# Games, communication complexity and tensor norms

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Functional Analysis Jaca 2011

# XOR games

A XOR game $G = (f, \pi)$ with $N$ inputs on each side is defined by a function

$$f : [N] \times [N] :\longrightarrow \{-1, 1\}$$

together with a probability distribution $\pi : [N] \times [N] :\longrightarrow [0, 1]$. Alice and Bob receive as inputs $x, y \in [N]$ respectively with probability $\pi(x, y)$ and each of them must answer a number $a, b \in \{-1, 1\}$, so that $f(x, y) = a \cdot b$.

# The players

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# Strategies

The *strategies* of the players are the probabilities $(p(a,b|x,y))_{a,b,x,y}$ with $a,b=\pm1$, $x,y=1,\ldots,N$

Since XOR games only use the product *ab*, we can characterize the stategies by their *correlations* $(\gamma_{x,y})_{x,y=1}^N$, with $\gamma_{x,y}=\mathbf{E}(ab|x,y)$.

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# Strategies

Every strategy $(\gamma_{x,y})_{x,y=1}^{N}$ belongs to the unit ball of $\ell_\infty^N \otimes_\epsilon \ell_\infty^N = \ell_\infty^{N^2}$

Vamos arrimando el ascua a nuestra sardina

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# Value of the game

We define the *value of the game* as the supremum of the numbers

$$\sum_{x,y} G(x,y)\gamma_{x,y}$$

where the supremum is considered over the set of *admissible* strategies.

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# The norm connection

In all the scenarios we will study, the sets of admissible strategies will be symmetric convex subsets of $\mathbb{R}^{N^2} = \mathbb{R}^N \otimes \mathbb{R}^N$ with non empty interior.

Therefore the different values of the game will be different tensor norms on $(\mathbb{R}^N \otimes \mathbb{R}^N)^*$.

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# Tensor norms

## Definition

Given $X$, $Y$ normed spaces, the projective ($\pi$) norm in $X \otimes Y$ is the norm whose unit ball is

$$co(B_X \otimes B_Y)$$

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Given $X$, $Y$, the injective ($\epsilon$) norm in $X \otimes Y$ is the dual norm of $\pi$.

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# Tensor norms for beginners

## Necessary (non sufficient) information to survive this talk

1. Today, normed spaces are finite dimensional.
   $\ell_\infty^N, \ell_1^N, \ell_2^N = \mathbb{C}_2^N, M_n, S_1^n, \dots$
2. $\ell_\infty^N \otimes_\epsilon X = \ell_\infty^N(X)$

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# Games as operators

- The strategies of the players can be seen as elements $\gamma_{x,y} \in \ell_\infty^N \otimes \ell_\infty^N$
- Games $G$ act linearly on strategies $\gamma$ by $\sum_{x,y} G_{x,y}\gamma_{x,y}$
- Hence, games can be seen as
  - $G : \ell_\infty^N \otimes \ell_\infty^N \longrightarrow \mathbb{R}$
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# More tensor norms

## More basic facts on tensor norms

1. $\|G : \ell_\infty^N \times \ell_\infty^N \longrightarrow \mathbb{R}\| = \|G : \ell_\infty^N \otimes_\pi \ell_\infty^N \longrightarrow \mathbb{R}\| = \|\tilde{G} : \ell_\infty^N \longrightarrow \ell_1^N\|$.
2. $\|G : \ell_\infty^N \otimes_\epsilon \ell_\infty^N \longrightarrow \mathbb{R}\| = \pi_1(\tilde{G}) = v(\tilde{G}) = \sup \sum_i \|\tilde{G}(A_i)\|$

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# The scenarios

We will consider different scenarios, using two parameters:

- The resources: They can be "classical" or "quantum"
- The communication: There can be no communication, classical comunication or quantum communication.
- Each of these types of communication can be two way, one way, or simultaneous message passing, or...

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# Classical resources, no communication

# Classical strategies

A *classical strategy* is a strategy of the form

$$p(a,b|x,y) = \int_\Lambda \alpha(a|x,\lambda)\beta(b|y,\lambda)d\lambda.$$

We view $\Lambda$ as the space of shared randomness.
It is very easy to see that the set of local correlations is exactly the unit ball of $\ell_\infty^N \otimes_\pi \ell_\infty^N$.
To maximize games, we need only consider correlations $\gamma_{x,y} = \alpha(x)\beta(y)$, with $\alpha(x), \beta(y) = \pm 1$.

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# Value of the game

- Value of the game$=\max \sum_{x,y} G_{x,y}\gamma_{x,y}$ over the classical strategies
- Classical strategies $=$ unit ball of $\ell_\infty^N \otimes_\pi \ell_\infty^N$
- Hence, Value of the game$= \|G\|$ when considering $G$ as an element of $(\ell_\infty^N \otimes_\pi \ell_\infty^N)^* = \ell_1^N \otimes_\epsilon \ell_1^N$

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# Classical resources, classical communication

# Two-Way classical communication

This one will be fast

We have no idea!

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# One way communication

We consider now the situation where one of the players can transmit $c$-bits of communication to the other one.

# Summing operators

Given a finite sequence with arbitrary length $(x_i)_{i=1}^n$ in a normed space $X$, we define the *weakly 1-summing* norm of $(x_i)_{i=1}^n$ by

$$\|(x_i)_{i=1}^n\|_1^w = \sup\{\sum_{i=1}^n |x^*(x_i)|, \text{ where } x^* \in B_{X^*}\}.$$

# Summing operators

Given an operator $T : X \longrightarrow Y$ between normed spaces, we define its 1*-summing* norm as

$$\pi_1(T) = \inf\{C \text{ such that } \sum_{i=1}^{n} \|T(x_i)\| \leq C\|(x_i)_{i=1}^n\|_1^w\}$$

for every sequence $(x_i)_{i=1}^n \subset X$.

# Summing operators

If we fix $r \in \mathbb{N}$ and restrict the previous definition to sequences $(x_i)_{i=1}^r$ of maximum length $r$ we obtain the definition of the 1*-summing with r vectors* norm of $T$, which we denote by $\pi_1^r(T)$.

We do not have a satisfactory description of the strategies in this case...

...but we do know

- They are a convex symmetric set with no empty interior, hence they are the unit ball of certain norm
- We can describe the dual norm.

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## Lemma

*Given* $T : \ell_\infty^N \longrightarrow \ell_1^N$,

$$\pi_1^r(T) = \sup \sum_i \| T(\chi_{A_i)} \|,$$

*where* $A_1, \dots A_r$ *is an* $r$*-length partition of* $\{1, \dots, N\}$ *and* $\chi_{A_i} = \sum_{j \in A_i} e_j$.

Consequence: Value of the game$= \pi_1^{2^c}(\tilde{G})$.

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# Simultaneous message passing

In the simultaneous message passing model, both players can not communicate, but they can each transmit $c_A, c_B$ bit of communication to a third party which does the computations. This model is weaker than the one way communication.

We are happy.

The value of the game here is the multiple (1,1)-summing norm with $(2^{c_A}, 2^{c_B})$ vectors.

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# Quantum resources, no communication

# Quantum faith

We pray you believe in the Quantum Commandments:

- The system formed by Alice and Bob is described by the norm one vectors of the Hilbertian tensor product $H_A \otimes H_B$ of Alice and Bob's local systems. Where $H_A = H_B = \mathbb{C}^n$
- That is, the state of their system is described by a *joint state* $|\varphi\rangle \in H_A \otimes H_B$
- It could well be that $|\varphi\rangle \neq |\varphi_A\rangle \otimes |\varphi_B\rangle$

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- For every question $x = 1, \ldots, N$ she is asked, Alice will do a *measurement* on her side of $|\varphi\rangle$. This will be nothing but a norm one selfadjoint operator $A_x \in B(H_A) = M_n$ with eigenvalues $\pm 1$. Indeed, think of $A_x = P - P^\perp$
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We would be happy with new tools!!

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# Operator Spaces

## Definition

An operator space is...

- ...A Banach/normed space $E$ together with an isometric embedding $E \hookrightarrow B(H)$, or...
- ... a Banach/normed space together with a "well behaved" sequence of norms on $M_n \otimes E$, or ...

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# Completely bounded maps

...or instead of thinking of the objects (Operator Spaces) we can think of the morphisms

**Definition**

Given two operator spaces $E, F$, a linear map $T : E \longrightarrow F$. is *completely bounded* if

$$\|T\|_{cb} := \sup_n \|Id_n \otimes T : M_n \otimes E \longrightarrow M_n \otimes F\| < \infty$$

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# One Operator Space

Today, just one example.

## Single Example

$\ell_\infty^N$.

We consider the operator space it inherits from the embedding $\ell_\infty^N \hookrightarrow M_N$ taking $(a_1, \ldots, a_N)$ to

$$\begin{matrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ & \ldots & \ldots & \\ 0 & \ldots & 0 & a_N \end{matrix}$$

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# The associated norms

$$u = (A_1, \ldots, A_N) \in M_n \otimes \ell_\infty^N,$$

$$\|u\| = \sup_i \|A_i\|_{M_n}$$

# Bilinear completely bounded operators

## Definition

A bilinear operator $G : \ell_\infty^N \times \ell_\infty^N \longrightarrow \mathbb{C}$ or $\mathbb{R}$ is *completely bounded* if

$$\|G\|_{cb} := \sup_n \|Id_n \otimes Id_n \otimes G : M_n \otimes \ell_\infty^N \times M_n \otimes \ell_\infty^N \longrightarrow M_{n^2}\| < \infty$$

# Bilinear norm vs game value

## Bilinear norm

If $G : \ell_\infty^N \times \ell_\infty^N \longrightarrow \mathbb{R}$ is given by $G(e_x, e_y) = G_{x,y}$ then

$$\|G\|_{cb} = \sup_n \sup_{A_x, B_y \in M_n} \|\sum_{x,y} G_{x,y} A_x \otimes B_y\|_{M_{n^2}}$$

## Value of the game

The value of the game

$$\sup_{A_x, B_y} \|\sum_{x,y} G_{x,y} A_x \otimes B_y\|_{M_{n^2}}$$

# Bilinear norm vs game value

## Bilinear norm

If $G : \ell_\infty^N \times \ell_\infty^N \longrightarrow \mathbb{R}$ is given by $G(e_x, e_y) = G_{x,y}$ then

$$\|G\|_{cb} = \sup_n \sup_{A_x, B_y \in M_n} \|\sum_{x,y} G_{x,y} A_x \otimes B_y\|_{M_{n^2}}$$

## Value of the game

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# Quantum resources, classical communication

# Classical resources and communication revisited

**Lemma**

*The value of the game in this case is the norm of the operator*

$$Id \otimes G : \ell_1^k \otimes_\epsilon \ell_\infty^N \longrightarrow \ell_1^k \otimes_\pi \ell_1^N$$

# Classical resources, quantum communication

**Lemma**

*The value of the game is the norm of the operator*

$$Id \otimes G : S_1^k \otimes_\epsilon \ell_\infty^N \longrightarrow S_1^k \otimes_\pi \ell_1^N$$

## Proof.

Alice receives $x$. She picks a $2^c$-dimensional state $\rho_x$ and sends it to Bob.

Bob receives $\rho_x$. He measures with a $\pm 1$ valued Hermitian operator $B_y$ to it.

The final correlation is $\gamma_{x,y} = tr(\rho_x B_y)$.

The value of the game is

$$\sup_{\gamma} \langle G, \gamma \rangle = \sup_{\rho_x, B_y} \sum_{x,y} G_{x,y} tr(\rho_x B_y) = \sup_{\rho_x} \sum_{y} \| \sum_{x} G_{x,y} \rho_x \|_{S_1^k}.$$

To see that this is what we were looking for, note that $(\rho_x)_x$ is a bounded sequence; that is, a norm one element in $S_1^k \otimes_\epsilon \ell_\infty^N$. □

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# Quantum resources, classical communication

**Lemma**

*The value of the game is is the completely bounded norm of the operator*

$$Id \otimes G : \ell_1^k \otimes_{\min} \ell_\infty^N \longrightarrow \ell_1^k \otimes_\wedge \ell_1^N$$

## Proof.

Alice and Bob share an arbitrary dimensional state $\rho$

Alice receives her input $x$ and she performs a measurement with $k$ possible outputs $P_x = (P_x^i)_{i=1}^k$ on her part of the state.

If she obtains output $i$, she will send the word $i$ to Bob.

Bob performs a $\pm 1$ valued measurement $B_y^i$ dependant on $y$ and $i$.

Then,

$$\gamma_{x,y} = tr(P_x \rho B_y^i).$$

Hence,

$$\sup_{\gamma} \langle G, \gamma \rangle = \sup_{\rho, P_x, B_y} \sum_{x,y} G_{x,y} tr(P_x \rho B_y^i) =$$

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