPaTo (99) CaLoPa(02)	DiGoLo (08)
Lipschitz-like of $\mathcal{F}$	CaDoLoPa (05)[1] CaGoPa (09)	CaMoLoPa,I (09)[4]
Lipschitz-like of $\mathcal{S}$	CaKlaLoPa (07)[2] CaGoPa (08)[3]	
Coderivative of $\mathcal{F}$		CaMoLoPa,I-II (09)[4]

[1]Math. Program 104B, 329-346, [2]SIOPT 18, 717-732, [3]SVAN 16, 511-538, [4]submitted

Definition

X, Y metric spaces. $\mathcal{M} : X \rightrightarrows Y$ is Lipschitz-like (Aubin, pseudo-Lipschitz) at (⇒around) $(\overline{x}, \overline{y}) \in \mathrm{gph}\mathcal{M}$ if $\exists U$ neigh. of $\overline{x}$, $\exists V$ neigh. of $\overline{y}$, $\exists \kappa \geq 0$ s.t.

$$d\left(y, \mathcal{M}\left(x\right)\right) \leq \kappa d\left(x, x'\right) \ \forall x, x' \in U, \ \forall y \in \mathcal{M}\left(x'\right) \cap V.$$

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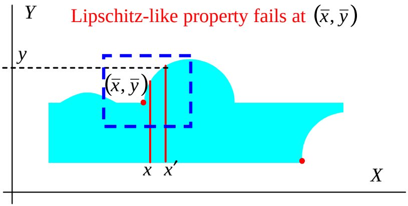

Definition (see Mordukhovich (2006))

X, Y Banach spaces, $\mathcal{M} : X \rightrightarrows Y$. The coderivative of $\mathcal{M}$ at $(\bar{x}, \bar{y}) \in \text{gph}\mathcal{M}$, $D^*\mathcal{M}(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ is given by

$$D^*\mathcal{M}(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N((\bar{x}, \bar{y}); \text{gph}\mathcal{M})\},$$

where $N(\cdot; \Omega)$ is the basic, or limiting, or Mordukhovich normal cone (the usual normal cone of convex analysis if $\text{gph}\mathcal{M}$ is locally convex around $(\bar{x}, \bar{y})$).

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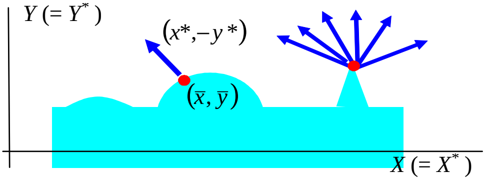
Illustration in finite dimension

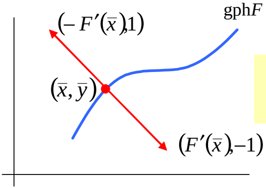


Example (Coderivatives of smooth functions in $\mathbb{R}^n$)

The coderivative of a smooth (single-valued) function $F : \mathbb{R}^n \to \mathbb{R}^m$, $D^*F(\overline{x}, \overline{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, is given by

$$D^*F(\overline{x}, \overline{y})(y) = \nabla F(\overline{x})^* y, \text{ for all } y \in \mathbb{R}^m,$$

where $\nabla F(\overline{x})^*$ represents the adjoint (transpose) of the Jacobian.



$D^*F(\overline{x}, \overline{y})(1) = F'(\overline{x});$
$D^*F(\overline{x}, \overline{y})(-1) = -F'(\overline{x});$

Theorem (Mordukhovich (1993))

Let X and Y be finite-dimensional Banach spaces, and $\mathcal{M} : X \rightrightarrows Y$ closed-graph mapping. Then $\mathcal{M}$ is Lipschitz-like at $(\bar{x}, \bar{y}) \in \operatorname{gph} \mathcal{M}$ if and only if

$$D^* \mathcal{M}(\bar{x}, \bar{y})(0) = \{0\}.$$

Moreover

$$\operatorname{lip} \mathcal{M}(\bar{x}, \bar{y}) = \|D^* \mathcal{M}(\bar{x}, \bar{y})\|,$$

where

$$\|D^* \mathcal{M}(\bar{x}, \bar{y})\| := \sup\{\|x^*\| \mid x^* \in D^* \mathcal{M}(\bar{x}, \bar{y})(y^*), \ \|y^*\| \leq 1\}.$$

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- Recall that we consider the feasible set mapping $\mathcal{F} : l_\infty (T) \rightrightarrows X$,

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- Observe that if T is infinite , $l_\infty(T)$ is never Asplund.

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1.4. Preliminary results

Lemma (extended Farkas lemma, [Dinh,Goberna, López (2006)])

Let $p \in \mathrm{dom}\mathcal{F}$, *and let* $(x^*, \alpha) \in X^* \times \mathbb{R}$. *The following are equivalent:*

(i) The inequality $\langle x^*, x\rangle \leq \alpha$ *is a consequence of* $\mathcal{F}(p)$, *i.e.,*

$$[\langle a_t^*, x\rangle \leq b_t + p_t \text{ for all } t \in T] \Longrightarrow [\langle x^*, x\rangle \leq \alpha].$$

(ii) The pair (x^*, α) *satisfies the inclusion*

$$(x^*, \alpha) \in \mathrm{cl}^*\mathrm{cone}[\{(a_t^*, b_t + p_t) \mid t \in T\} \cup \{(0, 1)\}], \text{ with } 0 \in X^*.$$

Lemma (Dinh, Goberna, López (2008))

Let $p \in \mathrm{dom}\mathcal{F}$. *Then the following properties are equivalent:*

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$$C(p) := \mathrm{co}\{(a_t^*, b_t + p_t) \mid t \in T\};$$

(iii) $p \in \mathrm{int}(\mathrm{dom}\mathcal{F})$;

(iv) $\mathcal{F}$ *satisfies the strong Slater condition (SSC) at* p
(i.e., $\exists \widehat{x} \in X \mid \sup_{t\in T} [\langle a_t^*, \widehat{x}\rangle - b_t - p_t] < 0$*)*;

2.1 Lipschitz-like property of $\mathcal{S}$. A motivating example: Nominal problem:

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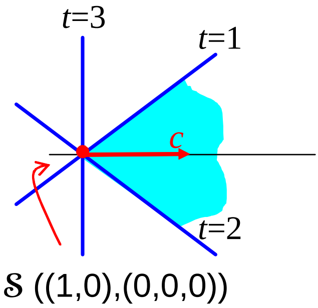
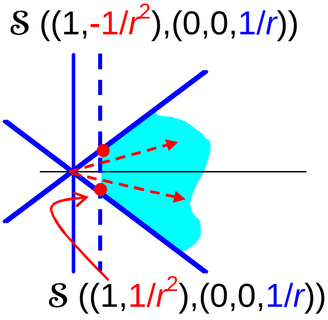
$$\text{s. t.} \quad -x_1 + x_2 \leq 0, \; -x_1 - x_2 \leq 0, \; -x_1 \leq 0.$$

Perturbed probems:

$$\pi_r \; (\pi_r'): \quad \text{Inf} \quad x_1 \pm (1/r^2) x_2$$
$$\text{s. t.} \quad -x_1 + x_2 \leq 0, \; -x_1 - x_2 \leq 0, \; -x_1 \leq -1/r.$$


t=3
t=1
c
t=2
S ((1,0),(0,0,0))


S ((1,-1/r2),(0,0,1/r))
S ((1,1/r2),(0,0,1/r))

A Karush-Kuhn-Tucker (KKT) type condition

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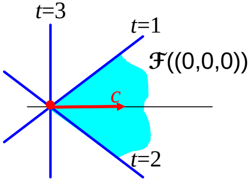
$$x \in \mathcal{F}(b) \text{ and } -c \in A_b(x) \underset{\underset{SSC}{\Leftarrow}}{\Rightarrow} x \in \mathcal{S}(c,b)$$

Definition

The *Nürnberger condition* (*NC*) is satisfied at $(\overline{\pi}, \overline{x}) \in \text{gph}(\mathcal{S})$ if

$$SSC \text{ holds at } \overline{b} \text{ and there is no } D \subset T_{\overline{b}}(\overline{x}) \\ \text{with } |D| < n \text{ such that } -\overline{c} \in cone(\{a_t,\ t \in D\}).$$

Exa. $\mathcal{S}$ not Lips.-like at $((\bar{c},\bar{b}),\bar{x})$), **not** NC



$x=(0,0)$, $b=(0,0,0)$, $T_b(x)=\{1,2,3\}$

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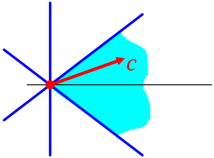

$T_b(x)=\{1,2,3\}$, $c=(1,1/2)$

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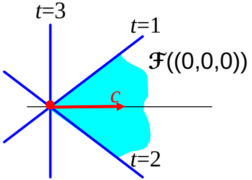

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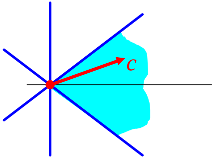

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Theorem (see [CKLP07, Th. 16])

Let $((\bar{c},\bar{b}),\bar{x})\in gph(\mathcal{S})$. The following conditions are equivalent:

(i) $\mathcal{S}$ is Lipschitz-like at $((\bar{c},\bar{b}),\bar{x})$;

(ii) NC is satisfied at $((\bar{c},\bar{b}),\bar{x})\in \mathrm{gph}(\mathcal{S})$

Lipschitz-like property of $\mathcal{S}$ in CSIP

We consider the parameterized convex optimization problem

$$\begin{aligned} P(c,b): \quad & \text{Inf } f(x) + \langle c, x \rangle \\ & \text{s. t. } g_t(x) \le b_t, \;\; t \in T. \end{aligned}$$

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- $f : \mathbb{R}^n \to \mathbb{R}$ and $g_t : \mathbb{R}^n \to \mathbb{R}$, $t \in T$, are convex functions,
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- Parameter: $(c, b) \in \mathbb{R}^n \times C(T, \mathbb{R})$ (canonical perturbations)

Definition ([CKLP06])

Extended Nürnberger Condition, ENC at $\left(\left(\bar{c},\bar{b}\right),\bar{x}\right) \in \mathrm{gph}\mathcal{S}$:

SSC $(\exists \widehat{x} \in \mathbb{R}^n \mid g_t(\widehat{x}) < \bar{b}_t,\ \ t \in T)$ and there is no $D \subset T_{\bar{b}}(\bar{x})$ with $|D| < n$ such that $(\bar{c} + \partial f(\bar{x})) \cap \mathrm{cone}(\bigcup_{t\in D} (-\partial g_t(\bar{x}))) \neq \emptyset$,

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KKT conditions: $(\bar{c} + \partial f(\bar{x})) \cap \mathrm{cone}(\bigcup_{t \in T_{\bar{b}}(\bar{x})} (-\partial g_t(\bar{x}))) \neq \emptyset.$

ENC at $\left(\left(\bar{c},\bar{b}\right),\bar{x}\right)\in \operatorname{gph}\mathcal{S}$ $\Rightarrow$ / $\not\Leftarrow$ $\left(\bar{c},\bar{b}\right)\in \operatorname{int}\left\{ \left(c,b\right) \mid P\left(c,b\right) \text{ has a strongly unique min.}\right\}$

$\Downarrow$ $\not\Uparrow$

$\left\{ \mathcal{S} \text{ is Lips. like} \right\}$ $\Longleftrightarrow$ $\left\{ \mathcal{S} \text{ is single valued and Lipschitz in a neighb.of } \left(\bar{c},\bar{b}\right) \text{ (strongly Lipschitz stable)} \right\}$

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$\left\{ \mathcal{S} \text{ is single valued and continuous in a neighborhood of } \left(\bar{c},\bar{b}\right) \text{ (Kojima's type stability)} \right\}$

All conditions are equivalent in the linear case

On the exact Lipschitzian bound of $\mathcal{S}$ in LSIP/CSIP. Some references.

- M. J. Cánovas, F.J. Gómez–Senent, and J. Parra (2008), On the Lipschitz modulus of the argmin mapping in linear semi-infinite optimization, Set-Valued Anal. 16, 511-538.
- M. J. Cánovas, A. Hantoute, M. A. López, and J. Parra (2008) , Lipschitz behavior of convex optimization problems: A variational approach, *J. of Global Optim*, 41, pp. 1-13
- M. J. Cánovas, A. Hantoute, M. A. López, and J. Parra (2008), Stability of indices in KKT conditions and metric regularity in convex semi-infinite optimization, *J. Optim. Th. Appl.*, 139, pp. 485-500
- M. J. Cánovas, A. Hantoute, M. A. López, and J. Parra (2008), Lipschitz modulus in convex semi-infinite optimization via d.c. functions, *ESAIM: Contr. Optim. Calc. Var.*, online.

3. Coderivatives and Lipschitz-like in LIP/CIP
3.1. Coderivative analysis in LIP

Let us consider $\mathcal{F} : l_\infty(T) \rightrightarrows X$,

$$\mathcal{F}(p) := \{x \in X \mid \langle a_t^*, x \rangle \leq b_t + p_t \ , \ t \in T\} .$$

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Without loss of generality the stability analysis will be made at $p = 0$. $\mathcal{F}$ has a closed and convex graph.

Definition

The coderivative of $\mathcal{F}$ at $(0, \bar{x})$, $D^*\mathcal{F}(0, \bar{x}) : X^* \rightrightarrows (l_\infty(T))^*$, is given by

$$D^*\mathcal{F}(0, \bar{x})(x^*) := \{p^* \in (l_\infty(T))^* \mid (p^*, -x^*) \in N((0, \bar{x}); \text{gph}\mathcal{F})\}.$$

- $N\left((0,\overline{x});\text{gph}\mathcal{F}\right)$ is the normal cone (of convex analysis) to gph$\mathcal{F}$ at $(0,\overline{x})$.

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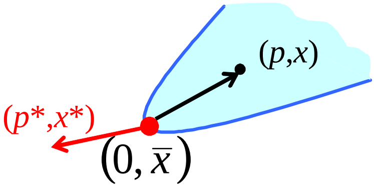
- Specifically, for $(p^*,x^*) \in (l_\infty(T))^* \times X^*$,

$$(p^*,x^*) \in N\left((0,\overline{x});\mathrm{gph}\mathcal{F}\right) \Leftrightarrow \begin{cases} \langle (p^*,x^*),(p,x)-(0,\overline{x})\rangle \leq 0, \\ \text{for all } (p,x) \in \mathrm{gph}\mathcal{F}. \end{cases}$$



Remark: gph$\mathcal{F}$ may be seen as the feasible set of the linear inequality system $\{\langle a_t^*, x\rangle \leq b_t + p_t\ ,\ t \in T\}$ with respect to the variable $(p, x) \in l_\infty(T) \times X$, which may be rewritten as

$$\{\langle -\delta_t, p\rangle + \langle a_t^*, x\rangle \leq b_t\ ,\ t \in T\}\ ,$$

where $\delta_t \in l_\infty(T)^*$ is the classical Dirac measure at t.

Via the extended Farkas Lemma:

Proposition (computing the normal cone)

Let $(0,\bar{x}) \in \operatorname{gph}\mathcal{F}$*, and let* $(p^*, x^*) \in l_\infty(T)^* \times X^*$. *Then we have* $(p^*, x^*) \in N((0,\bar{x}); \operatorname{gph}\mathcal{F})$ *if and only if*

$$(p^*, x^*, \langle x^*, \bar{x}\rangle) \in \mathrm{cl}^*\mathrm{cone}\left[\{(-\delta_t, a_t^*, b_t) \mid t \in T\} \cup \{(0,0,1)\}\right],$$

where δ_t *denotes the classical Dirac measure at* $t \in T$ *satisfying* $\langle \delta_t, p\rangle = p_t,\ t \in T$ *for* $p = (p_t)_{t\in T} \in l_\infty(T)$.

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Theorem (coderivative of the feasible solution map)

Let $\overline{x}\in \mathcal{F}(0)$. *Then* $p^* \in D^*\mathcal{F}(0,\overline{x})(x^*)$ *if and only if*

$$(p^*,-x^*,-\langle x^*,\overline{x}\rangle) \in \text{cl}^*\textit{cone}\left\{(-\delta_t,a_t^*,b_t) \mid t\in T\right\}.$$

Theorem (computing the coderivative norm)

Let $\overline{x} \in \mathcal{F}(0)$*. Assume that SSC is satisfied for* $p = 0$ *and that* $\{a_t^* \mid t \in T\}$ *is bounded in* X^**. One has:*

(i) If $\overline{x}$ *is a SS point for* $p = 0$*, then* $\|D^*\mathcal{F}(0,\overline{x})\| = 0$.

(ii) If $\overline{x}$ *is not a SS point for* $p = 0$*, then*

$$\|D^*\mathcal{F}(0,\overline{x})\| = \max\left\{\|x^*\|^{-1} \mid (x^*, \langle x^*, \overline{x}\rangle) \in \mathrm{cl}^* C(0)\right\}$$

via the w^**-closure of the characteristic set*

$$C(0) := co\left\{(a_t^*, b_t) \mid t \in T\right\}.$$

3.2. Arbitrary Banach spaces of decision variables

Theorem (coderivative criterion for Lipschitz-like property)

Let $\overline{x} \in \mathcal{F}(0)$. *Then* $\mathcal{F}$ *Lipschitz-like at* $(0, \overline{x})$ *if and only if*

$$D^*\mathcal{F}(0, \overline{x})(0) = \{0\}.$$

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Theorem (computing the exact Lipschitzian bound)

Let $\overline{x} \in \mathcal{F}(0)$, SSC hold at $b = 0$ and $\{a_t^* \mid t \in T\}$ be bounded:

(i) If $\overline{x}$ is a SS point for $p = 0$, then $\operatorname{lip}\mathcal{F}(0,\overline{x}) = 0$.

(ii) If $\overline{x}$ is not a SS point for $p = 0$, then

$$\begin{aligned} \operatorname{lip}\mathcal{F}(0,\overline{x}) &= \|D^*\mathcal{F}(0,\overline{x})\| \\ &= \max\left\{ \|x^*\|^{-1} \mid (x^*, \langle x^*, \overline{x}\rangle) \in \operatorname{cl}^* C(0) \right\}. \end{aligned}$$

The assumption of boundedness can be removed if X is reflexive.

In the proof the following fact is used:

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$$\operatorname{lip} \mathcal{F}(\overline{p},\overline{x}) = \operatorname{reg} \mathcal{F}^{-1}(\overline{x} \mid \overline{p}) = \limsup_{(x,p)\to(\overline{x},\overline{p})} \frac{d(x,\mathcal{F}(p))}{d(p,\mathcal{F}^{-1}(x))}$$

(Under the convention $\frac{0}{0} = 0$)

Lemma

Assume that SSC is satisfied for $p = 0$. Then for any $x \in X$ and $p \in l_\infty(T)$ we have the extended Ascoli formula

$$d(x,\mathcal{F}(p)) = \max_{(x^*,\alpha)\in \mathrm{cl}^* C(p)} \frac{[\langle x^*, x\rangle - \alpha]_+}{\|x^*\|}.$$

If X is reflexive we can remove cl^* *in the formula above, but replacing* max *with* sup.

Moreover, for any $x \in X$ and $p \in l_\infty(T)$,

Remark: For X arbitrary Banach, T arbitrary, and $\{a_t^* \mid t \in T\}$ bounded,the set

$$S := \{x^* \mid (x^*, \langle x^*, \overline{x} \rangle) \in \mathrm{cl}^* C(0)\}$$

is non-empty and w^*-compact provided that $\overline{x}$ is not a SS point for $p = 0$. Under SSC, $0 \notin S$, and then the (w^*-)usc mapping $x \mapsto \|x^*\|^{-1}$ attains its maximun on S.

Remark: In the continuous case considered in CaDoLoPa05 (T compact Hausdorff, $X = \mathbb{R}^n$ and $t \mapsto (a_t^*, b_t)$ continuous on T, under SSC at $p = 0$,

$$S = \mathrm{co}\left\{(a_t^*, b_t) : \ t \in T_0(\overline{x})\right\},$$

and, hence,

$$\begin{aligned} \mathrm{lip}\, \mathcal{F}(0, \overline{x}) &= \max\left\{\|x^*\|^{-1} \mid x^* \in \mathrm{co}\left\{a_t^* : \ t \in T_0(\overline{x})\right\}\right\} \\ &= d\left(0_n, \mathrm{co}\left\{a_t^* : \ t \in T_0(\overline{x})\right\}\right)^{-1}. \end{aligned}$$

Basic bibliography

Linear semi-infinite programming

- M. A. Goberna and M. A. López, *Linear Semi-Infinite Optimization*, John Wiley & Sons, Chichester (UK), 1998

Convex analysis

- R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ,.1970.
- C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific, 2002

Variational analysis

- D. Klatte and Kummer, *Nonsmooth Equations in Optimization. Regularity, Calculus, Methods and Applications*, Kluwer Academic Publ., Dordrecht, 2002.
- B.S. Mordukhovich*Variational Analysis and Generalized Differentiation I*, Springer-Verlag, Berlin, 2006.
- R.T. Rockafellar and R.J.-B. Wets (1998), *Variational Analysis*, Springer-Verlag, Berlín.

References

- M. J. Cánovas, A. L. Dontchev, M. A. López and J. Parra (2005), Metric regularity of semi-infinite constraint systems Math. Programming 104B, 329-346.
- M. J. Cánovas, D. Klatte, M. A. López and J. Parra (2007), Metric regularity in convex semi-infinite optimization under canonical perturbations, SIAM J. Optim. 18, 717-732.
- M. J. Cánovas, F.J. Gómez–Senent, and J. Parra (2008), On the Lipschitz modulus of the argmin mapping in linear semi-infinite optimization, Set-Valued Anal. 16, 511-538.
- M. J. Cánovas, B. Mordukhovich, M. A. López, and J. Parra (2009), Variational analysis in semi-infinite and infinite programming, I: Stability of linear inequality systems of feasible solutions, SIAM J. Optim., to appear.
- ________, ________, II: Necessary optimality conditions, submitted.