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Some Remarks on Banach Limits (**).

Summary. – *We define generalized limits as continuous linear functionals on the Banach space l_∞ which are not shift-invariant like the classical Banach limits, but have stronger geometric properties. Moreover, these generalized limits are precisely the multiplicative functionals on l_∞ which vanish on c_0.*

In his classical treatise [2], S. Banach mentioned a remarkable fact: there exist continuous linear functionals l on the space l_∞ of all bounded real sequences $x = (x_n)_n$ (now called «Banach limits») with the properties

$$\liminf_{n\to\infty} x_n \leq l(x) \leq \limsup_{n\to\infty} x_n \tag{1}$$

and

$$l(Sx) = l(x), \tag{2}$$

where S denotes the left shift operator

$$S(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots). \tag{3}$$

From (1) it follows that for convergent sequences x the Banach limit $l(x)$ coincides with the usual limit; in particular, $l(x) = 0$ for $x \in c_0$, and

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hence the Banach limit $l(x)$ is not affected by changing finitely many terms in the sequence $x \in l_\infty$. The property (2) in turns means that one may add or omit finitely many terms at the beginning of a sequence x without changing $l(x)$.

The existence of Banach limits is a consequence of the Hahn-Banach theorem on the extension of continuous linear functionals. Moreover, they provide an example (not explicit, of course) of a continuous linear functional on l_∞ which is not representable in «integral form»

$$l(x) = \sum_{k=1}^{\infty} x_k y_k \qquad (x = (x_n)_n \in l_\infty)$$

with some $y = (y_n)_n \in l_1$. One the other hand, the existence of such functionals is a consequence on the Alaoglu theorem [1] on the weak* compactness of the unit ball in l_∞^* (and of the obvious fact that the unit ball of l_1, as a subset of l_∞^*, is not closed in the weak*-topology).

The purpose of this note is to show that, building directly on the Alaoglu theorem, one may construct other «generalized limits» on l_∞ which are different from the classical Banach limits and have «nice» properties. In particular, it came out as a surprise for us that in this way we obtained a new proof for characterizing the multiplicative functionals on l_∞ vanishing on c_0.

Consider the «remainders» $\{e_n, e_{n+1}, \dots\}$ of the sequence $e = (e_n)_n$ of the usual basis elements $e_n = (\delta_{nk})_k$ in l_1. Obviously, the sets

$$F_n = \mathrm{cl}_{l_1} \mathrm{co}\{e_n, e_{n+1}, \dots\} \tag{4}$$

(strong closure in l_1) form a decreasing sequence of closed convex sets with empty intersection. However, if we consider l_1 as a subspace of l_∞^*, equipped with the weak* topology, the sets

$$\widehat{F}_n = \mathrm{cl}_{l_\infty^*} \mathrm{co}\{e_n, e_{n+1}, \dots\} \tag{5}$$

(weak* closure in l_∞^*) have a nonempty intersection. The sets (5) will be of importance in the construction of generalized limits below.

Another way of pointing out the difference between the sets F_n and $\widehat{F}_n$ is as follows: the sequence $e = (e_n)_n$ of unit vectors has cluster points (i.e. points which are arbitrarily close to e_n for infinitely many n), considered as a sequence in the unit ball of l_∞^*, but has no cluster points, considered as a sequence in the unit ball of l_1.

Given a bounded real sequence $x = (x_n)_n$, we denote by $\mathrm{Lim}(x)$ the set of all limit points of x, i.e. limits of all possible convergent subsequences. Of course, the set $\mathrm{Lim}(x)$ is always nonempty; in particular, it contains the lower limit $\liminf x_n$ and the upper limit $\limsup x_n$ as minimal and maximal element, respectively. The notion of limit point may be defined as well in the general setting of topological spaces. Recall, however, that the limit points and the cluster points of a sequence coincide in metric spaces, but may not coincide in non-metrizable spaces. For instance, the above sequence $e = (e_n)_n$, considered as a sequence in the unit ball of l_∞^*, has no limit points at all in the weak* topology.

LEMMA 1. *Let $l \in l_\infty^*$. Then l has the property* (1) *if and only if l belongs to the intersection of all sets $\widehat{F}_n$ $(n = 1, 2, \dots)$.*

PROOF. Recall that the sets $\{f: f \in l_\infty^*, |f(x)| < \varepsilon\}$ $(x \in l_\infty, \varepsilon > 0)$ form a basis of neighbourhoods of zero in the weak* topology on l_∞^*. $l \in l_\infty^*$ and $x \in l_\infty$. Then l belongs to the intersection of all sets $\widehat{F}_n (n = 1, 2, \dots)$ if and only if, for all $\varepsilon > 0$ and $n \in \mathbb{N}$, one can find an $f \in \mathrm{co}\{e_n, e_{n+1}, \dots\}$ such that $|l(x) - f(x)| < \varepsilon$. Any such f may be written in the form

$$f(x) = \sum_{k=n}^{\infty} \lambda_k e_k(x),$$

where the summation actually runs only over a finite number of scalars $\lambda_k > 0$ whose sum is equal to 1. But $e_k(x) = x_k$, and thus $l(x)$ is in the ε-neighbourhood of $[\liminf x_n, \limsup x_n]$. Since $\varepsilon > 0$ is arbitrary, the assertion follows. ■

LEMMA 2. *Let $l \in l_\infty^*$. Then l is a cluster point of $e = (e_n)_n$ if and only if*

$$l(x) \in \mathrm{Lim}(x) \qquad (x \in l_\infty). \tag{6}$$

PROOF. Using again the same neighbourhoods of zero as in Lemma 1, we see that l is a cluster point of $e = (e_n)_n$ if and only if, for any $x \in l_\infty$ and $\varepsilon > 0$, one has $|l(x) - x_n| < \varepsilon$ for infinitely many $n \in \mathbb{N}$. But this simply means that $l(x)$ belongs to the closure of $\mathrm{Lim}(x)$, and the assertion follows from the fact that $\mathrm{Lim}(x)$ is closed. ■

From the Alaoglu theorem and the preceding lemma we obtain the following

THEOREM. 1. *There exist functionals $l \in l_\infty^*$ such that* (6) *holds for all $x \in l_\infty$.*

Observe that (6) implies (1) but not vice versa. Indeed, (6) states that $l(x)$ is always a limit point of $x = (x_n)_n$, and not just an element of the interval $[\liminf x_n, \limsup x_n]$. On the other hand, the condition (2) is now incompatible with the requirement (6): in fact, for the sequence $x_n = (-1)^n$ one has $Sx = -x$ and hence $l(x) = 0$, but $\mathrm{Lim}\,(x) = \{-1, +1\}$.

The above Theorem 1 is essentially a consequence of the Alaoglu theorem applied to the unit ball of l_∞^*. We do not know whether or not the existence of classical Banach limits may also be deduced only from the Alaoglu theorem. We point out, however, that it may be obtained by combining the Alaoglu theorem with the Tykhonov fixed point principle. To see this, observe that the shift operator S leaves the remainders $\{e_n, e_{n+1}, \ldots\}$ invariant, and its adjoint S^* leaves the intersection of all sets $\bar{F}_n$ given in (5) invariant. Since the operator S^* is weakly* continuous, the Tykhonov principle guarantees the existence of a fixed point l of the operator S, *i.e.* the existence of a Banach limit.

When extending the notion of «limit» from the space c of convergent sequences to the space l_∞ of bounded sequences, it is natural to try to preserve also the algebraic properties of the usual limit. Unfortunately, the classical Banach limit l having the properties (1) and (2) is not multiplicative (see, e.g., [4]). Interestingly, the generalized limits described in Theorem 1 are not only multiplicative: they are essentially the only limits on l_∞ with such a property:

THEOREM 2. *Let $l \in l_\infty^*$, $l \neq 0$. Then the following two statements are equivalent:*

(*a*) *l is multiplicative on l_∞, i.e.*

$$l(xy) = l(x)l(y) \qquad (x, y \in l_\infty) \tag{7}$$

and vanishes on c_0;

(*b*) *l satisfies* (6).

PROOF. Suppose first that (*a*) holds. Then l is positive and satisfies $l(u) = 1$ on $u = (1, 1, \ldots)$. Denote by χ_D the characteristic function (sequence) of an arbitrary subset D of $\mathbb{N}$. Since $l(\chi_D)^2 = l(\chi_D^2) = l(\chi_D)$, the functional l may assume on χ_D only the value 0 or 1. If $\mathrm{Lim}\,(\chi_D) = \{0, 1\}$ then (6) follows. If $\mathrm{Lim}\,(\chi_D) = \{0\}$ or $\mathrm{Lim}\,(\chi_D) = \{1\}$, then D or $\mathbb{N}/D$, respectively, is finite. In this case (6) follows from the vanishing of l on c_0 and the identity $l(\chi_D) + l(\chi_{\mathbb{N}/D}) = 1$. Before proving (6) in general, we like to point out

that the family $\mathcal{F}$ of all sets $D \subseteq \mathbb{N}$ with $l(\chi_D) = 1$ is an ultrafilter on $2^{\mathbb{N}}$. Moreover, $\mathcal{F}$ contains the Fréchet filter (see, *e.g.*, [3]).

Let $x = (x_n)_n$ be an arbitrary sequence in l_∞. Given $\varepsilon > 0$, let $\{z_1, \ldots, z_m\}$ be a finite ε-net in $\mathrm{Lim}\,(x)$. Choose sequences $(n_k^{(1)})_k, \ldots, (n_k^{(m)})_k$ of natural numbers such that

$$|x_{n_k^{(j)}} - z_j| < \varepsilon \qquad (k \in \mathbb{N};\ j = 1, \ldots, m). \tag{8}$$

Without loss of generality we may assume here that the sets $D_j = \{n_1^{(j)}, n_2^{(j)}, \ldots, n_k^{(j)}, \ldots\}$ $(j = 1, \ldots, m)$ are mutually disjoint. Now, the element x may be represented in the form

$$x = \sum_{j=1}^{m} z_j \chi_{D_j} + v + w, \tag{9}$$

where v is a finite sequence and $\|w\| < \varepsilon$. Since $\mathcal{F}$ is an ultrafilter and the union of the sets D_j is equal to $\mathbb{N}$, except for a finite set, only one of the sets $D_1, \ldots, D_m$ (say D_s) belong to $\mathcal{F}$. But this implies that, by (9), $|l(x) - z_s| < \varepsilon$. We conclude that $l(x) \in \mathrm{Lim}\,(x)$, since $\mathrm{Lim}\,(x)$ is closed, and thus (*b*) holds. Conversely, suppose that $l \in l_\infty^*$ satisfies (6). The fact that l is zero on c_0, positive on l_∞, and normalized on $u = (1, 1, \ldots)$ is a consequence of (6); it remains to show that l is multiplicative.

The equality $l(\chi_{A \cup B}) + l(\chi_A \cdot \chi_B) = l(\chi_A) + l(\chi_B)$ implies that

$$l(\chi_A \cdot \chi_B) = l(\chi_A)l(\chi_B) \tag{10}$$

for all $A, B \subseteq \mathbb{N}$.

Now define $\mathcal{F}$ as in the first part of the proof; it is again straightforward to prove that $\mathcal{F}$ is an ultrafilter. To prove (7) in the general case, consider first sequences of the form

$$x = \sum_{i=1}^{p} \xi_i \chi_{A_i}, \qquad y = \sum_{j=1}^{q} \eta_j \chi_{B_j}, \tag{11}$$

where the sets $\{A_1, \ldots, A_p\}$ and $\{B_1, \ldots, B_q\}$ are partions of $\mathbb{N}$. Again by the ultrafilter property of $\mathcal{F}$, only one of these sets (say A_r and B_s, respectively) belongs to $\mathcal{F}$. Consequently, we have $l(x) = \xi_r$, $l(y) = \eta_s$, and

$$l(xy) = l\left(\sum_{i=1}^{p} \sum_{j=1}^{q} \xi_i \eta_j \chi_{A_i \cap B_j}\right) = \xi_r \eta_s = l(x)l(y).$$

The assertion (7) follows now from the fact that the sequences of the form (11) form a dense subset of l_∞, and thus we have proved (*a*).

Let us make some remarks on Theorem 2. As is well-known [6], there is a 1-1 correspondence between the multiplicative functionals $l \in l_\infty^*$ and the set Spec(l_∞) of all maximal ideals of the Banach algebra l_∞. One may therefore consider Theorem 2 as a purely analytic characterization of all maximal ideals in l_∞ containing c_0.

We also want to point the reader's attention to the paper [7], where it is shown that $l \in l_\infty^*$ is multiplicative if and only if $l(x) \in \sigma(x)$ for all $x \in l_\infty$. Since the latter condition means precisely that $|l(x) - x_n|$ is bounded away from zero, and Lim(x) is always conteined in $\sigma(x)$, we could have used [7] for proving the implication $(b) \Rightarrow (a)$ in Theorem 2. Nevertheless, our proof is completely elementary and different from the proof in [7] which is purely algebraic.

Observe that the proof of Theorem 2 bears a strong resemblance to one of the usual approaches to nonstandard analysis. In fact, if $\mu: 2^{\mathbb{N}} \to \{0, 1\}$ is any measure such that $\mu(\mathbb{N}) = 1$ and $\mu(D) = 0$ for finite $D \subseteq \mathbb{N}$, one may define an equivalence relation $\sim$ on l_∞ by requiring $(x_n)_n \sim (y_n)_n$ if and only if $\mu(\{n: x_n = y_n\}) = 1$. The set $l_\infty/\sim$ is then a model of the nonstandard hull $^*\mathbb{N}$ which contains $\mathbb{R}$ via the constant sequences.

A beautiful survey on the connections between nonstandard hulls and generalized limits may be found in the recent book [5].

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I-Density Continuous Transformations on $\Re^2$ (**).

Abstract. – *We shall investigate various classes of I-density continuous transformations from $\Re^2$ into $\Re^2$. We are interested primarily in studying the connections between I-density (or strong I-density) continuity of f, g: $\Re^2 \to \Re$ and I-density (or strong I-density) continuity of $F = (f, g)$, $F: \Re^2 \to \Re^2$. Similarly as in the case of metric density (compare [3]) one cannot expect too much because $F(x, y) = (x, x + y)$ is not strongly I-density continuous and $F(x, y) = (x, y^3)$ is not I-density continuous. On the other hand, if $F: \Re^2 \to \Re^2$ is I-density (or strongly I-density) continuous and $F(x, y) = (f(x, y), g(x, y))$, then f, g: $\Re^2 \to \Re$ are I-density (or strongly I-density) continuous. Also, if $F(x, y) = (f(x), g(y))$ and f, g: $\Re \to \Re$ are I-density continuous homeomorphisms such that f^{-1}, g^{-1} are I-density continuous, then F is strongly I-density continuous.*

1. – Introduction.

In the paper $\Re$ will denote the real line and $\Re^2$ the plane, τ_1 and τ_2 the natural topology on the real line and on the plane, respectively, d_I

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