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ANALYTIC PROOFS OF THE “HAIRY BALL THEOREM” AND THE BROUWER FIXED POINT THEOREM

JOHN MILNOR

This note will present strange but quite elementary proofs of two classical theorems of topology, based on a volume computation in Euclidean space and the observation that the function $(1+t^2)^{n/2}$ is not a polynomial when n is odd. The argument was suggested by the methods of Asimov [1]. Familiar proofs of these theorems all use either combinatorial arguments, homology theory, differential forms, or methods from geometric topology. Compare [2], [3], [4], [6], [7].

Here is a preliminary version of the first one.

THEOREM 1. *An even-dimensional sphere does not possess any continuously differentiable field of unit tangent vectors.*

By definition, the sphere S^{n-1} is the set of all vectors $\mathbf{u} = (u_1, \dots, u_n)$ in the Euclidean space \mathbf{R}^n such that the Euclidean length $\|\mathbf{u}\|$ equals 1. A vector $\mathbf{v}(\mathbf{u})$ in \mathbf{R}^n is *tangent* to S^{n-1} at \mathbf{u} if the Euclidean inner product $\mathbf{u} \cdot \mathbf{v}(\mathbf{u})$ is zero.

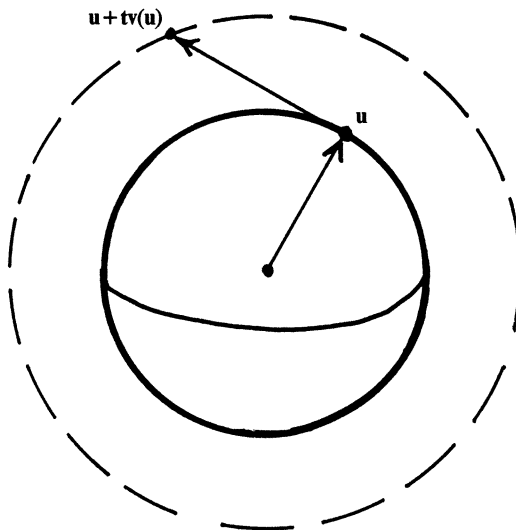


FIG. 1

The hypothesis that the dimension $n-1$ is even is essential. For if $n-1$ is odd then the formula

$$\mathbf{v}(u_1, \dots, u_n) = (u_2, -u_1, \dots, u_n, -u_{n-1})$$

defines a differentiable field of unit tangent vectors on S^{n-1} .

The proof of Theorem 1 will depend on two lemmas. The first involves a volume computation. Let A be a compact region in \mathbf{R}^n , and let $\mathbf{x} \mapsto \mathbf{v}(\mathbf{x})$ be a continuously differentiable vector field which is defined throughout a neighborhood of A . The values $\mathbf{v}(\mathbf{x})$ can be arbitrary vectors in \mathbf{R}^n . For each real number t , consider the function

$$\mathbf{f}_t(\mathbf{x}) = \mathbf{x} + t\mathbf{v}(\mathbf{x})$$

which is defined for all \mathbf{x} in A .

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LEMMA 1. *If the parameter t is sufficiently small, then this mapping \mathbf{f}_t is one-to-one and transforms the region A onto a nearby region $\mathbf{f}_t(A)$ whose volume can be expressed as a polynomial function of t .*

Proof. Since A is compact, and since the function $\mathbf{x} \mapsto \mathbf{v}(\mathbf{x})$ is continuously differentiable, there exists a Lipschitz constant c so that

$$\|\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y})\| \leq c \|\mathbf{x} - \mathbf{y}\| \quad (*)$$

for all \mathbf{x} and \mathbf{y} in A . [This is proved as follows. First consider the special case where A is a cube with edges parallel to the coordinate axes. Passing from \mathbf{x} to \mathbf{y} in n steps by changing one coordinate at a time, and applying the Mean Value Theorem of differential calculus, one sees that $|v_i(\mathbf{x}) - v_i(\mathbf{y})| \leq \sum_j c_{ij} |x_j - y_j|$, where

$$c_{ij} = \sup_A |\partial v_i / \partial x_j|.$$

Therefore

$$\|\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y})\| \leq \sum_i |v_i(\mathbf{x}) - v_i(\mathbf{y})| \leq \sum_{i,j} c_{ij} |x_j - y_j| \leq \sum_{i,j} c_{ij} \|\mathbf{x} - \mathbf{y}\|,$$

as required. Now an arbitrary compact set A in \mathbf{R}^n can be covered by finitely many open cubes I_α , chosen so that a Lipschitz condition (*) holds whenever \mathbf{x} and \mathbf{y} belong to the same cube. But if \mathbf{x} and \mathbf{y} in A do not belong to any common cube I_α , then the distance $\|\mathbf{x} - \mathbf{y}\|$ is bounded away from zero. In fact this expression $\|\mathbf{x} - \mathbf{y}\|$ can be thought of as a continuous and nowhere zero function on the compact set $A \times A - \cup I_\alpha \times I_\alpha$. It is now easy to choose a constant c so that the Lipschitz condition (*) holds uniformly for all \mathbf{x} and \mathbf{y} in A .]

Choose any t with $|t| < c^{-1}$. Then \mathbf{f}_t is one-to-one; for if $\mathbf{f}_t(\mathbf{x}) = \mathbf{f}_t(\mathbf{y})$ then $\mathbf{x} - \mathbf{y} = t(\mathbf{v}(\mathbf{y}) - \mathbf{v}(\mathbf{x}))$, hence the inequality $\|\mathbf{x} - \mathbf{y}\| \leq |t|c \|\mathbf{x} - \mathbf{y}\|$ implies that $\mathbf{x} = \mathbf{y}$.

The matrix of first derivatives of \mathbf{f}_t can be written as $I + t[\partial v_i / \partial x_j]$, where I is the identity matrix. Hence its determinant is a polynomial function of t , of the form $1 + t\sigma_1(\mathbf{x}) + \cdots + t^n \sigma_n(\mathbf{x})$, the coefficients being continuous functions of \mathbf{x} . This determinant is strictly positive for $|t|$ sufficiently small. Integrating over A , we see that the volume of the image region can be expressed as a polynomial function of t ,

$$\text{volume } \mathbf{f}_t(A) = a_0 + a_1 t + \cdots + a_n t^n,$$

with coefficients $a_k = \int \cdots \int_A \sigma_k(\mathbf{x}) dx_1 \cdots dx_n$. ■

Now suppose that the sphere S^{n-1} has a continuously differentiable field $\mathbf{u} \mapsto \mathbf{v}(\mathbf{u})$ of unit tangent vectors. For any real number t , note that the vector $\mathbf{u} + t\mathbf{v}(\mathbf{u})$ has length $\sqrt{1 + t^2}$.

LEMMA 2. *If the parameter t is sufficiently small, then the transformation $\mathbf{u} \mapsto \mathbf{u} + t\mathbf{v}(\mathbf{u})$ maps the unit sphere in \mathbf{R}^n onto the sphere of radius $\sqrt{1 + t^2}$.*

Assuming this Lemma for the moment, we can now prove Theorem 1. As region A we take the region between two concentric spheres, defined by the inequalities $a \leq \|\mathbf{x}\| \leq b$. We extend the vector field \mathbf{v} throughout this region by setting $\mathbf{v}(r\mathbf{u}) = r\mathbf{v}(\mathbf{u})$ for $a \leq r \leq b$. It follows that the mapping $\mathbf{f}_t(\mathbf{x}) = \mathbf{x} + t\mathbf{v}(\mathbf{x})$ is defined throughout the region A , and maps the sphere of radius r onto the sphere of radius $r\sqrt{1 + t^2}$ providing that t is sufficiently small. (Note that $\mathbf{f}_t(r\mathbf{u}) = r\mathbf{f}_t(\mathbf{u})$.) Hence it maps A onto the region between spheres of radius $a\sqrt{1 + t^2}$ and $b\sqrt{1 + t^2}$. Evidently

$$\text{volume } \mathbf{f}_t(A) = (\sqrt{1 + t^2})^n \text{volume } (A).$$

Thus, if n is odd, this volume is not a polynomial function of t . Comparing Lemma 1, we obtain a contradiction which proves Theorem 1.

Now we must prove Lemma 2. Here are two alternative arguments, the first based on the "Shrinking Lemma" (see [5]), and the second based on elementary point set topology.

First Proof. Let the region A considered above be defined by the inequalities $1/2 \leq \|x\| \leq 3/2$. Choose t small enough so that $|t| < 1/3$ and $|t| < c^{-1}$, where c is a Lipschitz constant for v . Then for each fixed u_0 in S^{n-1} the auxilliary mapping

$$x \mapsto u_0 - tv(x)$$

carries the complete metric space A into itself (since $|tv(x)| < 1/2$); and satisfies a Lipschitz condition with Lipschitz constant less than 1. Hence, by the Shrinking Lemma, this auxiliary map has a unique fixed point. In other words, the equation $f_t(x) = u_0$ has a unique solution x . Multiplying x and u_0 by $\sqrt{1+t^2}$, Lemma 2 follows.

Second Proof. We may assume that $n \geq 2$. If t is sufficiently small, then the matrix of first derivatives of f_t is non-singular throughout the compact region A . (Compare the proof of Lemma 1.) Using the Inverse Function Theorem, it follows that f_t maps open sets in the interior of A to open sets. Hence the image $f_t(S^{n-1})$ is a relatively open subset of the sphere of radius $\sqrt{1+t^2}$. But this image $f_t(S^{n-1})$ is also compact and hence closed. Since S^{n-1} is connected, an open and closed subset must be the entire sphere, and the conclusion follows. ■

A slightly sharper version of Theorem 1, which does not mention differentiability or unit vectors, follows as an immediate corollary.

THEOREM 1'. *An even dimensional sphere does not admit any continuous field of non-zero tangent vectors.*

Proof. Suppose that the sphere S^{n-1} possesses a continuous field of non-zero tangent vectors $v(u)$. Let $m > 0$ be the minimum of $\|v(u)\|$. By the Weierstrass Approximation Theorem [5], there exists a polynomial mapping p from S^{n-1} to \mathbb{R}^n satisfying

$$\|p(u) - v(u)\| < m/2$$

for all u . Defining a differentiable vector field $w(u)$ by the formula

$$w = p - (p \cdot u)u$$

for every u , the computation $w \cdot u = 0$ shows that $w(u)$ is tangent to S^{n-1} at u , while the computation

$$\|w - p\| = \|p \cdot u\| < m/2,$$

together with the triangle inequality, shows that $w \neq 0$. Therefore, the quotient $w(u)/\|w(u)\|$ is an infinitely differentiable field of unit tangent vectors on the sphere S^{n-1} . If $n-1$ is even, this is impossible by Theorem 1. ■

Starting with Theorem 1', it is quite easy to prove the Brouwer Fixed Point Theorem:

THEOREM 2. *Every continuous mapping f from the disk D^n to itself possesses at least one fixed point.*

Here D^n is defined to be the set of all vectors x in \mathbb{R}^n with $\|x\| \leq 1$.

Proof. If $f(x) \neq x$ for all x in D^n , then the formula $v(x) = x - f(x)$ would define a non-zero vector field v on D^n which points *outward* everywhere on the boundary, in the sense that $u \cdot v(u) > 0$ for every point u in S^{n-1} .

With a little care, we can modify this definition to obtain a non-zero vector field w on D^n which points *directly outward* on the boundary, in the sense that $w(u) = u$ for every u in S^{n-1} . For example, set

$$w(x) = x - y(1 - x \cdot x)/(1 - x \cdot y),$$

where $y = f(x) \neq x$. Evidently $w(x) = x$ whenever $x \cdot x = 1$. This expression depends continuously on x , since the denominator never vanishes. It is clearly non-zero whenever x and y are linearly independent; while if x and y are linearly dependent the identity $(x \cdot x)y = (x \cdot y)x$ implies that $w(x) = (x - y)/(1 - x \cdot y) \neq 0$.

Let us transplant this hypothetical vector field $\mathbf{w}(\mathbf{x})$ to the southern hemisphere of the unit sphere S^n in \mathbb{R}^{n+1} . Identifying \mathbb{R}^n with the hyperplane $x_{n+1}=0$ which passes through the "equator" of S^n , we will use stereographic projection from the north pole $(0, \dots, 0, 1)$ to map each point \mathbf{x} of D^n to a point $\mathbf{s}(\mathbf{x}) = \mathbf{u}$ of the southern hemisphere $u_{n+1} < 0$. The precise formula is

$$\mathbf{s}(\mathbf{x}) = (2x_1, \dots, 2x_n, \mathbf{x} \cdot \mathbf{x} - 1) / (\mathbf{x} \cdot \mathbf{x} + 1).$$

Applying the derivative of the mapping \mathbf{s} at \mathbf{x} to the vector $\mathbf{w}(\mathbf{x})$, we obtain a corresponding tangent vector $\mathbf{W}(\mathbf{u})$ to S^n at the image point $\mathbf{s}(\mathbf{x}) = \mathbf{u}$. (The vector $\mathbf{W}(\mathbf{u})$ can be described as the velocity vector $d\mathbf{s}(\mathbf{x} + t\mathbf{w}(\mathbf{x}))/dt$ of the spherical curve $t \mapsto \mathbf{s}(\mathbf{x} + t\mathbf{w}(\mathbf{x}))$, evaluated at $t=0$.) In this way, we obtain a non-zero tangent vector field \mathbf{W} on the southern hemisphere. At every point $\mathbf{u} = \mathbf{s}(\mathbf{u})$ of the equator, since $\mathbf{w}(\mathbf{u}) = \mathbf{u}$ points directly outward, computation shows that the corresponding vector $\mathbf{W}(\mathbf{u}) = (0, \dots, 0, 1)$ points due north (i.e., away from the southern hemisphere).

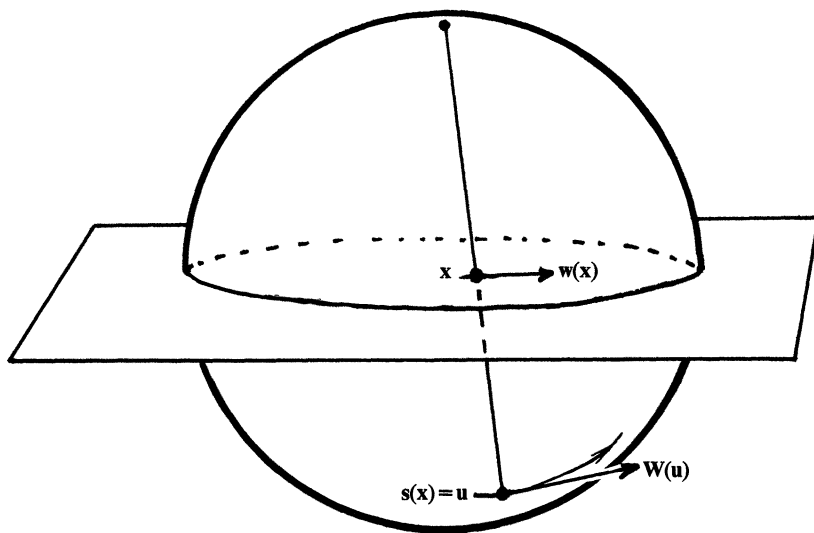


FIG. 2

Similarly, using stereographic projection from the south pole, the vector field $-\mathbf{w}(\mathbf{x})$ corresponds to a vector field on the northern hemisphere which also points due north on the equator. Piecing these two vector fields together, we obtain a non-zero tangent vector field \mathbf{W} which is defined and continuous everywhere on S^n . If n is even, this is impossible by Theorem 1'.

This contradiction proves the Brouwer Fixed Point Theorem for even values of n . But this suffices to prove the Theorem for an odd value $n = 2k - 1$ also. For any map \mathbf{f} from D^{2k-1} to itself without fixed point would give rise to a map $\mathbf{F}(x_1, \dots, x_{2k}) = (\mathbf{f}(x_1, \dots, x_{2k-1}), 0)$ from D^{2k} to itself without fixed point. ■

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