A Version of The Lomonosov Invariant Subspace Theorem for Real Banach Spaces

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Abstract. It is known that for real Banach spaces the famous Lomonosov Invariant Subspace Theorem may fail. In our paper we will give a complete characterization of operators on real Banach spaces for which this theorem holds true.

1. Introduction

Let T be a continuous linear operator on a Banach space X, and Y be a linear subspace of X. Recall that Y is *T-invariant* if $T(Y) \subseteq Y$, and Y is *T-hyperinvariant* if it is invariant under every continuous operator that commutes with T. The *Invariant Subspace Problem* is the problem of finding nontrivial closed invariant subspaces for continuous operators.

In 1973 Victor Lomonosov [7] obtained a remarkable result that has caused enormous progress on the existence of nontrivial invariant and hyperinvariant subspaces for compact-related operators. This result has become known as the "Lomonosov invariant subspace theorem." A survey on these results and many applications can be found in [2, 3, 10].

Theorem 1.1 (Lomonosov, [7])**.** *If a non-scalar operator T on a complex Banach space commutes with a non-zero compact operator, then T has a non-trivial closed hyperinvariant subspace.*

The fact that this theorem in general is valid for complex Banach spaces only was addressed by N.D. Hooker in [4] and recently by Y. Abramovich and

C. Aliprantis in [1, Chapter 10]. Hooker showed that if an operator $T : X \to X$ on a real Banach space X does not satisfy any irreducible polynomial equation, then the Lomonosov theorem remains valid for T. In the present paper we will show that the condition above actually characterizes when the Lomonosov theorem works in real Banach spaces. Some other sufficient conditions that are easier to check will be provided. It will follow that the Lomonosov theorem is always true for positive operators. We will also provide a straightforward proof of the Hooker's result.

We begin with several definitions that might be found, for instance, in [1]. If X is a real vector space, then the *complexification* of X is the complex vector space

$$X_{\mathbb{C}} = X \oplus \imath X = \{x + \imath y \mid x, y \in X\},$$

whose vector space operations are defined by

$$\begin{aligned}(x_1 + \imath y_1) + (x_2 + \imath y_2) &= (x_1 + x_2) + \imath(y_1 + y_2),\\ (\alpha + \imath\beta)(x + \imath y) &= (\alpha x - \beta y) + \imath(\beta x + \alpha y).\end{aligned}$$

If X is also a normed space with norm $\|\cdot\|$, then we can extend the norm $\|\cdot\|$ to a norm on $X_{\mathbb{C}}$. Moreover, if X is a Banach space, then $X_{\mathbb{C}}$ is also a Banach space.

Every operator $T : X \to Y$ between two real vector spaces gives rise naturally to a complex linear operator $T_{\mathbb{C}} : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ defined via the formula

$$T_{\mathbb{C}}(x + \imath y) = Tx + \imath Ty.$$

Moreover, if X and Y are normed spaces and $T : X \to Y$ is a bounded operator, then the operator $T_{\mathbb{C}} : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ is also bounded and satisfies $\|T_{\mathbb{C}}\| = \|T\|$.

Let X be a Banach space. For an operator $T \in \mathcal{L}(X)$, the following set

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is not invertible in } \mathcal{L}(X)\}$$

is known as the *spectrum* of the operator. As usual we may write $\lambda - T$ instead of $\lambda I - T$. Clearly, if X is a real Banach space, then the spectrum of T consists of real numbers. Moreover, in that case it is not hard to notice that

$$\sigma(T) = \sigma(T_{\mathbb{C}}) \cap \mathbb{R}.$$

The *spectral radius* of T is defined as usual by $r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$. An algebra $\mathcal{A}$ of operators is said to be *transitive*, if it has no non-trivial closed $\mathcal{A}$-invariant subspaces. All terminology employed but not defined here can be found in [1].

2. A Modification of the Lomonosov Theorem for Real Banach Spaces

Our main goal is to prove the following modification of the Lomonosov theorem for real Banach spaces.

Theorem 2.1. *Consider a real Banach space X and a non-scalar operator $T \in \mathcal{L}(X)$ commuting with a non-zero compact operator. Then the following two statements are equivalent:*

(a) *T has a non-trivial closed hyperinvariant subspace.*
(b) *For each pair of real numbers α, β with $\beta \neq 0$ we have $(\alpha - T)^2 + \beta^2 \neq 0$.*

Proof. First, we prove that (a) implies (b). Let $T : X \to X$ be a non-scalar operator acting on a real Banach space. Let us assume that for some real numbers α, β with $\beta \neq 0$ the operator $(\alpha - T)^2 + \beta^2$ is zero. Notice that without loss of the generality we may assume that

$$T^2 + I = 0. \tag{2.1}$$

Indeed, letting $T_1 = (\alpha - T)/\beta$ we see that the commutant of T_1 is equal to the commutant of T and, thus, any T-hyperinvariant subspace of X will be T_1-hyperinvariant. In addition, T_1 satisfies (2.1).

Next we will use the operator T to define a complex structure on X. To do so we need to define a multiplication by $\imath$ on X. We let

$$\imath x = Tx$$

for each $x \in X$. It is easy to check that equation (2.1) is enough to guarantee that the multiplication introduced is well-defined, and with respect to this multiplication X becomes a vector space over $\mathbb{C}$. Let us denote the space X equipped with this complex structure by X_T. It is also easy to check that the formula

$$\|x\|_{\mathbb{C}} = \sup\{\|(a + \imath b)x\| : a, b \in \mathbb{R},\ |a + \imath b| = 1\}$$

defines a norm on the complex vector space X_T, and for every $x \in X$ the following inequality holds

$$\|x\| \leq \|x\|_{\mathbb{C}} \leq (1 + \|T\|)\|x\|. \tag{2.2}$$

In particular, it follows that $(X_T, \|\cdot\|_{\mathbb{C}})$ is a complex Banach space. Moreover, if Y is a closed subspace of X such that $TY \subset Y$ (that is, Y is closed with respect to the multiplication by $\imath$), then the complex space Y_T is a closed subspace of X_T.

Denote by $\mathcal{L}(X_T)$ the space of continuous complex linear operators on X_T. We claim that

$$\{T\}' = \mathcal{L}(X_T),$$

where, as usual, $\{T\}' = \{S : X \to X \mid ST = TS\}$. Fix $S \in \{T\}'$; then $S : X_T \to X_T$ is clearly additive. To see that S is complex linear we need the complex homogeneity. We have $S(\imath x) = S(Tx) = TSx = \imath Sx$, i.e., we get what is needed. The continuity of $S : X_T \to X_T$ follows from (2.2). Hence we have obtained the inclusion $\{T\}' \subset \mathcal{L}(X_T)$.

To show the converse inclusion, fix $S \in \mathcal{L}(X_T)$. Then $S : X \to X$ is clearly a linear operator that is bounded in view of (2.2) and commutes with T because $STx = S(\imath x) = \imath Sx = TSx$ for all $x \in X$. Thus, $\{T\}' = \mathcal{L}(X_T)$.

Assume now that T has a non-trivial closed hyperinvariant subspace Y. Then, in particular, Y is T-invariant and, therefore, Y_T is a non-trivial closed subspace of X_T. Moreover, since Y is $\{T\}'$-invariant and $\{T\}' = \mathcal{L}(X_T)$, we can conclude that Y_T is $\mathcal{L}(X_T)$-invariant. But this is impossible, and so we get a contradiction.

The implication (b)⇒(a) has been proved in [4]. Here we present a straightforward proof. For this we need the following fact known as "Lomonosov's lemma."

Theorem 2.2 (Lomonosov, [7]). *If $\mathcal{A}$ is a transitive algebra of operators on a real or complex Banach space, then for each non-zero compact operator K there exists some $A \in \mathcal{A}$ such that the compact operator AK has a non-zero fixed point, i.e., $AKu = u$ for some $u \neq 0$.*

Assume that $T \in \mathcal{L}(X)$ is a non-scalar operator and that for some non-zero compact operator $K \in \mathcal{L}(X)$ we have $TK = KT$. To establish the existence of a non-trivial closed T-hyperinvariant subspace, it suffices to show that the commutant of T, $\{T\}' = \{S \in \mathcal{L}(X) \mid ST = TS\}$, is non-transitive.

To verify this assume by way of contradiction that the commutant $\{T\}'$ is a transitive algebra. So, by Theorem 2.2, there exists some $A \in \{T\}'$ such that the operator AK has a non-zero fixed point. Let

$$F = \{x \in X \mid AKx = x\}.$$

Clearly, F is a non-zero closed AK-hyperinvariant subspace. Since AK is a compact operator and it is the identity on F, it follows that F is finite dimensional. Since T commutes with AK and F is AK-hyperinvariant, we see that F is T-invariant.

Consider next the complexification $X_{\mathbb{C}}$ of X and the operator $T_{\mathbb{C}} : X_{\mathbb{C}} \to X_{\mathbb{C}}$. Notice that since the subspace $F \subset X$ is T-invariant, the subspace $F_{\mathbb{C}} \subset X_{\mathbb{C}}$ is $T_{\mathbb{C}}$-invariant. From this and from the fact that $F_{\mathbb{C}}$ is a finite dimensional complex Banach space, it follows that the operator $T_{\mathbb{C}} : F_{\mathbb{C}} \to F_{\mathbb{C}}$ has an eigenvalue $\lambda = \alpha + \imath\beta \in \mathbb{C}$. Let $x + \imath y \neq 0$ be an eigenvector of $T_{\mathbb{C}}$ corresponding to λ, i.e., $T_{\mathbb{C}}(x + \imath y) = \lambda(x + \imath y)$ for some $x, y \in X$.

Consider first the case $\beta = 0$. Then $Tx + \imath Ty = T_{\mathbb{C}}(x + \imath y) = \alpha(x + \imath y)$. Therefore, $Tx = \alpha x$ and $Ty = \alpha y$. Since at least one of the vectors x, y is non-zero, it follows that the operator T has an eigenvector, too. And since T is non-scalar, the eigenspace corresponding to this real eigenvalue α is a non-trivial closed T-hyperinvariant subspace of X. This contradicts our assumption, and so the case $\beta = 0$ is impossible.

So, we may assume that $\beta \neq 0$. Notice that if $x + \imath y \in X_{\mathbb{C}}$ is an eigenvector of $T_{\mathbb{C}}$ having the eigenvalue λ, then $x \neq 0$ and $T_{\mathbb{C}}(x + \imath y) = \lambda(x + \imath y) = (\alpha + \imath\beta)(x + \imath y)$, whence

$$(*) \qquad \begin{cases} Tx = \alpha x - \beta y, \\ Ty = \beta x + \alpha y. \end{cases}$$

Note that, under the assumption $\beta \neq 0$, we may eliminate vector y from the system (∗) and obtain that the equation

$$(**) \qquad ((\alpha - T)^2 + \beta^2)x = 0$$

has a non-trivial solution. That is, if $T_{\mathbb{C}}(x + \imath y) = (\alpha + \imath\beta)(x + \imath y)$, then the null space

$$X_\lambda = \operatorname{Ker}((\alpha - T)^2 + \beta^2)$$

of the operator $(\alpha - T)^2 + \beta^2$ is non-zero. In addition, X_λ is closed and $((\alpha - T)^2 + \beta^2)$-hyperinvariant. Obviously, $\{T\}' \subset \{((\alpha - T)^2 + \beta^2)\}'$, and thus X_λ is a non-zero closed $\{T\}'$-invariant subspace of X. However, by our assumption the algebra $\{T\}'$ is transitive, and so we must have $X_\lambda = X$. That is, the equation (∗∗) is satisfied by all $x \in X$. This contradicts our hypothesis that the operator $(\alpha - T)^2 + \beta^2$ is non-zero. ◻

A somewhat weaker sufficient condition that nevertheless might be easier to check is given next.

Corollary 2.3. *Let X be a real Banach space, and let a non-scalar operator $T \in \mathcal{L}(X)$ commutes with a non-zero compact operator. If the spectrum $\sigma(T)$ is non-empty or, in other words, $\sigma(T_{\mathbb{C}}) \cap \mathbb{R} \neq \emptyset$, then T has a non-trivial closed hyperinvariant subspace.*

Proof. For the conclusion of our corollary it is enough to show that for each pair of real numbers α, β with $\beta \neq 0$ we have $(\alpha - T)^2 + \beta^2 \neq 0$. Assume to the contrary that there are $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$ such that $(\alpha - T)^2 + \beta^2 = 0$. Then by the Spectral Mapping Theorem, the equation $(\alpha - \lambda)^2 + \beta^2 = 0$ holds for every $\lambda \in \sigma(T_{\mathbb{C}})$. In particular, it must hold for a real number $\lambda \in \sigma(T_{\mathbb{C}}) \cap \mathbb{R}$ that exists according to the hypothesis of the corollary. It follows that $\alpha = \lambda$ and $\beta = 0$. This contradicts the assumption that $\beta \neq 0$. ◻

The hypothesis of the last corollary is always true for positive operators. That gives us another corollary of Theorem 2.1.

Corollary 2.4. *Let $T : E \to E$ be a non-scalar positive operator on a real Banach lattice. If T commutes with a non-zero compact operator, then there exists a non-trivial T-hyperinvariant subspace of E.*

Proof. It is well known (see, for instance [1, Theorem 7.9]) that the spectral radius of a positive operator necessarily belongs to its spectrum. Therefore, the previous corollary is applicable. ◻

Let us mention several things in conclusion.

(1) The result of H.W. Kim, C. Pearcy, and A. Shields [6] can be proved for real Banach spaces in the form of Theorem 2.1;

(2) For any real Banach space X with a symmetric basis there exists an operator $T : X \to X$ that commutes with a compact operator and does not have a non-trivial hyperinvariant subspace (see [4]);
(3) For any operator $T : J \to J$ on the James's space J [5] the Lomonosov theorem remains valid.

The latter is true since J is a real Banach space which does not admit complex structure.

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