4. For a bounded operator $T: X \to X$ on an infinite dimensional real Banach space establish the following.
(a) The operator T has a non-zero finite dimensional invariant subspace if and only if the operator $T_c: X_c \to X_c$ likewise has a non-zero finite dimensional invariant subspace.
(b) The operator T has a non-zero finite dimensional hyperinvariant subspace if and only if the operator $T_c: X_c \to X_c$ likewise has a non-zero finite dimensional hyperinvariant subspace.

5. Let $T: X \to X$ be a bounded operator on a Banach space. If X is not reflexive, then show that the double adjoint $T^{**}: X^{**} \to X^{**}$ has a non-trivial norm closed T^{**}-invariant subspace.

6. Let Ω be a compact Hausdorff space. Fix some $\phi \in C(\Omega)$ and consider the multiplication operator M_ϕ on $C(\Omega)$. If Ω_0 is a non-empty subset of Ω and $V = \{f \in C(\Omega): f(\omega) = 0 \text{ for all } \omega \in \Omega_0\}$, then show that V is a closed M_ϕ-invariant subspace of $C(\Omega)$.

7. Let Ω be a compact Hausdorff space. Fix a continuous mapping $\tau: \Omega \to \Omega$ and consider the composition operator C_τ on $C(\Omega)$ defined, as usual, by $C_\tau f = f\circ\tau$. If Ω_0 is a nonempty τ-invariant subset of Ω (i.e., $\tau(\Omega_0) \subseteq \Omega_0$), then show that the vector subspace

$$V = \{f \in C(\Omega): f(\omega) = 0 \text{ for all } \omega \in \Omega_0\}$$

is a closed C_τ-invariant subspace of $C(\Omega)$.

8. Prove Lemma 10.14.

9. If X is a Banach space, then show that the only I-hyperinvariant subspaces are $\{0\}$ and X.

10. Let $\mathcal{A}$ be an arbitrary algebra of operators on a vector space. Show that $\mathcal{A}_1 = \{\alpha I + A: \alpha \text{ is a scalar and } A \in \mathcal{A}\}$ is the unital algebra generated by $\mathcal{A}$. (This implies that any algebra of operators $\mathcal{A}$ and the unital algebra generated by $\mathcal{A}$ have the same invariant subspaces.)

11. Show that an algebra of operators $\mathcal{A} \subseteq \mathcal{L}(X)$ has no non-trivial closed invariant subspaces if and only if for any pair $x, y \in X$ with $x \neq 0$ and every $\epsilon > 0$ there exists some operator $A \in \mathcal{A}$ such that $\|Ax - y\| < \epsilon$.

12. Let $\mathcal{A}$ be a unital algebra of continuous operators on a Banach space X and, as usual, let $\mathcal{A}x = \{Ax: A \in \mathcal{A}\}$. Show that $\mathcal{A}$ is non-transitive if and only if $\mathcal{A}$ has a non-trivial closed vector subspace of the form $\overline{\mathcal{A}x}$.

13. Let $T: X \to X$ be a bounded operator on a Banach space and consider the set

$$V = \Big\{x \in X: \lim_{n\to\infty} \|T^n x\| = 0\Big\}.$$

Establish the following.
(a) V is a T-hyperinvariant subspace.
(b) If T is power bounded, then V is also closed.

14. Let $S, T: X \to X$ be a pair of bounded operators on a Banach space. As usual, the **direct sum** of S and T is the operator $S\oplus T: X\oplus X \to X\oplus X$ defined by $(S\oplus T)(x\oplus y) = Sx \oplus Ty$. If a bounded operator $R: X \to X$ satisfies $RS = TR$, then show that its graph $G_R = \{x \oplus Rx: x \in X\}$ is a closed $S\oplus T$-invariant subspace of $X \oplus X$.

15. Show that the shift operator on any ℓ_p-space has a non-trivial closed hyperinvariant subspace.

16. Show that the converse of Corollary 10.13 is not true. That is, show that the dual algebra $\mathcal{A}^*$ of a transitive algebra of operators $\mathcal{A}$ can be non-transitive.

17. Let $\phi: \Omega \to \mathbb{R}$ be a continuous function separating the points of a compact Hausdorff space Ω and let M_ϕ denote the usual multiplication operator on $C(\Omega)$ defined by $M_\phi(f) = \phi f$. Show that $\{M_\phi\}' = \{M_g: g \in C(\Omega)\}$. [HINT: Let $T \in \{M_\phi\}'$ and put $h = T(1)$. Since $TM_\phi = M_\phi T$ implies $T(\phi f) = \phi T(f)$ for each $f \in C(\Omega)$, it follows that $T(\phi) = h\phi$ and by induction we see that $T(\phi^n) = h\phi^n$ for each $n \geq 0$. The latter implies $T(p(\phi)) = hp(\phi)$ for each polynomial p. By the Stone–Weierstrass Approximation Theorem the algebra $\{p(\phi): p \text{ a polynomial}\}$ is norm dense in $C(\Omega)$ and from this we conclude that $Tf = hf$ for each $f \in C(\Omega)$.]

10.2. The Lomonosov Invariant Subspace Theorem

Many well-known invariant subspace theorems are for operators that are related in one way or another with the compact operators. It is generally acclaimed that the most famous of these results is Lomonosov's Invariant Subspace Theorem [211]. It asserts that:

- *If an operator $T: X \to X$ on a complex Banach space commutes with a non-scalar operator $S \in \mathcal{L}(X)$ which in turn commutes with a non-zero compact operator, then T has a non-trivial closed invariant subspace.*

The objective of this section is to prove this theorem. To do so, we need the following result which is of importance in its own right.

Theorem 10.18 (Lomonosov [211]). *If $\mathcal{A}$ is a transitive* algebra of operators on a real or complex Banach space, then for each non-zero compact operator K there exists some $A \in \mathcal{A}$ such that the compact operator AK has a non-zero fixed point, i.e., $AKu = u$ for some $u \neq 0$.*

Proof. Let $\mathcal{A} \subseteq \mathcal{L}(X)$ be a transitive algebra and let K be a non-zero compact operator. We can assume $\|K\| = 1$.

Pick some $x_0 \in X$ such that $\|Kx_0\| > 1$ and let U_0 denote the closed unit ball centered at x_0. That is, $U_0 = \{x \in X: \|x - x_0\| \leq 1\}$. An easy

*Transitive ≡ with no non-trivial common closed invariant subspaces argument shows that

$$0 \notin U_0 \quad \text{and} \quad 0 \notin \overline{K(U_0)}\,. \tag{$\star$}$$

Since $\mathcal{A}$ is an algebra, $\overline{\mathcal{A}x}$ (the norm closure of $\mathcal{A}x$) is $\mathcal{A}$-invariant for each $x \in X$. We claim that $\overline{\mathcal{A}x} \neq \{0\}$ for each $x \neq 0$. Indeed, if $\overline{\mathcal{A}x} = \{0\}$ for some $x \neq 0$, then the non-trivial closed subspace $\{\alpha x : \alpha \text{ scalar}\}$ is clearly $\mathcal{A}$-invariant, contrary to the hypothesis. Moreover, the absence of non-trivial closed $\mathcal{A}$-invariant subspaces guarantees that $\overline{\mathcal{A}x} = X$ for each $x \neq 0$.

In particular, for each $x \in \overline{K(U_0)}$ there exists some $A \in \mathcal{A}$ such that $\|Ax - x_0\| < 1$. So, the family of open sets $\{y \in X : \|Ay - x_0\| < 1\}_{A \in \mathcal{A}}$ is an open cover of $\overline{K(U_0)}$, that is,

$$\overline{K(U_0)} \subseteq \bigcup_{A \in \mathcal{A}} \{y \in X : \|Ay - x_0\| < 1\}\,.$$

Since (in view of the compactness of K) the set $\overline{K(U_0)}$ is compact, there exist operators $A_1, \ldots, A_m \in \mathcal{A}$ such that

$$\overline{K(U_0)} \subseteq \bigcup_{i=1}^{m} \{y \in X : \|A_i y - x_0\| < 1\}\,. \tag{$\star\star$}$$

Next, for each i define the continuous function $f_i : X \to [0, \infty)$ by

$$f_i(z) = \max\{0, 1 - \|A_i z - x_0\|\}\,.$$

Clearly, $f_i(z) > 0$ if and only if $\|A_i z - x_0\| < 1$. From ($\star\star$), we see that $f(z) = \sum_{i=1}^{m} f_i(z) > 0$ for each $z \in \overline{K(U_0)}$, and so the formulas $g_i(z) = \frac{f_i(z)}{f(z)}$ define non-negative continuous functions on $\overline{K(U_0)}$ such that $\sum_{i=1}^{m} g_i(z) = 1$ for each $z \in \overline{K(U_0)}$. In particular, $\sum_{i=1}^{m} g_i(Kx) = 1$ for each $x \in U_0$.

Now consider the function $\phi : U_0 \to X$ defined by

$$\phi(x) = \sum_{i=1}^{m} g_i(Kx) A_i K x\,.$$

Since $x \in U_0$ and $g_i(Kx) > 0$ imply $A_i K x \in U_0$, it follows from the convexity of U_0 that $\phi(U_0) \subseteq U_0$. In addition, because $\phi(x)$ is a convex combination of the vectors $A_1 K x, \ldots, A_m K x$ and $A_i K x \in A_i K(U_0)$, we see that $\phi(x)$ belongs to $C = \overline{\text{co}}\left[\bigcup_{i=1}^{m} A_i K(U_0)\right]$. But $\bigcup_{i=1}^{m} A_i K(U_0)$ is a norm totally bounded set (since each $A_i K$ is a compact operator), and so by Mazur's theorem its closed convex hull C is likewise a compact set; see, for example [30, Theorem 9.4, p. 131].[2] Hence, $\phi(U_0) \subseteq C \cap U_0$ and $C \cap U_0$ is a non-empty convex compact subset of X.

[2] If X is a complex Banach space, then when taking the closed convex hull of C, we consider X as a real vector space.

By Tychonoff's Fixed Point Theorem, the map $\phi : C \cap U_0 \to C \cap U_0$ has a fixed point, say u. Since $u \in U_0$, it follows from ($\star$) that $u \neq 0$. To complete the proof, consider the operator $A = \sum_{i=1}^{m} g_i(Ku) A_i \in \mathcal{A}$, and note that $AKu = \sum_{i=1}^{m} g_i(Ku) A_i K u = \phi(u) = u$. ■

And now we are ready to state and prove the two famous Lomonosov invariant subspace theorems.

Theorem 10.19 (Lomonosov [211]). *If a non-scalar operator T on a complex Banach space commutes with a non-zero compact operator, then T has a non-trivial closed hyperinvariant subspace.* F (F invariant for every $S \in \mathcal{L}(X)$ s.t. $ST = TS$)

Proof. Assume that $T \in \mathcal{L}(X)$ is a non-scalar operator and that for some non-zero compact operator $K \in \mathcal{L}(X)$ we have $TK = KT$. To establish the existence of a non-trivial closed T-hyperinvariant subspace, according to
* Lemma 10.14, it suffices to show that the commutant of T is non-transitive.

To verify this, assume by way of contradiction that the commutant $\{T\}'$ is a transitive algebra. So, by Theorem 10.18, there exists some $A \in \{T\}'$ such that the operator AK has a non-zero fixed point. Let

$$F = \{x \in X : AKx = x\}\,.$$

Clearly, F is a non-zero closed AK-hyperinvariant subspace. Since AK is a compact operator and it is the identity on F, it follows that F is finite dimensional. Since T commutes with AK and F is AK-hyperinvariant, we see that F is T-invariant. So, from $T(F) \subseteq F$ and the fact that F is a finite dimensional complex Banach space, it follows that the operator $T : F \to F$ has an eigenvalue $\lambda \in \mathbb{C}$.[3] Now let

$$N_\lambda = \{x \in X : Tx = \lambda x\}\,,$$

and note that N_λ is a non-zero closed $\{T\}'$-invariant subspace which is different from X since T is a non-scalar operator. However, this contradicts our assumption and the proof of the theorem is complete. ■

The assumption that the Banach space X is complex is essential. As we know from Corollary 10.7, on a finite dimensional real Banach space of even dimension there exist operators with non-trivial invariant subspaces but without non-trivial hyperinvariant subspaces. It should be noted that for complex Banach spaces Theorem 10.19 implies immediately Theorem 10.15 and Corollary 10.16.

[3] This is the place where we use that X is a complex Banach space. When X is a real Banach space, there is no guarantee that the operator $T : F \to F$ has a real eigenvalue.

* $\{T\}' = \{S ; ST = TS\}$ is the commutant. Hence $\{T\}'$ transitive $\Leftrightarrow$ T has a non-trivial closed hyperinvariant subspace F