Nuevos resultados sobre e-convexidad

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Outline

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- Notation: $\operatorname{conv} C$, $\operatorname{cone} C$, $\operatorname{rint} C$, $\operatorname{cl} C$ (*weak*-closure* if $C \subset X^*$).

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For a given function $f : X \to \overline{\mathbb{R}}$ we consider the following notions:

- The effective domain, the sublevel set ($r \in \mathbb{R}$) and the epigraph of f:

$$\begin{aligned} \operatorname{dom} f &:= \{x \in X \mid f(x) < +\infty\}, \\ L(f, r) &:= \{x \in X \mid f(x) \leq r\}, \\ \operatorname{epi} f &:= \{(x, a) \in X \times \mathbb{R} \mid f(x) \leq a\}. \end{aligned}$$

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- The (Fenchel) conjugate of f is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) := \sup_{x \in X} \{\langle x^*, x \rangle - f(x)\}.$$

- f is lsc at $\bar{x} \in X$ if for each $\lambda \in \mathbb{R}$ such that $\lambda < f(\bar{x})$ there exists a neighbourhood of $\bar{x}$, $V_{\bar{x}}$, such that $\lambda < f(x)$ for all $x \in V_{\bar{x}}$.

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$$\partial_\varepsilon f(\bar{x}) := \{x^* \in X^* \mid f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle - \varepsilon, \forall\, x \in X\};$$

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Lower semicontinuous hull of f

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Infimal convolution of $f, g : X \to \overline{\mathbb{R}}$

- $f \,\square\, g : X \to \overline{\mathbb{R}}$ with $(f \,\square\, g)(x) := \inf_{u \in X} \{f(u) + g(x-u)\}$.
- For f, g proper, convex and lsc with $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$,

$$\text{Moreau-Rockafellar formula}: \ (f+g)^* = \operatorname{cl}(f^* \,\square\, g^*).$$

The Hahn-Banach theorem

Separation of convex sets

Theorem

Suppose A and B are disjoint, nonempty, convex sets in a topological (real) vector space X.

(i) *If A is open, there exist $v^* \in X^*$ and $\alpha \in \mathbb{R}$ such that*

$$\langle v^*, x\rangle < \alpha \leq \langle v^*, y\rangle \quad \text{for all } x \in A,\ y \in B.$$

(ii) *If A is compact, B is closed and X is locally convex, then there exist $v^* \in X^*$, $\alpha \in \mathbb{R}$ and $\varepsilon > 0$ such that*

$$\langle v^*, x\rangle \leq \alpha - \varepsilon < \alpha + \varepsilon \leq \langle v^*, y\rangle \quad \text{for all } x \in A,\ y \in B.$$

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- $X^* \neq \{0\} \Rightarrow$ there exist open and closed halfspaces.
- X^* separates points on X, i.e., $\forall\, x_1, x_2 \in X$, $\exists\, x^* \in X^*$ such that
$$\langle x^*, x_1\rangle \neq \langle x^*, x_2\rangle.$$
- $C \subset X$ is a closed convex set $\Leftrightarrow$ C is the intersection of some family of closed halfspaces.

Outline

1. Introduction
   - Notation and basic definitions
   - On even convexity
2. Evenly convex functions
   - Introduction: Motivation
   - Basic properties of e-convex functions
   - Functional operations preserving even convexity
3. Duality for evenly convex functions
   - A new support function for e-convex sets
   - New characterizations of e-convex functions
   - A conjugation scheme for e-convex functions
4. Fenchel duality in evenly convex optimization problems
   - Introduction
   - Main results

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Definition (Fenchel, 1952)

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Main Properties

Proposition (Goberna et al. 2003)

Given $\emptyset \neq C \subsetneq \mathbb{R}^n$, *the following conditions are equivalent to each other:*

(*i*) C *is e-convex.*

(*ii*) C *is a convex set and for each* $x \notin C$ *there exists a hyperplane* H *such that* $x \in H$ *and* $H \cap C = \emptyset$.

(*iii*) C *is the result of eliminating from a closed convex set the union of a certain family of its exposed faces.*

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The *e-convex hull* of $C \subset X$, $\operatorname{eco} C$, is the intersection of all the open halfspaces containing C, i.e., the smallest e-convex set that contains C.

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If $C \subset \mathbb{R}^m$ and $A : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, then

$$A(\operatorname{eco} C) \subset \operatorname{eco} AC.$$

Main Properties

- $\operatorname{conv} C \subset \operatorname{eco} C \subset \operatorname{cl} \operatorname{conv} C$.
- $\bar{x} \notin \operatorname{eco} C \Leftrightarrow \exists\, x^* \in X^*$ such that $\langle x^*, x \rangle < \langle x^*, \bar{x} \rangle$ for all $x \in C$.
  In particular, $0 \notin \operatorname{eco} C \Leftrightarrow \{\langle z, x \rangle < 0,\ x \in C\}$ is consistent.
- If $C_1 \subset X$ and $C_2 \subset Y$, then

  $$\operatorname{eco}(C_1 \times C_2) = (\operatorname{eco} C_1) \times (\operatorname{eco} C_2).$$

  Therefore, C_1 and C_2 are e-convex $\Leftrightarrow C_1 \times C_2$ is e-convex.
- Given a family of nonempty sets in X, $\operatorname{eco}(\cap_{i \in I} C_i) \subset \cap_{i \in I} (\operatorname{eco} C_i)$.
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- If $C_1, C_2 \subset X$, then $\operatorname{eco} C_1 + \operatorname{eco} C_2 \subset \operatorname{eco}(C_1 + C_2)$.
- For any $D \subset X \times Y$, $\operatorname{proj}_X(\operatorname{eco} D) \subset \operatorname{eco}(\operatorname{proj}_X D)$.

Outline

Definition

A function $f : X \to \overline{\mathbb{R}}$ is said to be *evenly convex* (or, in brief, *e-convex*), if its epigraph, $\operatorname{epi} f$, is an e-convex set in $X \times \mathbb{R}$.

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Outline

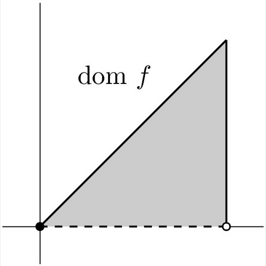
On the effective domain

- The effective domain of an e-convex function is not necessarily an e-convex set in X.

Example

Consider the function $f : \mathbb{R}^2 \to \overline{\mathbb{R}}$ defined by

$$f(x_1, x_2) = \begin{cases} x_1 \ln \frac{x_1}{x_2} & \text{if } 0 < x_1 \leq 1,\ 0 < x_2 \leq x_1, \\ 0 & \text{if } x_1 = x_2 = 0, \\ +\infty & \text{otherwise.} \end{cases}$$



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Proposition

Let f be an e-convex function. If either

- *f is improper, or*
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Let f be an improper function such that $f(x_0) = -\infty$ for some $x_0 \in X$. If f is e-convex, then $f(x) = -\infty$ for all $x \in \operatorname{dom} f$.

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Theorem

Let f be an improper function s.t. $f(x_0) = -\infty$ for some $x_0 \in X$. Then,

$$f \text{ is e-convex} \Leftrightarrow \begin{array}{c} \operatorname{dom} f \text{ is e-convex and} \\ f(x) = -\infty \;\; \forall x \in \operatorname{dom} f. \end{array}$$

Characterization

Theorem

Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be a proper function. Then,*

$$f \text{ is e-convex } \Leftrightarrow f \text{ is convex and lsc on } \operatorname{eco}(\operatorname{dom} f).$$

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Sketch of the Proof:

($\Rightarrow$) It is well-known that f is convex and lsc on $\operatorname{rint}(\operatorname{dom} f)$.

- We prove that f is lsc on $\operatorname{eco}(\operatorname{dom} f) \setminus \operatorname{rint}(\operatorname{dom} f) \subset \operatorname{rbd}(\operatorname{dom} f)$.

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($\Leftarrow$) For any $(\bar{x}, \bar{a}) \notin \operatorname{epi} f$, $\exists\, H$ such that $(\bar{x}, \bar{a}) \in H$ and $H \cap \operatorname{epi} f = \emptyset$?

- $\bar{x} \notin \operatorname{eco}(\operatorname{dom} f)$: Easy!
- $\bar{x} \in \operatorname{eco}(\operatorname{dom} f) \setminus \operatorname{rint}(\operatorname{dom} f)$: $(\bar{x}, \bar{a}) \notin \operatorname{cl}(\operatorname{epi} f)$.
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- $\bar{x} \in \operatorname{rint}(\operatorname{dom} f)$: We consider the following result:

Theorem (Rockafellar, 1970)

Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be a proper convex function and* $\bar{x} \in \operatorname{rint}(\operatorname{dom} f)$. *Then,* $\exists\, u \in \mathbb{R}^n$ *such that* $a - f(\bar{x}) \geq \langle u, x - \bar{x} \rangle$ *for all* $(x, a) \in \operatorname{epi} f$.

On the strict epigraph

- The strict epigraph of an e-convex function is not necessarily an e-convex set.

Example

Consider the function $f : \mathbb{R} \to \overline{\mathbb{R}}$ defined by

$$f(x) = \begin{cases} -\sqrt{1-x^2} & \text{if } -1 \leq x \leq 1, \\ +\infty & \text{otherwise.} \end{cases}$$

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Let $\emptyset \neq C \subset X \times \mathbb{R}$ be an e-convex set such that $(0,1) \in 0^+C$. Then, the function $f_C : X \to \overline{\mathbb{R}}$ is e-convex.

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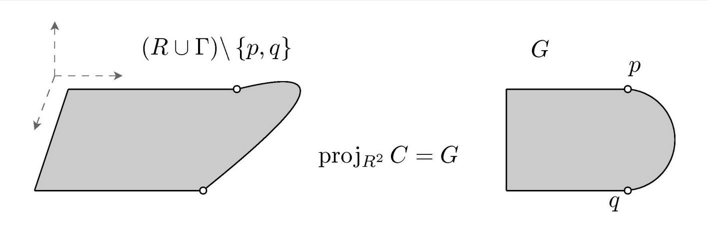
Example (Klee et al. 2007)

Consider the e-convex set $C := \text{conv}(R \cup \Gamma) \setminus \{p,q\} \subset \mathbb{R}^2 \times \mathbb{R}$.



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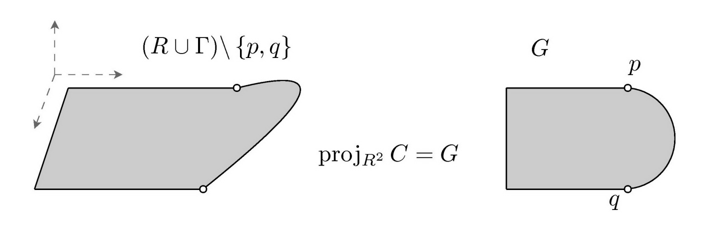

Observe that $(0,1) \notin 0^+C = \{0_n\}$ (C is bounded), but the function $f_C : \mathbb{R}^2 \to \overline{\mathbb{R}}$ is not e-convex.

Outline

Main operations

Proposition

(i) f *is e-convex,* $\alpha > 0 \Rightarrow \alpha f$ *is e-convex.*

(ii) $\{f_i,\ i \in I\}$ *are e-convex* $\Rightarrow \sup_{i \in I} f_i$ *is e-convex.*

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Proposition

Let f and g be e-convex functions and assume that f is improper. Then,

$$f + g \text{ is e-convex} \Leftrightarrow \operatorname{dom}(f+g) \text{ is an e-convex set.}$$

E-convex hull function

Definition

The *e-convex hull* of f, $\operatorname{eco} f$, is the largest e-convex minorant of f.

f is said to be *e-convex at* $x_0 \in X$ if $(\operatorname{eco} f)(x_0) = f(x_0)$.

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- f is e-convex at $x_0 \in X \Leftrightarrow (x_0, a) \notin \operatorname{eco}(\operatorname{epi} f)$ for all $a < f(x_0)$.
- f is e-convex $\Leftrightarrow f$ is e-convex at x_0, for every $x_0 \in X$.

Outline

The e-support function τ_C

$L := \overline{\mathbb{R}} \times \{0, 1\}$ with the lexicographic order $\leq_L$ is a complete chain.

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- For any $C \subset X$ and $(\alpha, \beta) \in L$, one has

$$C \subset \{x \in X \mid (\langle x^*, x\rangle, 1) \leq_L (\alpha, \beta)\} \;\Leftrightarrow\; \tau_C(x^*) \leq_L (\alpha, \beta).$$

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- Geometric interpretation: τ_C describes all the closed and the open halfspaces containing C.

$$\{x \in X \mid (\langle x^*, x\rangle, 1) \leq_L (\alpha, 1)\} = \{x \in X \mid \langle x^*, x\rangle \leq \alpha\}$$
$$\{x \in X \mid (\langle x^*, x\rangle, 1) \leq_L (\alpha, 0)\} = \{x \in X \mid \langle x^*, x\rangle < \alpha\}$$

Relationship between τ_C and σ_C

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Proposition

For any $C \subset X$ and $x^* \in X^*$, one has

$$\tau_C(x^*) = (\sigma_C(x^*), \eta_C(x^*)),$$

where $\eta_C : X^* \to \{0, 1\}$ is the function defined by

$$\eta_C(x^*) := \begin{cases} 0 & \text{if } \langle x^*, x\rangle < \sigma_C(x^*) \ \ \forall x \in C, \\ 1 & \text{if } \exists\, x \in C \mid \langle x^*, x\rangle = \sigma_C(x^*). \end{cases}$$

Relationship between C and $\mathcal{T}_{\tau_C}$

- For any $g : X^* \to L$, we define the e-convex set

$$\mathcal{T}_g := \{x \in X \mid (\langle x^*, x \rangle, 1) \leq_L g(x^*),\ \forall x^* \in X^*\}.$$

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Theorem

For any $C \subset X$, one has

$$\operatorname{eco} C = \mathcal{T}_{\tau_C}.$$

Corollary

Given $C, D \subset X$, the following statements hold:

- *C is e-convex $\Leftrightarrow$ $C = \mathcal{T}_{\tau_C}$.*
- *C is e-convex $\Leftrightarrow$ C is the solution set of the general linear system*

$$\{\langle x^*, x\rangle < \sigma_C(x^*),\ \forall x^* \mid \eta_C(x^*) = 0;\ \langle x^*, x\rangle \leq \sigma_C(x^*),\ \forall x^* \mid \eta_C(x^*) = 1\}.$$

- $\operatorname{eco} C \subset \operatorname{eco} D \Leftrightarrow \tau_C \leq_L \tau_D$.
- $\tau_C = \tau_{\operatorname{eco} C}$.

Characterization of τ_C

Theorem (Rockafellar, 1970)

The functions which are the support functions of non-empty (closed) convex sets are the closed proper sublinear functions.

- What conditions should satisfy a function $g : X^* \to L$ for being the e-support function of some non-empty set?

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- What conditions should satisfy a function $g : X^* \to L$ for being the e-support function of some non-empty set?

Theorem

Let $g : \mathbb{R}^n \to L$ be a function such that $g = (\sigma, \eta)$. Then, g is the e-support function of some non-empty e-convex set C ($g = \tau_C$) if and only if the following conditions hold:

(i) σ *is sublinear, lsc and does not take* $-\infty$.

(ii) *if* $\sigma(x^*) = -\sigma(-x^*)$ *then* $\eta(x^*) = 1$.

(iii) *if* $\partial\sigma(x^*) = \emptyset$ *then* $\eta(x^*) = 0$.

(iv) *if there exists* $\hat{x} \in \mathbb{R}^n$ *such that* $\eta(\hat{x}) = 0$ *and* $\partial\sigma(x^*) \subset \partial\sigma(\hat{x})$, *then* $\eta(x^*) = 0$.

Consequences

- The mapping $C \mapsto \tau_C$ is a bijection from the family of non-empty e-convex sets in $\mathbb{R}^n$, to the family of functions $g = (\sigma, \eta) : \mathbb{R}^n \to L$ satisfying conditions (i) to (iv).

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- If $C \subset \mathbb{R}^n$ is e-convex, then

$$C \text{ is closed } \Leftrightarrow \eta_C(x^*) = 1 \ \ \forall\, x^* \in \mathbb{R}^n \mid \partial\sigma_C(x^*) \neq \emptyset.$$

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- If $C \subset \mathbb{R}^n$ is convex, then

$$C \text{ is open } \Leftrightarrow \eta_C(x^*) = 0 \quad \forall\, x^* \in \mathbb{R}^n \setminus \{0\}.$$

Consequences

- The mapping $C \mapsto \tau_C$ is a bijection from the family of non-empty e-convex sets in $\mathbb{R}^n$, to the family of functions $g = (\sigma, \eta) : \mathbb{R}^n \to L$ satisfying conditions (i) to (iv).

  The converse bijection is the mapping $g \mapsto \mathcal{T}_g$.

- If $C \subset \mathbb{R}^n$ is e-convex, then

$$C \text{ is closed } \Leftrightarrow \eta_C(x^*) = 1 \;\; \forall\, x^* \in \mathbb{R}^n \,|\, \partial\sigma_C(x^*) \neq \emptyset.$$

- If $C \subset \mathbb{R}^n$ is convex, then

$$C \text{ is open } \Leftrightarrow \eta_C(x^*) = 0 \;\; \forall\, x^* \in \mathbb{R}^n \backslash \{0\}.$$

- If $C \subset \mathbb{R}^n$ is convex, then

$$C \text{ is relatively open } \Leftrightarrow \eta_C(x^*) = 0 \;\; \forall\, x^* \in \mathbb{R}^n \,|\, \sigma_C(x^*) \neq -\sigma_C(-x^*).$$

Outline

Characterization (1)

Definition

Let $C \subset X$. A function $a : X \to \overline{\mathbb{R}}$ is called *C-affine* if there exist $y^* \in X^*$ and $\beta \in \mathbb{R}$ such that

$$a(x) = \begin{cases} \langle y^*, x \rangle - \beta & \text{if } x \in C, \\ +\infty & \text{if } x \notin C. \end{cases}$$

- If C is e-convex, then every C-affine function is e-convex.

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For any $f : X \to \overline{\mathbb{R}}$, if $M_f := \operatorname{eco}(\operatorname{dom} f)$, we define the set $\mathcal{H}_f$ as

$$\mathcal{H}_f := \left\{ a : X \to \overline{\mathbb{R}} \mid a \textit{ is } M_f\textit{-affine, } a \leq f \right\}.$$

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Lemma

For any $f : X \to \overline{\mathbb{R}}$, one has

$$\mathcal{H}_f = \mathcal{H}_{\operatorname{eco} f}.$$

Characterization (1)

Proposition

Let $f : X \to \overline{\mathbb{R}}$. *The following statements are equivalent:*

(i) $\mathcal{H}_f \neq \emptyset$.

(ii) $\operatorname{eco} f$ *is proper or* $f \equiv +\infty$.

(iii) f *has a proper e-convex minorant.*

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Theorem

Let $f : X \to \overline{\mathbb{R}}$ be a function such that $f \not\equiv -\infty$ and $f \not\equiv +\infty$. Then

$$f \text{ is proper and e-convex} \Leftrightarrow f = \sup \{a \mid a \in \mathcal{H}_f\}.$$

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- If f has a proper e-convex minorant, then $\operatorname{eco} f = \sup\{a \mid a \in \mathcal{H}_f\}$.

Characterization (2)

Definition

Let $\mathcal{C}$ be the family of all e-convex sets in X. A function $a : X \to \overline{\mathbb{R}}$ is called *$\mathcal{C}$-affine* if there exists $C \in \mathcal{C}$ such that a is C-affine.

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$$f \textit{ is proper and e-convex} \;\Leftrightarrow\; f = \sup\left\{a \,|\, a \in \mathcal{E}_f\right\}.$$

- $\operatorname{eco}(\operatorname{dom} f) = \bigcap_{a\in\mathcal{E}_f} \operatorname{dom} a$, for any proper e-convex function f.

Outline

Generalized convex conjugation

Moreau (1970), Martínez-Legaz (2005)

- X, W arbitrary sets.
- $c : X \times W \to \overline{\mathbb{R}}$ is called the coupling function.
- $c' : W \times X \to \overline{\mathbb{R}}$ is given by $c'(w, x) = c(x, w) \;\; \forall x \in X, \, w \in W$.
- Conventions: $+\infty - (+\infty) = -\infty + (+\infty) = -\infty$.

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- The c-conjugate of $f : X \to \overline{\mathbb{R}}$ is the function $f^c : W \to \overline{\mathbb{R}}$ defined by

$$f^c(w) := \sup_{x \in X} \{c(x,w) - f(x)\}.$$

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- Fenchel conjugate: $X,\ W = X^*,\ c(x,x^*) = \langle x^*, x\rangle,\ f^c = f^*$.

$$f^*(x^*) = \sup_{x \in X} \{\langle x^*, x\rangle - f(x)\}.$$

A new conjugation scheme

- Consider X and $W = X^* \times X^* \times \mathbb{R}$, and the coupling function

$$c\left(x,(y^*,z^*,\alpha)\right) := \begin{cases} \langle y^*, x\rangle & \text{if } \langle z^*, x\rangle < \alpha, \\ +\infty & \text{if } \langle z^*, x\rangle \geq \alpha. \end{cases}$$

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$$f^{cc'} = \begin{cases} f^{**} + \delta_{\operatorname{eco}(\operatorname{dom} f)} & \textit{if } \operatorname{dom} f^* \neq \emptyset, \\ -\infty & \textit{if } \operatorname{dom} f^* = \emptyset. \end{cases}$$

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- c-elementary functions: $x \in X \mapsto c(x, (y^*, z^*, \alpha)) - \beta \in \overline{\mathbb{R}}$.
  c'-elementary functions: $(y^*, z^*, \alpha) \in W \mapsto c'((y^*, z^*, \alpha), x) - \beta \in \overline{\mathbb{R}}$.
- The c-elementary functions are the e-affine functions.
- Φ_c $(\Phi_{c'})$: the set of c-elementary (c'-elementary) functions.

- A function $f : X \to \overline{\mathbb{R}}$ is called Φ-convex if it is the pointwise supremum of a subset of Φ.
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$$f \text{ is e-convex} \Leftrightarrow f = f^{cc'}.$$
$$g \text{ is e}'\text{-convex} \Leftrightarrow g = g^{c'c}.$$

Outline

Generalized Optimization

Consider the primal problem

$$(GP) \quad \operatorname*{Inf}_{x\in X} F(x),$$

where $F : X \to \overline{\mathbb{R}}$ is a proper function, and the pertubation function $\Phi : X \times \Theta \to \overline{\mathbb{R}}$ having the property that, for every $x \in X$,

$$\Phi(x, 0_\Theta) = F(x).$$

The infimum value function $p : \Theta \to \overline{\mathbb{R}}$ is defined by

$$p(u) := \inf_{x\in X} \Phi(x, u).$$

The dual problem of (GP) associated to Φ is

$$(GD) \quad \operatorname*{Sup}_{u^*\in\Theta^*} -\Phi^*(0, u^*) = \operatorname*{Sup}_{u^*\in\Theta^*} -p^*(u^*).$$

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One has ***weak duality****, i.e.,*

$$v(GD) = p^{**}(0_\Theta) \leq p(0_\Theta) = v(GP).$$

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We have ***strong duality*** *if*

$v(GD) = v(GP)$ *and the dual problem is solvable when* $v(GP)$ *is finite.*

Convex Optimization

Let us consider the primal problem

$$(P) \quad \operatorname*{Inf}_{x \in A} f(x)$$

where $f : X \to \overline{\mathbb{R}}$ is proper closed convex and $\emptyset \neq A \subset X$ is closed convex, and the perturbation function $\Phi : X \times X \to \overline{\mathbb{R}}$ defined by

$$\Phi(x,u) := \begin{cases} f(x+u) & \text{if } x \in A, \\ +\infty & \text{otherwise.} \end{cases}$$

The Fenchel dual problem of (P) is

$$(D) \quad \operatorname*{Sup}_{u^* \in X^*} -p^*(u^*)$$

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Theorem (Burachik & Jeyakumar, 2005)

If $A \cap \operatorname{dom} f \neq \emptyset$ and the set $\operatorname{epi} f^* + \operatorname{epi} \delta_A^*$ is weak-closed, then **strong duality** holds for (P)–(D), i.e.,*

$$\inf_{x \in A} f(x) = \max_{u^* \in X^*} \{-f^*(u^*) - \delta_A^*(-u^*)\}.$$

Outline

Definition

A set $D \subset W \times \mathbb{R}$ is called e$'$-convex if there exists an e$'$-convex function $k : W \to \overline{\mathbb{R}}$ such that $D = \operatorname{epi} k$. The e$'$-convex hull of $D \subset W \times \mathbb{R}$, $\mathrm{e}'\mathrm{co}\, D$, is the smallest e$'$-convex containing D.

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Definition

Given $f, g : X \to \overline{\mathbb{R}}$, a function $a : X \to \overline{\mathbb{R}}$ belongs to $\widetilde{\mathcal{E}}_{f+g}$ if there exist $a_1 \in \mathcal{E}_f$ and $a_2 \in \mathcal{E}_g$ such that, if

$$a_i(\cdot) := \begin{cases} \langle y_i^*, \cdot\rangle - \beta_i & \text{if } \langle z_i^*, \cdot\rangle < \alpha_i, \\ +\infty & \text{otherwise,} \end{cases}$$

for $i = 1, 2$, then

$$a(\cdot) = \begin{cases} \langle y_1^* + y_2^*, \cdot\rangle - (\beta_1 + \beta_2) & \text{if } \langle z_1^* + z_2^*, \cdot\rangle < \alpha_1 + \alpha_2, \\ +\infty & \text{otherwise.} \end{cases}$$

- $\widetilde{\mathcal{E}}_{f+g} \subset \mathcal{E}_{f+g}$.

Evenly Convex Optimization

Let us consider the primal problem

$$(P) \quad \operatorname*{Inf}_{x \in A} f(x)$$

where $f : X \to \overline{\mathbb{R}}$ is proper e-convex and $\emptyset \neq A \subset X$ is e-convex, and the perturbation function $\Phi : X \times X \to \overline{\mathbb{R}}$ defined by

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Let us consider the primal problem

$$(P) \quad \operatorname*{Inf}_{x \in A} f(x)$$

where $f : X \to \overline{\mathbb{R}}$ is proper e-convex and $\emptyset \neq A \subset X$ is e-convex, and the perturbation function $\Phi : X \times X \to \overline{\mathbb{R}}$ defined by

$$\Phi(x,u) := \begin{cases} f(x+u) & \text{if } x \in A, \\ +\infty & \text{otherwise.} \end{cases}$$

The Fenchel dual problem of (P) is

$$(D) \quad \operatorname*{Sup}_{\substack{u^*, v^* \in X^* \\ \alpha > 0}} -p^c(u^*, v^*, \alpha) = \operatorname*{Sup}_{\substack{u^*, v^* \in X^* \\ \alpha_1 + \alpha_2 > 0}} -f^c(u^*, v^*, \alpha_1) - \delta_A^c(-u^*, -v^*, \alpha_2).$$

Evenly Convex Optimization

Let us consider the primal problem

$$(P) \quad \operatorname*{Inf}_{x \in A} f(x)$$

where $f : X \to \overline{\mathbb{R}}$ is proper e-convex and $\emptyset \neq A \subset X$ is e-convex, and the perturbation function $\Phi : X \times X \to \overline{\mathbb{R}}$ defined by

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The Fenchel dual problem of (P) is

$$(D) \quad \operatorname*{Sup}_{\substack{u^*, v^* \in X^* \\ \alpha > 0}} -p^c(u^*, v^*, \alpha) = \operatorname*{Sup}_{\substack{u^*, v^* \in X^* \\ \alpha_1 + \alpha_2 > 0}} -f^c(u^*, v^*, \alpha_1) - \delta_A^c(-u^*, -v^*, \alpha_2).$$

Theorem

*If $A \cap \operatorname{dom} f \neq \emptyset$, the set $\operatorname{epi} f^c + \operatorname{epi} \delta_A^c$ is e'-convex and $f + g = \sup\left\{a \mid a \in \widetilde{\mathcal{E}}_{f+g}\right\}$, then **strong duality** holds for (P)–(D).*

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