James boundaries and copies of $\ell_1(\mathfrak{c})$

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Introduction and preliminaries

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• For instance, K and the set of extreme points $Ext(K)$ are boundaries of K.

• If B is a boundary of K, then $\overline{\mathrm{co}}^{w^*}(B) = \overline{\mathrm{co}}^{w^*}(K)$ but, in general, $\overline{\mathrm{co}}(B) \neq \overline{\mathrm{co}}^{w^*}(K)$. Even $\overline{\mathrm{co}}(K) \neq \overline{\mathrm{co}}^{w^*}(K)$.

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• We are interested in studying the conditions under which $\overline{\mathrm{co}}(B) = \overline{\mathrm{co}}^{w^*}(K)$ and the consequences of the inequality $\overline{\mathrm{co}}(B) \neq \overline{\mathrm{co}}^{w^*}(K)$.

• Let us say that a subset A of a dual Banach space X^* has the **property** (P) (**also,** A **is a Pettis set**) if $\overline{\mathrm{co}}(K) = \overline{\mathrm{co}}^{w^*}(K)$ for every w^*-compact subset K of A.

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• X^* is **super-**(P) if $\overline{\mathrm{co}}(B) = \overline{\mathrm{co}}^{w^*}(K)$ for every w^*-compact subset $K \subset X^*$ and every boundary $B \subset K$.

Theorem

[Haydon, 1976] For a Banach space X the following are equivalent:

(1) X fails to have an isomorphic copy of ℓ_1.

(2) X^* has property (P).

(3) For every w^*-compact subset K of X^*, the set of extreme points $Ext(K)$ of K satisfies $\overline{\mathrm{co}}(Ext(K)) = \overline{\mathrm{co}}^{w^*}(K)$.

(4) Every $z \in X^{**}$ is universally measurable on $(B(X^*), w^*)$.

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Theorem

[Godefroy] For a separable Banach space TFAE:

(a) X fails to have a copy of ℓ_1.

(b) X^* is super-(P).

$\mathcal{S}eq(X^{**})$ and 1-Baire functions

- For a subset $A \subset X^*$, let $Seq(X^{**}; A)$ be the subspace of functionals $\psi \in X^{**}$ such that there exists a sequence $(x_n)_{n\geq 1} \subset X$ with $\langle a, x_n \rangle \underset{n\to\infty}{\longrightarrow} \langle \psi, a \rangle$ for every $a \in A$.

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• We put $\mathcal{S}eq(X^{**}) := \mathcal{S}eq(X^{**}; X^*)$. $\mathcal{S}eq(X^{**})$ is a closed subspace of X^{**} (McWilliams, 1962).

$\mathcal{S}eq(X^{**})$ and 1-Baire functions

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• We put $\mathcal{S}eq(X^{**}) := \mathcal{S}eq(X^{**}; X^*)$. $\mathcal{S}eq(X^{**})$ is a closed subspace of X^{**} (McWilliams, 1962).

• Let (T, τ) be a Hausdorf topological space. A real function $f : T \to \mathbb{R}$ is said to be **an** 1-**Baire function** if there exists a sequence $\{f_n : n \geq 1\}$ in the space of continuous real functions $C(T)$ such that $f_n \to f$ pointwise on T. Let $\mathcal{B}_{1b}(T)$ denote the family of real bounded 1-Baire functions.

- The fact $\overline{\mathrm{co}}(K) \neq \overline{\mathrm{co}}^{w^*}(K)$ implies the existence of a functional $\psi \in X^{**}$ not universally measurable and so not 1-Baire on $\overline{\mathrm{co}}^{w^*}(K)$.

• The fact $\overline{\mathrm{co}}(K) \neq \overline{\mathrm{co}}^{w^*}(K)$ implies the existence of a functional $\psi \in X^{**}$ not universally measurable and so not 1-Baire on $\overline{\mathrm{co}}^{w^*}(K)$.

• When B is a boundary of K, the fact $\overline{\mathrm{co}}(B) \neq \overline{\mathrm{co}}^{w^*}(K)$ generally does not imply the existence of a functional $\psi \in X^{**}$ not universally measurable on $\overline{\mathrm{co}}^{w^*}(K)$, but always implies the existence of a functional $\psi \in S(X^{**})$ such that $\psi \notin \mathcal{B}_{1b}(\overline{\mathrm{co}}^{w^*}(K))$ and $\psi \notin \mathcal{S}eq(X^{**})$. We calculate in the sequel an estimation of the distances $dist(\psi, Seq(X^{**}))$, $dist(\psi, Seq(X^{**}; \overline{\mathrm{co}}^{w^*}(K)))$ and $dist(\psi, \mathcal{B}_{1b}(\overline{\mathrm{co}}^{w^*}(K)))$ with respect to the distance $dist(\overline{\mathrm{co}}^{w^*}(K), \overline{\mathrm{co}}(B))$.

Distance to $\mathcal{B}_{1b}(H)$

Proposition

Let X be a Banach space, H a convex w^*-compact subset of X^*, B a boundary of H, $w_0 \in H$, $d > 0$ and $\psi \in S(X^{**})$ fulfilling $\langle \psi, w_0 \rangle > \sup \langle \psi, B \rangle + d$. Then $\operatorname{dist}(\psi \restriction H, \mathcal{B}_{1b}(H)) \geq \frac{1}{6} d$ in $\ell_\infty(H)$.

[An idea of the proof]

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[An idea of the proof] **Part A.** Let $T : X \to C(H)$ be such that $Tx = x \restriction H, \forall x \in X$. If $\varphi \in \mathcal{B}_{1b}(H)$, let $\tilde{\varphi} \in \mathcal{S}eq((C(H)^{**})$ be such that

$$\langle \tilde{\varphi}, \mu \rangle = \int_H \varphi \cdot d\mu, \forall \mu \in C(H)^*.$$

Then

$$\|T^{**}\psi - \tilde{\varphi}\| \leq 3\|\psi \restriction H - \varphi\|,$$
$$dist(T^{**}\psi, \mathcal{S}eq(C(H)^{**})) \leq 3 dist(\psi \restriction H, \mathcal{B}_{1b}(H)).$$

Part B. If $\psi \in S(X^{**})$ satisfies

$$\langle \psi, w_0 \rangle > \sup \langle \psi, \overline{\mathrm{co}}(B) \rangle + d,$$

then $dist(T^{**}\psi, \mathcal{S}eq(C(H)^{**})) \geq \frac{1}{2} d$.

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Proposition (Simons equality, 1995)

Let E be a Banach space and $B \subset G \subset E^*$ subsets such that every element of E attains on B its maximum on G. Then if $(x_n)_{n\geq 1} \subset E$ is a bounded sequence, we have

$$\sup_{b \in B} \limsup_{n \to \infty} \langle b, x_n \rangle = \sup_{g \in G} \limsup_{n \to \infty} \langle g, x_n \rangle.$$

Distances $Seq(X^{**})$ and $Seq(X^{**}; H)$

Corollary

Let X be a Banach space, H a convex w^*-compact subset of $B(X^*)$, B a boundary of H and $d > 0$ such that $dist(H, \overline{\mathrm{co}}(B)) > d$. Then there exist $w_0 \in H$ and a functional $\psi \in S(X^{**})$ fulfilling

$$\langle \psi, w_0 \rangle > \sup \langle \psi, B \rangle + d$$

such that $dist(\psi, Seq(X^{**})) \geq dist(\psi, Seq(X^{**}; H)) \geq \frac{d}{2}$.

Distances $Seq(X^{**})$ and $Seq(X^{**}; H)$

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such that $dist(\psi, Seq(X^{**})) \geq dist(\psi, Seq(X^{**}; H)) \geq \frac{d}{2}$.

- **Proof.** Let $T : X \to C(H)$ be the restriction operator such that $Tx = x \restriction H$, $\forall x \in X$. Since $\|T\| \leq 1$ (because $H \subset B(X^*)$) and $T^{**}(\mathcal{S}eq(X^{**}, H)) \subset \mathcal{S}eq(C(H)^{**})$ by Part A, then

$$dist(\psi, Seq(X^{**}, H)) \geq dist(T^{**}\psi, \mathcal{S}eq(C(H)^{**})).$$

- Now an application of Part B gives that

$$dist(\psi, Seq(X^{**}, H)) \geq dist(T^{**}\psi, \mathcal{S}eq(C(H)^{**})) \geq \frac{d}{2}.$$

Finally, the inequality $dist(\psi, Seq(X^{**})) \geq dist(\psi, Seq(X^{**}, H))$ is obvious because $Seq(X^{**})$ is a subspace of $Seq(X^{**}, H)$.

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Finally, the inequality $dist(\psi, Seq(X^{**})) \geq dist(\psi, Seq(X^{**}, H))$ is obvious because $Seq(X^{**})$ is a subspace of $Seq(X^{**}, H)$.

Corollary

For a Banach space X always $(1) \Rightarrow (2) \Rightarrow (2')$, where
(1) $X^{**} = \mathcal{S}eq(X^{**})$.
(2) X^* is ultra-(P), i.e., Y^* is super-(P), for every subspace $Y \subset X$.
(2') X^* is super-(P), i.e., $\overline{\mathrm{co}}(B) = \overline{\mathrm{co}}^{w^*}(K)$, for every w^*-compact subset $K \subset X^*$ and every boundary B of K.

On the equality $X^{**} = \mathcal{S}eq(X^{**})$

Proposition

Let X be a Banach space. Consider the following statements:
(0) $(B(X^{**}), w^*)$ *is angelic; (1)* $X^* \in (C)$*.*
(2) X^* *fails to have an uncountable basic sequence of type* ℓ_1^+*.*
(3) $X^{**} = Seq(X^{**})$*.*
(4) X^* *is ultra-*(P) *;(4')* X^* *is super-*(P)*.*
(5) $X \in (C)$ *and* X *fails to have a copy of* ℓ_1*.*
Then always $(0) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (4') \Rightarrow (5)$*.*

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Then always $(0) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (4') \Rightarrow (5)$.

Corollary

If X *is a Banach space and* X^* *has the property* (C), *then* X *has the property* (C).

Positive results

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Let K be a Hausdorf compact space. TFAE:
(1) K is scattered countable; (2) $C(K)^* \in (C)$.
(3) $Seq(C(K)^{**}) = C(K)^{**}$.
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Proposition

Let X be either a σ-complete Banach lattice or a dual Banach lattice. TFAE:
(1) $X^* \in (C)$; (2) $X^{**} = Seq(X^{**})$; (3) X^* is ultra-(P); (3') X^* is super-(P).

Proposition

Let V be a Banach space with a projective generator. TFAE:

(1) V^* is super-(P) ; (2) $V^{**} = Seq(V^{**})$.

(3) V^* is ultra-(P); (4) V^* is $\aleph_1$-super-(P).

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(1) X^* has the property (C); (2) $X^{**} = Seq(X^{**})$.
(3) X^* is ultra-(P).(3') X^* is super-(P).

Martin's Axiom and $X^{**} = \mathcal{S}eq(X^{**})$

Proposition (MM)

Let X be a Banach space. TFAE:

(1) $X^{**} = Seq(X^{**})$; (2) X^{*} is ultra-(P); (3) X^{*} is $\aleph_1$-super-(P).

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Proposition (MM)

Let X be a Banach space such that $Dens(X) = \aleph_1$. TFAE

(1) $X^{**} = \mathcal{S}eq(X^{**})$. (2) X^{*} is super-(P).

The Martin's Maximum Axiom MM

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• $\mathcal{M}$ =the family of Čech-complete spaces K with the que CCC property fulfilling that, given a sequence of regular open subsets $\{O_\alpha : \alpha < \omega_1\}$ of K, there exists a "club" $\Gamma \subset \omega_1$ such that $O_{[\alpha\beta)}$ is constant for every pair $\alpha, \beta \in \Gamma, \alpha < \beta$, where

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$\mathfrak{mm}$ satisfies $\omega_1 \leq \mathfrak{mm} \leq \omega_2$.

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$\mathfrak{mm}$ satisfies $\omega_1 \leq \mathfrak{mm} \leq \omega_2$.

- The Martin's Maximum Axiom MM is the claim $\omega_1 < \mathfrak{mm}$.

El Teorema de Talagrand

Theorem

(Talagrand) Let τ be a cardinal with cofinality $cf(\tau) > \aleph_0$, X a Banach space and A a subset of X. The following are equivalent
(1) A has a copy of the basis of $\ell_1(\tau)$.
(2) $\overline{\mathrm{co}}(A)$ has a copy of the basis of $\ell_1(\tau)$.
(3) $\overline{[A]}$ has a copy of $\ell_1(\tau)$.

Characterization of the fact $K \in (P)$

Proposition

[G. and S.] For a w^*-compact subset K of a dual Banach space X^* TFAE:
(1) $K \notin (P)$.
*(2) There exists in K a w^*-$\mathbb{N}$-**family** and a copy of the basis of $\ell_1(\mathfrak{c})$.*
(3) There exists $z \in X^{**}$ which is not universally measurable on K.

- A. S. Granero and M. Sánchez, *Distances to convex sets*, Studia Math.,182 (2007), 165-181.

- *Convex w^*-closures versus convex norm-closures*, J. Math. Anal. Appl., 350 (2009), 485-497.

w^*-$\mathbb{N}$-families

(1) A subset $\mathcal{F}$ of X^* is said to be a w^*-$\mathbb{N}$-**family of width** $d > 0$ if $\mathcal{F}$ is bounded and has the form

$$\mathcal{F} = \{\eta_{M,N} : M, N \text{ disjoint subsets of } \mathbb{N}\},$$

and there exist a number $r_0 \in \mathbb{R}$ and a sequence $\{x_m : m \geq 1\} \subset B(X)$ such that for every pair of disjoint subsets M, N of $\mathbb{N}$ we have

$$\eta_{M,N}(x_m) \geq r_0 + d, \ \forall m \in M, \quad \text{and} \quad \eta_{M,N}(x_n) \leq r_0, \ \forall n \in N.$$

We say that $Width(\mathcal{F}) \geq d$.

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We say that $Width(\mathcal{F}) \geq d$.

(2) We define the $Width(Y)$ of a subset Y of X^* as follows:

$$Width(Y) := \sup\{d \geq 0 : \exists K \subset Y \ w^*\text{-compact}$$
$$\text{and a } w^*\text{-}\mathbb{N}\text{-family } \mathcal{A} \subset K \text{ of width } \geq d\}.$$

• J. DIESTEL, *Sequences and Series in Banach Spaces*, Springer-Verlag, New-York, 1984, pag. 206.

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• If $\mathcal{F} \subset X^*$ is a w^*-$\mathbb{N}$-family, a standard argument proves that a subset of $\mathcal{F}$ is equivalent to the basis of $\ell_1(\mathfrak{c})$. Moreover, the same argument yields that the sequence $\{x_n : n \geq 1\} \subset B(X)$ associated to $\mathcal{F}$ is equivalent to the basis of ℓ_1.

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• If $\mathcal{F} \subset X^*$ is a w^*-$\mathbb{N}$-family, a standard argument proves that a subset of $\mathcal{F}$ is equivalent to the basis of $\ell_1(\mathfrak{c})$. Moreover, the same argument yields that the sequence $\{x_n : n \geq 1\} \subset B(X)$ associated to $\mathcal{F}$ is equivalent to the basis of ℓ_1.

• So, if $\mathcal{A} \subset K \subset X^*$ is a w^*-$\mathbb{N}$-family, K has a copy of the basis of $\ell_1(\mathfrak{c})$ and X has an isomorphic copy of ℓ_1. And vice versa, if X has a copy of ℓ_1, then X^* contains a w^*-$\mathbb{N}$-family.

Question. Let $K \subset X^*$ be a w^*-compact subset and $B \subset K$ a boundary:

(Q1) If $\overline{\mathrm{co}}(B) \neq \overline{\mathrm{co}}^{w^*}(K)$, does K contain a w^*-$\mathbb{N}$-family (and a copy of the basis of $\ell_1(\mathfrak{c})$)? And B?

(Q2) Does B contain a w^*-$\mathbb{N}$-family if K does?

(Q3) Does B contain a copy of the basis of $\ell_1(\mathfrak{c})$ if $\overline{\mathrm{co}}^{w^*}(K)$ does?

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• The answer to Q1 is, in general, negative (see the following Counterexample).

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- The answer to Q1 is, in general, negative (see the following Counterexample).

- The answers to Q2 and Q3 are affirmative in many cases. We do not know Counterexamples for these two questions.

• **Counterexample .** Let Y be the isometric predual of the long James space $J(\omega_1)$ and $X := Y^* = J(\omega_1)$. Then:

(i) Y and all its successive dual spaces are Asplund. So, $X^* = Y^{**} = J(\omega_1)^*$ does not have a copy of $\ell_1(\mathfrak{c})$.

(ii) Let $K := B(X^*)$ and $B_0 := Y_c \cap K$, where

$$Y_c := \bigcup \{\overline{[A]}^{w^*} : A \subset Y \text{ countable } \}.$$

It is easy to see that Y_c is a norm-closed subspace of X^* and that B_0 is a boundary of K such that $\overline{\mathrm{co}}(B_0) \subset Y_c$.

(iii) There is a vector e_{ω_1} that satisfies $e_{\omega_1} \in B(X^*)$ but $e_{\omega_1} \notin Y_c$ and so $e_{\omega_1} \notin \overline{\mathrm{co}}(B_0)$. Thus $\overline{\mathrm{co}}(B_0) \neq \overline{\mathrm{co}}^{w^*}(K)$. $\square$

R. D. Bourgin, *Geometric Aspects of Convex Sets with the Radon-Nikodým Property*, Lect. Notes in Math., Springer-Verlag, Vol. 993(1983), p.346.

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- **Step 1.** The metrizable case.

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- **Step 1.** The metrizable case.
- **Step 2.** The general case.

The metrizable case

- Let (H, τ) be a topological space. The **index of fragmentation** $Frag(f, H)$ of a function $f : H \to \mathbb{R}$ is the infimum of the family of numbers $\epsilon \geq 0$ such that for every $\eta > \epsilon$ and every non-empty subset $F \subset H$, there exists an open set $V \subset H$ such that $V \cap F \neq \emptyset$ and $diam(f(V \cap F)) \leq \eta$.

The metrizable case

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Proposition

If H is a separable metric space and $f \in \ell_\infty(H)$ then $dist(f, \mathcal{B}_{1b}(H)) \leq \frac{1}{2} Frag(f, H)$.

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Proposition

Let (H, τ) be a hereditarily Baire space, $\epsilon \geq 0$ and $f \in \ell_\infty(H)$. TFAE:
(1) $Frag(f, H) \leq \epsilon$.
(2) For every non-empty closed subset $F \subset H$ and every pair of real numbers $s < t$ such that $t - s > \epsilon$ we have either $\overline{F \cap \{f \leq s\}} \neq F$ or $\overline{F \cap \{f \geq t\}} \neq F$.

Solution of the metrizable case.

Proposition

Let X be a Banach space, $H \subset X^*$ a convex w^*-compact subset and B a boundary of H such that $dist(H, \overline{\mathrm{co}}(B)) > d > 0$. If H is w^*-metrizable, H has a w^*-$\mathbb{N}$-family $\mathcal{A}$ of $width(\mathcal{A}) \geq \frac{d}{3}$ and a copy of the basis of $\ell_1(\mathfrak{c})$. So $Width(H) \geq \frac{1}{3} dist(H, \overline{\mathrm{co}}(B))$.

Sketch of the proof.

• As $dist(H, \overline{\mathrm{co}}(B)) > d$, we can choose $w_0 \in H$ with $dist(w_0, \overline{\mathrm{co}}(B)) > d > 0$ and $\psi \in S(X^{**})$ such that

$$\langle \psi, w_0 \rangle > \sup \langle \psi, \overline{\mathrm{co}}(B) \rangle + d$$

Thus $dist(\psi \restriction H, \mathcal{B}_{1b}(H)) > \frac{1}{6} d$ in $\ell_\infty(H)$.

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Thus $dist(\psi \restriction H, \mathcal{B}_{1b}(H)) > \frac{1}{6} d$ in $\ell_\infty(H)$.

• As H is w^*-compact and metrizable, $dist(\psi \restriction H, \mathcal{B}_{1b}(H)) \leq \frac{1}{2} Frag(\psi \restriction H, H)$. Thus $Frag(\psi \restriction H, H) > \frac{1}{3} d$. Hence there exists a non-empty w^*-compact subset $F \subset H$ and two real numbers $s < t$ with $t - s > \frac{1}{3} d$ such that $\overline{F \cap \{\psi \leq s\}}^{w^*} = F = \overline{F \cap \{\psi \geq t\}}^{w^*}$. From this fact we deduce the existence in F of a w^*-$\mathbb{N}$-family $\mathcal{F}$ such that $width(\mathcal{F}) > \frac{1}{3} d$.

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• As H is w^*-compact and metrizable, $dist(\psi \restriction H, \mathcal{B}_{1b}(H)) \leq \frac{1}{2} Frag(\psi \restriction H, H)$. Thus $Frag(\psi \restriction H, H) > \frac{1}{3} d$. Hence there exists a non-empty w^*-compact subset $F \subset H$ and two real numbers $s < t$ with $t - s > \frac{1}{3} d$ such that $\overline{F \cap \{\psi \leq s\}}^{w^*} = F = \overline{F \cap \{\psi \geq t\}}^{w^*}$. From this fact we deduce the existence in F of a w^*-$\mathbb{N}$-family $\mathcal{F}$ such that $width(\mathcal{F}) > \frac{1}{3} d$.

A. S. Granero and M. Sánchez, *Convex w^*-closures versus convex norm-closures*, J. Math. Anal. Appl., 350 (2009), 485-497.

The general case

Definition

Let X be a Banach space and K a w^*-compact subset of X^*.

(A) The $Bindex(K)$ is

$$Bindex(K) = \sup\{dist(\overline{\mathrm{co}}^{w^*}(W), \overline{\mathrm{co}}(B)) : W \subset K \ w^*\text{-compact} \\ \text{and } B \text{ a boundary of } W\}.$$

(B) The $Bindex_c(K)$ is **the supremum of the** *$Bindex(i^*(K))$, where $i : Y \to X$ is the canonical inclusion mapping and $Y \subset X$ is a separable subspace.*

Proposition

Let X be a Banach space and H a w^*-compact subset of X^*. Then

(A) $Width(H) \leq Bindex_c(H)$.

(B) If H is convex then $Width(H) \leq Bindex_c(H) \leq 3Width(H)$.

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Let X be a Banach space and H a w^*-compact subset of X^*. Then

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Corollary

Let X be a Banach space and $K \subset X^*$ a w^*-compact subset of X^*. TFAE

(1) $Width(\overline{\mathrm{co}}^{w^*}(K)) = 0$.

(2) $Width(K) = 0$.

(3) $Bindex_c(K) = 0$.

(4) $Bindex_c(\overline{\mathrm{co}}^{w^*}(K)) = 0$.

w^*-$\mathcal{CD}$ boundaries

Proposition

Let X be a Banach space, $K \subset X^*$ a w^*-compact subset and $B \subset K$ a w^*-$\mathcal{CD}$ boundary such that $\overline{\mathrm{co}}(B) \neq \overline{\mathrm{co}}^{w^*}(K)$. Then K contains a w^*-$\mathbb{N}$-family and a copy of the basis of $\ell_1(\mathfrak{c})$.

The boundary $Ext(K)$

Proposition

Let X be a Banach space and K a w^*-compact metrizable subset of X^* such that $dist(\overline{\mathrm{co}}^{w^*}(K), \overline{\mathrm{co}}(Ext(K))) > d > 0$. Then $Ext(K)$ has a w^*-$\mathbb{N}$-family $\mathcal{A}$ of $width(\mathcal{A}) > d > 0$ and a copy of the basis of $\ell_1(\mathfrak{c})$. Thus $Width(Ext(K)) \geq dist(\overline{\mathrm{co}}^{w^*}(K), \overline{\mathrm{co}}(Ext(K)))$.

The boundary $Ext(K)$

Proposition

Let X be a Banach space and K a w^*-compact metrizable subset of X^* such that $dist(\overline{\mathrm{co}}^{w^*}(K), \overline{\mathrm{co}}(Ext(K))) > d > 0$. Then $Ext(K)$ has a w^*-$\mathbb{N}$-family $\mathcal{A}$ of $width(\mathcal{A}) > d > 0$ and a copy of the basis of $\ell_1(\mathfrak{c})$. Thus $Width(Ext(K)) \geq dist(\overline{\mathrm{co}}^{w^*}(K), \overline{\mathrm{co}}(Ext(K))$.

Proof. Since K is metrizable, $Ext(K)$ is a $\mathcal{G}_\delta$ subset and for every $w \in \overline{\mathrm{co}}^{w^*}(K)$ there exists a Radon Borel probability μ carried by $Ext(K)$ such that $w = r(\mu)$. This fact and the hypothesis $dist(\overline{\mathrm{co}}^{w^*}(K), \overline{\mathrm{co}}(Ext(K))) > d > 0$ imply that there exists a w^*-compact subset $H \subset Ext(K)$ such that $dist(\overline{\mathrm{co}}^{w^*}(H), \overline{\mathrm{co}}(H)) > d$. So, H contains a w^*-$\mathbb{N}$-family $\mathcal{A}$ with $width(\mathcal{A}) \geq d$.

Proposition

Let K be a w^*-compact subset of a dual Banach space X^* with $K \notin (P)$. Then $\mathrm{Ext}(K)$ has a w^*-$\mathbb{N}$-family and a copy of the basis of $\ell_1(\mathfrak{c})$.

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Proposition

Let K be a w^*-compact subset of a dual Banach space X^*. TFAE:

(1) $Ext(K)$ has a w^*-$\mathbb{N}$-family.

(2) $K \notin (P)$, i.e., K has a w^*-$\mathbb{N}$-family.

(3) $\overline{\mathrm{co}}(Ext(W)) \neq \overline{\mathrm{co}}^{w^*}(W)$ for some w^*-compact subset W of K.

Proposition

Let K be a w^*-compact subset of a dual Banach space X^*. TFAE:

(1) $Ext(K)$ has a copy of the basis of $\ell_1(\mathfrak{c})$.

(2) K has a copy of the basis of $\ell_1(\mathfrak{c})$.

Proposition

Let K be a w^*-compact subset of a dual Banach space X^*. TFAE:

(1) Ext(K) has a copy of the basis of $\ell_1(\mathfrak{c})$.

(2) K has a copy of the basis of $\ell_1(\mathfrak{c})$.

Proof. $(1) \Rightarrow (2)$ is obvious.

Proposition

Let K be a w^*-compact subset of a dual Banach space X^*. TFAE:

(1) $Ext(K)$ has a copy of the basis of $\ell_1(\mathfrak{c})$.

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Proof. $(1) \Rightarrow (2)$ is obvious.

$(2) \Rightarrow (1)$. There are two cases:

Proposition

Let K be a w^*-compact subset of a dual Banach space X^*. TFAE:

(1) $Ext(K)$ has a copy of the basis of $\ell_1(\mathfrak{c})$.

(2) K has a copy of the basis of $\ell_1(\mathfrak{c})$.

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). There are two cases:

Case 1. Suppose that $K \in (P)$. Then $\overline{\mathrm{co}}(Ext(K)) = \overline{\mathrm{co}}^{w^*}(K)$. From a result of Talagrand we obtain that $Ext(K)$ has a copy of the basis of $\ell_1(\mathfrak{c})$.

Proposition

Let K be a w^*-compact subset of a dual Banach space X^*. TFAE:

(1) $Ext(K)$ has a copy of the basis of $\ell_1(\mathfrak{c})$.

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Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). There are two cases:

Case 1. Suppose that $K \in (P)$. Then $\overline{\mathrm{co}}(Ext(K)) = \overline{\mathrm{co}}^{w^*}(K)$. From a result of Talagrand we obtain that $Ext(K)$ has a copy of the basis of $\ell_1(\mathfrak{c})$.

Case 2. Suppose that $K \notin (P)$. Then K has a w^*-$\mathbb{N}$-family and by the above Proposition we get that $Ext(K)$ has a w^*-$\mathbb{N}$-family, and so a copy of the basis of $\ell_1(\mathfrak{c})$.

$\mathcal{K}_\sigma$ boundaries

Let X be a Banach space, K a w^*-compact subset and B a boundary of K. If B is a $\mathcal{K}_\sigma$ subset, its behavior is analogous to that of $Ext(K)$.

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Let X be a Banach space, K a w^*-compact subset and B a boundary of K. If B is a $\mathcal{K}_\sigma$ subset, its behavior is analogous to that of $Ext(K)$.

Proposition

Let X be a Banach space, K a w^*-compact subset of X^* that has a w^*-$\mathbb{N}$-family and B a boundary of K which is a $\mathcal{K}_\sigma$ set. Then
(1) B has a w^*-$\mathbb{N}$-family iff K does.
(2) B has a copy of the basis of $\ell_1(\mathfrak{c})$ iff K does.

$w^*\mathcal{KA}$ boundaries

Lemma

Let X be a separable Banach space and E be a norm-closed $w^*\mathcal{KA}$ subspace of X^* such that $E \in (P)$. If $w_1^* = \sigma(E^*, E)$ then $(B(E^*), w_1^*)$ is angelic.

$w^*\mathcal{KA}$ boundaries

Lemma

Let X be a separable Banach space and E be a norm-closed $w^*\mathcal{KA}$ subspace of X^* such that $E \in (P)$. If $w_1^* = \sigma(E^*, E)$ then $(B(E^*), w_1^*)$ is angelic.

Lemma

Let X be a separable Banach space, K be a w^*-compact subset of X^* containing a w^*-$\mathbb{N}$-family and B a $w^*\mathcal{KA}$ boundary of K. Then B contains a w^*-$\mathbb{N}$-family.

$w^*\mathcal{KA}$ boundaries

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Let X be a separable Banach space and E be a norm-closed $w^*\mathcal{KA}$ subspace of X^* such that $E \in (P)$. If $w_1^* = \sigma(E^*, E)$ then $(B(E^*), w_1^*)$ is angelic.

Lemma

Let X be a separable Banach space, K be a w^*-compact subset of X^* containing a w^*-$\mathbb{N}$-family and B a $w^*\mathcal{KA}$ boundary of K. Then B contains a w^*-$\mathbb{N}$-family.

Proof. Suppose that B fails to contain a w^*-$\mathbb{N}$-family and let $E := \overline{[B]}$. Clearly, E is a $w^*\mathcal{KA}$ subspace of X^* such that $E \in (P)$ and so E fails to contain a w^*-$\mathbb{N}$-family. Then $(B(E^*), \sigma(E^*, E))$ is angelic by the previous Lemma. Thus $\overline{\mathrm{co}}(B) = \overline{\mathrm{co}}^{w^*}(K)$ by a Theorem of Godefroy and so E contains a w^*-$\mathbb{N}$-family, a contradiction that proves the statement.

Proposition

Let X be a Banach space and K a w^*-compact subset of X^*. Let $B \subset K$ be a $w^*\mathcal{K}\mathcal{A}$ boundary of K. Then

(A) If $\overline{\mathrm{co}}(B) \neq \overline{\mathrm{co}}^{w^*}(K)$, K has a w^*-$\mathbb{N}$-family.

(B) We have
(B1) K contains a w^*-$\mathbb{N}$-family if and only if B contains a w^*-$\mathbb{N}$-family.
(B2) K contains a copy of the basis of $\ell_1(\mathfrak{c})$ if and only if B does.

Proposition

Let X be a Banach space and K a w^*-compact subset of X^*. Let $B \subset K$ be a $w^*\mathcal{KA}$ boundary of K. Then

(A) If $\overline{\mathrm{co}}(B) \neq \overline{\mathrm{co}}^{w^*}(K)$, K has a w^*-$\mathbb{N}$-family.

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(B1) K contains a w^*-$\mathbb{N}$-family if and only if B contains a w^*-$\mathbb{N}$-family.
(B2) K contains a copy of the basis of $\ell_1(\mathfrak{c})$ if and only if B does.

Proof. (A) This is true for every w^*-$\mathcal{CD}$ boundary.

(B1) Suppose that K has a w^*-$\mathbb{N}$-family $\mathcal{A}$. Then X contains a copy of ℓ_1. Let $T : \ell_1 \to X$ be the corresponding isomorphism. If B is a $w^*\mathcal{KA}$ boundary of K, then it is easy to see that: (a) $T^*(B)$ is a $w^*\mathcal{KA}$ boundary of $T^*(K)$; (b) $T^*(\mathcal{A})$ is a w^*-$\mathbb{N}$-family inside $T^*(K)$.
Now we apply the previous Lemma.

(B2) We prove that B contains a copy of the basis of $\ell_1(\mathfrak{c})$ when K does. We consider two cases, namely:
Case 1. $\overline{\mathrm{co}}(B) = \overline{\mathrm{co}}^{w^*}(K)$. The cardinal $\mathfrak{c}$ satisfies $\mathrm{cf}(\mathfrak{c}) > \aleph_0$ because $\mathrm{cf}(2^\alpha) > \alpha$ for every infinite cardinal α and because $\mathfrak{c} = 2^{\aleph_0}$. Thus, we can apply Talagrand Theorem and so there exists a copy of the basis of $\ell_1(\mathfrak{c})$ inside B.

(B2) We prove that B contains a copy of the basis of $\ell_1(\mathfrak{c})$ when K does. We consider two cases, namely:
Case 1. $\overline{\mathrm{co}}(B) = \overline{\mathrm{co}}^{w^*}(K)$. The cardinal $\mathfrak{c}$ satisfies $\mathrm{cf}(\mathfrak{c}) > \aleph_0$ because $\mathrm{cf}(2^\alpha) > \alpha$ for every infinite cardinal α and because $\mathfrak{c} = 2^{\aleph_0}$. Thus, we can apply Talagrand Theorem and so there exists a copy of the basis of $\ell_1(\mathfrak{c})$ inside B.

Case 2. $\overline{\mathrm{co}}(B) \neq \overline{\mathrm{co}}^{w^*}(K)$. Then there exists a w^*-$\mathbb{N}$-family inside K and so inside B by part (A). Thus B contains a copy of the basis of $\ell_1(\mathfrak{c})$ because every w^*-$\mathbb{N}$-family does.

Conjecture 1. Let X be a Banach space such that $\ell_1 \subset X$. Then every boundary of $B(X^*)$ contains a w^*-$\mathbb{N}$-family.

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Proposition

The following are equivalent:

(a) The Conjecture 1 is true.

(b) If X is a Banach space **isomorphic** *to ℓ_1, then every boundary of $B(X^*)$ contains a w^*-$\mathbb{N}$-family.*

Conjecture 2. Let X be a Banach space such that $\ell_1(\mathfrak{c}) \subset X^*$. Then every boundary of $B(X^*)$ contains a copy of the basis of $\ell_1(\mathfrak{c})$.

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Proposition

The following are equivalent:

(a) The Conjecture 2 is true for every separable Banach space X.

(b) If X is a Banach space **isomorphic** *to ℓ_1, then every boundary of $B(X^*)$ contains a copy of the basis of $\ell_1(\mathfrak{c})$.*

THANKS YOU FOR YOUR ATTENTION