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Operadores No Expansivos, Monótonos, Acretivos y Cíclicos

University of Seville
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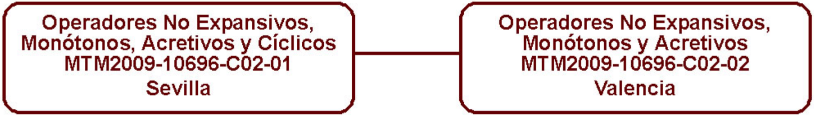
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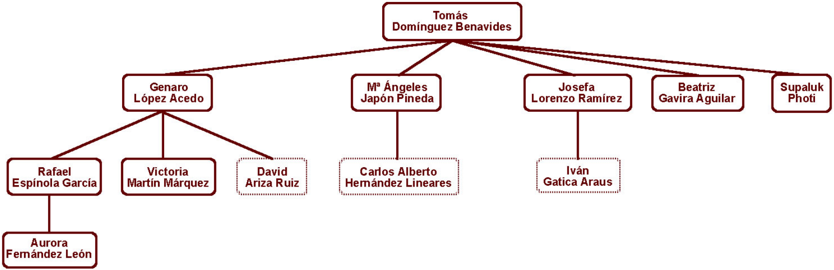
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Genaro
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Mª Ángeles
Japón Pineda
Josefa
Lorenzo Ramírez
Beatriz
Gavira Aguilar
Supaluk
Photi
Rafael
Espínola García
Victoria
Martín Márquez
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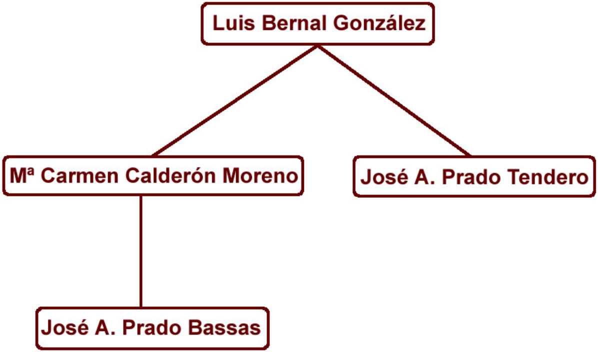
Luis Bernal González
Mª Carmen Calderón Moreno
José A. Prado Tendero
José A. Prado Bassas

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Banach Theorem

Theorem

S. Banach (1922). Let M a complete metric space and $T : M \to M$ a contractive mapping, i.e, there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for every $x, y \in M$. Then T has a (unique) fixed point. Furthermore, for every $x_0 \in M$ the iterates $\{T^n x_0\}$ converges to the fixed point.

Remark

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The Fixed Point Property

Definition

Let X be a Banach space. We say that X satisfies the Fixed Point Property (FPP) if every nonexpansive mapping T defined from a convex bounded closed subset C of X into C has a fixed point.

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Geometrical conditions implying the FPP

1. If the characteristic of noncompact convexity $\epsilon_\beta(X)$ for the separation measure of noncompactness is less than 1, then X has the FPP (T. Domínguez y G. López (1992))
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Stability of the FPP

STABILITY PROBLEM

Assume that X is a Banach space satisfying the FPP. Does there exists a positive number $d > 1$ such that if Y is another Banach space, which is isomorphic to X and the Banach-Mazur distance $d(X; Y) < d$, then Y satisfies the FPP?

Some stability bounds for ℓ_2

1. W. Bynum (1980). If $d(X, \ell_2) \leq \sqrt{2}$, then X satisfies the FPP.
2. Jiménez-Llorens (1992). X satisfies the FPP if $d(X, \ell_2) \leq \sqrt{\frac{3+\sqrt{5}}{2}}$ (the golden number).

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Reflexivity and the FPP

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The space ℓ_1 can be renormed to satisfy the FPP.

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There are more reflexive spaces satisfying the FPP (distinct of a renorming of ℓ_1, Nominally, they obtain a renorming of the space $\bigoplus_1 \sum_n \ell_1^n$ which satisfies the FPP).

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Renormings with the FPP

There are some spaces which cannot be renormed to satisfy the FPP. For instance, by using a refinement of James' Distortion Theorem, P. Dowling, C. Lennard and B. Turett (2002) have proved that every renorming of ℓ_∞ fails to satisfy the FPP.

Long time open Question

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A deeper problem

Problem

Let X be a reflexive Banach space. Do almost all renormings satisfy the FPP?

Genericity and Renormings

M. Fabian; L. Zajiček; V. Zizler (1982)

Let $(X, \|\cdot\|)$ a Banach space. Denote $\mathcal{P}$ the set of all equivalent norms with the metric $\rho(p,q) = \sup\{|p(x) - q(x)| : x \in B\}$. Then, $\mathcal{P}$ is a Baire space.

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T.D.B.; S. Phothi (2010). Let X be a reflexive Banach space. Then, almost all renormings of X satisfy the FPP.

The FPP for multivalued mappings

Problem

Assume that X is a Banach space which satisfies the FPP for single-valued non-expansive mappings. Does X satisfy the FPP for multivalued non-expansive mappings? In particular, assume that X is an NUC Banach space. Does X satisfy the FPP for multivalued non-expansive mappings?

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The FPP for affine mappings

Theorem

Let C be an convex closed subset of a nonreflexive Banach space. Then, C satisfies the FPP for affine mappings if and only if C is weakly compact (T. Domínguez, M.A. Japón and S. Prus 2004) .

Subsets of ℓ_1 with the FPP for continuous mappings

Theorem

Let C be an convex closed subset of ℓ_1. Then, C satisfies the FPP for continuous mappings with a center if and only if C has the compact proximinality property (T. Domínguez, J. García-Falset, E. Llorens and P. Lorenzo 2009) .

The failure of the FPP for unbounded sets in c_0

Theorem

Let C be an convex closed subset of c_0. Then, C satisfies the FPP for nonexpansive mappings if and only if C is weakly compact (T. Domínguez 2011) .

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The FPP in metric spaces

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Let H be a bounded hyperconvex metric space and $T : H \to H$ a limit compact mapping. Then T has a fixed point. (Espínola-López'00)

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In order to take up version of Ky-Fan approximating lemma as well as non-self fixed point results, different properties on the metric projection in hyperconvex spaces were studied.

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The FPP in geodesic spaces

$\mathbb{R}$-trees can be regarded both as hyperconvex spaces and CAT(0) spaces. An $\mathbb{R}$-tree does not need to be bounded to have the FPP.

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The FPP in CAT(0) spaces

Theorem

Let X be a CAT(0) space and $A \subseteq X$ nonempty closed and convex. Then any nonexpansive mapping from A into A has a fixed point. (Kirk'02)

Theorem

Let $k < 0$. If (X, d) is a bounded complete CAT(k), then every uniformly l-lipschitzian mapping $T : X \to X$ with $l < k(M_k^2)$ has a fixed point, where

$$k(M_k^2) = \frac{\operatorname{arccosh}(\cosh^2 \sqrt{-k})}{\sqrt{-k}}.$$

(Espínola-Fernández'09)

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Best approximation in CAT(κ) spaces

Theorem

Suppose that X is a geodesic metric space of curvature bounded below by $\kappa \in (-\infty, 0)$. Then the set $K(A)$ is a dense G_δ-set in $X \setminus A$, where $K(A)$ denotes the set of all points $x \in X \setminus A$ such that the minimization problem $\min(x, A)$ is well posed. (Espínola-Li-López'10)

Approximation in spaces of continuous functions

A geometrical characterization of spaces of continuous functions was obtained:

Theorem

A Banach space X is a Smith-Ward space if and only if it is isometric to a $C(K)$ or a $C_0(K)$ space, where X is a Smith-Ward space if

$$r_G(A) = r(A) + \lim_{\varepsilon \to 0^+} \operatorname{dist}(E^{\varepsilon}(A), G) \tag{1}$$

for each nonempty bounded subset A and each nonempty subset G of M. (Espínola-Wiśnicki-Wośko'00)

Projection in spaces of continuous functions

As a consequence of our work with selection of the metric projection in hyperconvex spaces the same problem was taken in spaces of continuous functions, leading to the following resul°t.

Consider subspaces defined next:

$$
\begin{aligned}
E &= \left\{ f \in C(K) :\ f_{|Z} = 0,\ \text{and } f_{|S_i}, f_{|S_j^1} = -f_{|S_j^2} \text{ are constant} \right\} \\
&= E_Z^0 \cap \left(\cap E_{S_i} \right) \left(\cap E_{S_j^1, S_j^2} \right).
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Theorem

Let E be a finite-codimensional subspace of $C(K)$ admiting a nonexpansive selection of the metric projection, then it is given by the above expression. (Benyamini-Espínola-López'06)

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Monotonicity and Accretivity in Banach spaces

Let X be a real Banach space, X^* its dual space.

$A : \mathcal{D}(A) \subseteq X \to 2^{X^*}$ set-valued operator is said to be

- **Monotone** if $\forall x, y \in \mathcal{D}(A), x^* \in A(x), y^* \in A(y)$

$$\langle x^* - y^*, x - y \rangle \geq 0$$

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- **Maximal monotone** if *monotone* and
$$\left. \begin{array}{l} (x, x^*) \in X \times X^* \\ \langle x^* - y^*, x - y \rangle \geq 0 \quad \forall y \in \mathcal{D}(A), y^* \in A(y) \end{array} \right\} \Rightarrow x^* \in A(x),$$
that is, there exists no monotone operator whose graph contains the graph of A.

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- **Maximal monotone** if *monotone* and
$$\left.\begin{array}{l}(x, x^*) \in X \times X^* \\ \langle x^* - y^*, x - y \rangle \geq 0 \quad \forall y \in \mathcal{D}(A), y^* \in A(y)\end{array}\right\} \Rightarrow x^* \in A(x),$$
that is, there exists no monotone operator whose graph contains the graph of A.

Monotonicity and Accretivity in Banach spaces

Let X be a real Banach space.

$A : \mathcal{D}(A) \subseteq X \to \underline{2^X}$ set-valued operator is said to be

- **Accretive** if $\forall x, y \in \mathcal{D}(A), u \in A(x), v \in A(y)$

$$d(x, y) \leq d(x + ru, y + rv), \quad \forall r \geq 0$$

or equivalently, if $\forall x, y \in \mathcal{D}(A), u \in A(x), v \in A(y)$

$$\langle u - v, j(x - y)\rangle \geq 0$$

where $j(x - y) \in J(x - y)$ is the duality mapping

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If H is a Hilbert space

$A : H \to 2^H$ set-valued operator

1. A is (maximal) monotone $\Leftrightarrow$ A is (m-) accretive
2. If A is monotone:
   A is maximal monotone $\Leftrightarrow$ A is norm-to-weak Upper Semicontinuous (USC) and $A(x)$ is closed and convex $\forall x \in H$

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Let M be a Hadamard manifold
$A : \mathcal{D}(A) \subseteq M \to 2^{TM}$ set-valued *vector field*

G. López, C. Li, V. Martín-Márquez, J.H. Wang (2010)

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If $\mathcal{D}(A) = M$

$$A \text{ maximal monotone} \Leftrightarrow J_\lambda \text{ firmly nonexp } \mathcal{D}(J_\lambda) = M$$

$$A^{-1}(0) = Fix(J_\lambda)$$

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Let $f : M \to (-\infty, +\infty]$ be proper, convex, LSC with $\mathcal{D}(f) = M$ then ∂f

$$\partial f(x) = \{u \in T_x M : \langle u, \exp_x^{-1} y\rangle \leq f(y) - f(x), \forall y \in M\}$$

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Definiciones

Sea X es un EVT y $L(X)$ el EV de los operadores (aplicaciones lineales y continuas) de X en X. Un $T \in L(X)$ se dice cíclico cuando existe un vector $x_0 \in X$ tal que $\overline{\mathrm{span}}\{T^n x_0 : n \geq 0\} = X$.

Aquí trataremos con una versión fuerte de ciclicidad, a saber, la hiperciclicidad, concepto acuñado por Beauzamy en 1980. Un operador $T \in L(X)$ se dice hipercíclico (HC) cuando $\exists x_0 \in X$ tal que $\overline{\{T^n x_0 : n \geq 0\}} = X$.

Conexión con el Problema del Subespacio [Subconjunto, resp.] Invariante: T es cíclico [HC, resp.] cuando $\exists x_0 \in X$ que no está contenido en ningún subespacio [subconjunto, resp.] cerrado T-invariante propio.

Antecedentes

Birkhoff (1929)

El operador de traslación traslación $f \mapsto f(\cdot + 1)$ es HC en $H(\mathbb{C})$.

MacLane (1952)

El operador de derivación $f \mapsto f'$ es HC en $H(\mathbb{C})$.

Rolewicz (1969)

Si $|\lambda| > 1$, λB es HC en ℓ_p ($p \in [1, \infty)$), donde $B : (x_n) \mapsto (x_{n+1})$ es el operador backward shift. Además, si $T \in L(X)$ es HC, entonces $\dim(X) = +\infty$.

El problema de partida

Rolewicz (1969)

¿Soporta cada E. de Banach X separable con $\dim(X) = \infty$ un operador HC? [Por ejemplo, $L^p(0,1)$ $(0 < p < 1)$, que no es Banach, admite operadores HCs, pero $L^p(0,1) \oplus \mathbb{R}^N$ no].

Ansari-Bernal (1997) (independientemente)

Todo espacio de Banach separable infinito dimensional soporta algún operador HC.

Observación

En 1998, Bonet y Peris probaron lo mismo para espacios de Fréchet.

Sobre el tamaño de $HC(T)$

Observación

Si T es HC y X es un F-espacio, entonces $HC(T) := \{$vectores hipercíclicos para $T\}$ es residual, es decir, topológicamente grande.

Herrero, Bourdon, Bès, Wengenroth (1991–2003)

$HC(T)$ es algebraicamente grande: si X es un EVT y $T \in L(X)$ es HC, existe un subespacio vectorial denso T-invariante M con $M \setminus \{0\} \subset HC(T)$.

Principales líneas

Buscar operadores HC con condiciones adicionales.

(a) Estructura lineal del conjunto $HC(T)$.

(b) Crecimiento de las funciones HC.

(c) Comportamiento caótico en la frontera.

Estructura lineal (a)

Bernal y Petersson (2006)

Cada espacio de Fréchet separable infinito dimensional que soporta una norma continua admite un operador T tal que $HC(T)$ es espaciable (existe un subespacio cerrado infinito dimensional M tal que $M \setminus \{0\} \subset HC(T)$).

Remark

El mismo año, Bès y Conejero lo demuestran para $\mathbb{K}^{\mathbb{N}}$.

Crecimiento de funciones (b)

Bernal, Calderón y Luh (2009)

Existe un subespacio vectorial denso M de funciones enteras HC para la traslación tal que cada $f \in M \setminus \{0\}$ tiende exponencialmente a 0 cuando $z \to \infty$ sobre un conjunto cuyo complemento tiene área arbitrariamente pequeña.

Comportamiento caótico y Estructura lineal (c)+(a)

Cluster Sets

$f \in H(G)$, y $\gamma \subset G$ es una curva que tiende a la frontera de G, el Cluster set de f a lo largo de γ es

$$C_\gamma(f) = \{\omega \in \partial G : \exists (z_n) \subset \gamma \text{ con } z_n \to \partial G \text{ y } f(z_n) \to \omega\}$$

Bernal, Calderón y Prado-Bassas (2010)

El conjunto de funciones $f \in H(G)$ con cluster set maximal a lo largo de una familia de curvas que tienden a la frontera y, además, son HCs para la C_{φ_n} (sucesión de op. composición con $\varphi_n : G \to G$ asintóticamente inyectivas y G región de Jordan) es espaciable y maximal-denso-lineable (existe un subespacio denso de dimensión maximal).

THANK YOU (Trabajo de fin de Grado)

UNIVERSIDAD DE SEVILLA

Grupo de análisis no lineal