On Schatten-Herz operators

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Introduction

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Schatten classes

Recall that a compact operator $T : A_2 \to A_2$ is said to belong to the Schatten class $S_p(A_2)$ if

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For positive operators T, i.e. $\langle Tf, f\rangle \geq 0$ for all $f \in A_2$,

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Also recall that if $T \geq 0$ then $T \in S_p$ if and only if $T^p \in S_1$.

Toeplitz operators and Berezin transform

A Toeplitz operator T_φ with symbol $\varphi \in L^1(\mathbb{D})$, is given by

$$T_\varphi(f)(z) = \int_{\mathbb{D}} \frac{f(w)\varphi(w)}{(1-\bar{w}z)^2} dA(w), \quad f = \sum_{n=0}^{m} \alpha_m e_m.$$

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$$\widetilde{\varphi}(z) = (1-|z|^2)^2 \int_{\mathbb{D}} \frac{\varphi(w)}{|1-z\overline{w}|^4} dA(w)$$

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A general question: **What can be said for other spaces of operators or other spaces of functions in terms of the Berezin transform?**

Mixed norm spaces

For $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$ the mixed norm space $L^{p,q,\alpha}$ is the space of all measurable complex functions f on $\mathbb{D}$ such that

$$\|f\|_{L^{p,q,\alpha}} = \left(\int_0^1 (1 - r^2)^{q\alpha - 1} M_p^q(f, r) dr \right)^{1/q} < \infty, \tag{5}$$

with

$$M_p(f, r) = \left(\int_0^{2\pi} \left| f(re^{i\theta}) \right|^p \frac{d\theta}{2\pi} \right)^{1/p}.$$

Of course $L^p(\mathbb{D}, dA) = L^{p,p,1/p}$ and $f \in L^p(\mathbb{D}, d\lambda)$ if and only if $(1 - |z|^2)^{-2/p} f \in L^{p,p,1/p}$.

Herz spaces

For $\alpha \in \mathbb{R}$ and $1 \leq p, q \leq \infty$ let $\mathscr{K}_q^{p,\alpha}$ be Herz spaces consisting of all measurable functions such that $\left(2^{-n\alpha} \|f\|_{L^p(A_n,dA)}\right)_{n\in\mathbb{N}} \in \ell^q$ where

$$A_n = \{z : 1 - 2^{-n} \leq |z| < 1 - 2^{-(n+1)}\}.$$

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We provide $\mathscr{K}_q^{p,\alpha}$ with the norm

$$\|f\|_{\mathscr{K}_q^{p,\alpha}} = \left\| \left(2^{-n\alpha} \|f\|_{L^p(A_n, dA)}\right)_{n\in\mathbb{N}} \right\|_{\ell^q} < \infty. \quad (6)$$

Of course $\mathscr{K}_p^{p,0} = L^p(\mathbb{D}, dA)$ and $\mathscr{K}_p^{p,-2/p} = L^p(\mathbb{D}, d\lambda)$
We will write $\mathscr{K}_q^p(\lambda) = \mathscr{K}_q^{p,-2/p}$.

Kellog spaces

Denote $I_k = [2^k - 1, 2^{k+1}) \cap (\mathbb{N} \cup \{0\})$ for $k \in \mathbb{N} \cup \{0\}$ and for $0 < p, q \leq \infty$ define Kellog's spaces $\ell(p,q)$ as the space of sequences $(\lambda_n)_{n \geq 0}$ such that

$$\|(\lambda_n)\|_{\ell(p,q)} = \Big(\sum_{k \geq 0} \big(\sum_{n \in I_k} |\lambda_n|^p \big)^{q/p} \Big)^{1/q},$$

and the obvious modifications for $p = \infty$ or $q = \infty$.
We have $\ell(p,p) = \ell^p$ and

$$\|(\lambda_n)\|_{\ell(p,q)} = \|(|\lambda_n|^p)\|_{\ell(1,q/p)}^{1/p}. \tag{7}$$

Two problems on operators

Problem 1: **Which are the new classes of Toeplitz operators whose symbols have Berezin transform in mixed norm spaces $L^{p,q,\alpha}$.?**

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Previous results for Herz spaces

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Problem 2 : **Find a definition of Schatten-Herz classes valid for general operators.**

Some new classes of compact operators

Definition

Let $0 < p, q < \infty$ and T be a bounded operator on A_2. Denote $\Delta_j : A_2 \to A_2$ given by

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Let T be a positive compact operator on A_2 and $0 < q < \infty$. The following are equivalent:

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The case $S(1,q)$

Lemma

For $0 < q \leq \infty$ *and any sequence* $(\alpha_n)_n$ *of nonegative real numbers, one has*

$$\|(\alpha_n)_n\|_{\ell(1,q)} \sim \|(1-r)\sum_{n=0}^{\infty}(n+1)\alpha_n r^n\|_{L^q([0,1],\frac{dr}{1-r})}. \tag{8}$$

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Corollary

Let T *be a positive compact operator on* A_2. *Then* $(1-|z|^2)^{-2}\widetilde{T} \in L^{1,q,1}$ *if and only if* $\langle Te_n, e_n\rangle \in \ell(1,q)$.