Ultrafilters

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1. INTRODUCTION: THREE PROBLEMS. In this article we introduce readers to a fascinating concept in mathematics: ultrafilters. We start with three problems concerning finite situations with similar conclusions. The first problem is very simple and is included solely in order to help us in the infinite case. By contrast, the other two have surprising conclusions. The analogue of the first problem for infinite sets leads us to ultrafilters, and we shall see that this is just the right concept needed to formulate the other problems in an infinite setting. The original problems in the finite case reflect the fact that every ultrafilter on a finite set is principal. Finally, we show how to apply ultrafilters to construct generalized limits in an elegant way.

The concept of an ultrafilter was introduced by Frederic Riesz [**9**] in 1908 in a talk that should have received more attention. Its systematic use in mathematics was started in the 1930s by the Polish school (Banach, Tarski, Ulam, etc.). Ultrafilters are standard tools in mathematical logic and set theory, but they are also widely used in combinatorics and topology. For example, the powerful ultraproduct construction in mathematical logic is based on them; in set theory and infinitary combinatorics they are used in Ramsey theory (partition relations). These applications go well beyond the level we target in this article. The books [**4**] and [**3**] treat many of these advanced applications.

As we have said, the first problem is very simple, but it will guide us in the infinite case:

Problem 1. In a land far away, game for the king's table is delivered by a number of royal hunters. The chancellor has labeled groups Y of hunters as "negligible" or "substantial" according to their contributions. It has turned out that if groups Y_1 and Y_2 are "negligible," then so is their union $Y_1 \cup Y_2$, and group Y is "negligible" precisely when the rest of the hunters form a "substantial" group. Show that there is a single hunter whose contribution is "substantial" and that the total contribution of all the others is "negligible."

The second problem is more surprising and is due to Greenwell, Lovász, and Lempert (see [**5**]):

Problem 2. Finitely many switches, each having three positions, are connected to a light with three states in such a way that if the positions of all switches are simultaneously changed, then the state of the light also changes. Show that there is a switch s such that the state of the light depends only on the position of s.

Our third problem is the celebrated theorem of Arrow [**1**, pp. 97–100] (see also [**6**] in this MONTHLY).

Problem 3. Suppose that in an election there are $n(\geq 3)$ candidates and finitely many voters, each of whom makes a ranking of the candidates, and the outcome is also a ranking of the candidates. There are two requirements for the outcome:

- if all the voters enter the same ranking, then this is the outcome;
- whether a candidate a precedes candidate b in the outcome depends only on their order on the different ranking lists of the individual voters (and it does not depend on where a and b are on those lists; i.e., on how the voters ranked other candidates).

Show that there is a "dictator" whose ranking gives the outcome.

The conclusion is to be understood in the following sense. As in any election, there is a rule (a machine) that decides (calculates) the outcome of the vote from the lists of the individual voters, and the rule obeys the two stated conditions. Then inherent in any such rule (machine) there is a dictator D such that the result will always be D's ranking (the ranking that the machine calculates will be the same as that of D) no matter how many times and how the voting is done.

2. ULTRAFILTERS. The solution to Problem 1 is very simple. Let X be the set of all hunters, a set that is clearly substantial. Note that if $Y_1, \dots, Y_k$ are negligible, then so is their union $Y_1 \cup \dots \cup Y_k$. This implies that not every hunter (one element group) is negligible, for then the union X of these singletons would be negligible. On the other hand, there cannot be two substantial hunters h_1 and h_2, for then both $X \setminus \{h_1\}$ and $X \setminus \{h_2\}$ would be negligible, as would be their union X, which is not the case. Therefore there is one and only one substantial hunter, and every other hunter is negligible.

Now what if the number of hunters is infinite? Let X be the set of hunters and $\mathcal{H}$ the collection of all substantial subsets of X. Thus, $X \in \mathcal{H}$, $\emptyset \notin \mathcal{H}$, and a subset Y of X is in $\mathcal{H}$ precisely if $X \setminus Y$ does not belong to $\mathcal{H}$. If Y_1 and Y_2 are elements of $\mathcal{H}$, then $X \setminus Y_1$ and $X \setminus Y_2$ are negligible, hence so is their union

$$(X \setminus Y_1) \cup (X \setminus Y_2) = X \setminus (Y_1 \cap Y_2).$$

It then follows by assumption that $Y_1 \cap Y_2 = X \setminus (X \setminus (Y_1 \cap Y_2))$ belongs to $\mathcal{H}$. It was implicit in Problem 1, but now we state explicitly that if Y is substantial, then so is every larger set Y' (i.e., if $Y \in \mathcal{H}$ and $Y \subset Y'$, then $Y' \in \mathcal{H}$).

Thus, $\mathcal{H}$ is a family of subsets of X with the following properties:

(i) $X \in \mathcal{H}, \emptyset \notin \mathcal{H}$;

(ii) if $Y_1 \in \mathcal{H}$ and $Y_1 \subset Y_2$, then $Y_2 \in \mathcal{H}$;

(iii) if $Y_1, Y_2 \in \mathcal{H}$, then $Y_1 \cap Y_2 \in \mathcal{H}$;

(iv) $Y \in \mathcal{H}$ if and only if $X \setminus Y \notin \mathcal{H}$.

Any family $\mathcal{H}$ of subsets of X with properties (i)–(iii) is called a *filter*, and if, in addition, (iv) is also true, then $\mathcal{H}$ is called an *ultrafilter*. Property (iii) implies by induction that if $Y_1, \dots, Y_k$ is any finite collection of sets in $\mathcal{H}$, then $Y_1 \cap \dots \cap Y_k$ also belongs to $\mathcal{H}$. We continue to refer to this trivial extension of (iii) as (iii).

Let a in X be fixed, and let $\mathcal{H}_a = \{Y \subset X \ : \ a \in Y\}$ be the collection of subsets containing a. Every such $\mathcal{H}_a$ is clearly an ultrafilter. These we call *principal ultrafilters*—they are the trivial ultrafilters generated by a single element. Are there nonprincipal ones? The solution of Problem 1 ensures that on a finite set every ultrafilter is principal. If $\{a\}$ belongs to an ultrafilter $\mathcal{H}$ for some a in X, then plainly $\mathcal{H}_a = \mathcal{H}$, so on an infinite set X an ultrafilter $\mathcal{H}$ is nonprincipal if and only if for each a in X it is the case that $X \setminus \{a\}$ lies in $\mathcal{H}$, hence by (iii) $\mathcal{H}$ is nonprincipal if and only if it contains the complements of all finite subsets (so-called cofinite sets) of X.

The cofinite sets themselves form a filter (check properties (i)–(iii)), and it is easy to show (see Exercise 1 at the end of this article) that a filter $\mathcal{H}$ is an ultrafilter (i.e., also has property (iv)) if and only if it is a maximal filter (i.e., there is no filter $\mathcal{H}'$ on X that properly contains $\mathcal{H}$). Now a statement called Zorn's lemma (which is equivalent to the Axiom of Choice) can be applied to show that every filter can be extended to a maximal filter, whence every filter is contained in an ultrafilter. Applying this fact to the filter of cofinite sets, we conclude that *on an infinite set X there is always a nonprincipal ultrafilter*.

Thus, when there are an infinite number of hunters, all we can say with respect to Problem 1 is that the substantial groups form an ultrafilter. It may not be principal (i.e., there may not be a substantial hunter).

There is an alternative way to introduce ultrafilters. Let X be a set, and let $\mathcal{P}(X)$ signify its collection of subsets. Call a mapping $\mu : \mathcal{P}(X) \to \{0, 1\}$ a *finitely additive measure* on X if $\mu(X) = 1$ and $\mu(Y_1 \cup Y_2) = \mu(Y_1) + \mu(Y_2)$ whenever Y_1 and Y_2 are disjoint subsets of X. Thus μ assigns the value 0 or 1 to each subset of X in an additive way. The term "finitely additive" comes from the fact that clearly

$$\mu(Y_1 \cup \cdots \cup Y_k) = \mu(Y_1) + \cdots + \mu(Y_k)$$

for any finite k and pairwise disjoint sets $Y_1, \ldots, Y_k$ in X. If a in X is fixed and $\mu_a(Y) = 1$ precisely if Y contains a, then μ_a is clearly a finitely additive measure. (We call such μ_a *trivial*.) From our perspective Problem 1 translates to the following:

Problem 1′. Show that on a finite set every finitely additive measure is trivial.

In general, ultrafilters and finitely additive measures correspond to each other in the sense that if we set

$$\mu(Y) = 1 \iff Y \in \mathcal{H}, \tag{1}$$

then μ is a finitely additive, $\{0, 1\}$-valued measure on X if and only if $\mathcal{H}$ is an ultrafilter on X. Moreover, in this correspondence nontrivial measures correspond to nonprincipal ultrafilters (see Exercise 2). Therefore, *on an infinite set there is always a nontrivial finitely additive,* $\{0, 1\}$*-valued measure*.

As pointed out earlier, on a finite set every ultrafilter is principal (Exercise 3). Therefore, we shall not solve Problems 2 and 3, for the solutions are consequences of their general versions discussed later.

In the proofs that follow we frequently use the following property of ultrafilters $\mathcal{H}$: if Y belongs to $\mathcal{H}$ and $Y = Y_1 \cup \cdots \cup Y_m$ is a decomposition of Y into pairwise disjoint sets, then one and only one Y_k belongs to $\mathcal{H}$. Indeed, if none of the Y_k were in $\mathcal{H}$, then each of the sets $X \setminus Y_k$ would be in $\mathcal{H}$, as would their intersection, which is the complement of $Y_1 \cup \cdots \cup Y_m = Y$. Thus, $\mathcal{H}$ would contain two disjoint sets (Y and $X \setminus Y$), which is not possible, for then it would also contain their intersection, the empty set. This contradiction shows that at least one of the Y_k must belong to $\mathcal{H}$, and since the Y_k are disjoint, it is not possible for two of them to belong to $\mathcal{H}$.

3. THE INFINITE SWITCH-LAMP PROBLEM.

For an arbitrary number of switches Problem 2 takes the following form:

> Consider a set X of switches, each having three positions $\{0, 1, 2\}$, and a light also with three states $\{L_0, L_1, L_2\}$ (like a traffic light). They are connected in such a way that if the positions of all switches are simultaneously changed, then the state of the light also changes. We suppose that if all the switches are in the ith

position, then the light is in the state L_i. Show that there is a (possibly principal) ultrafilter $\mathcal{H}$ on X that determines the state of the light, in the sense that it is L_i ($i = 0, 1, 2$) precisely when $\{S \in X : S \text{ is in the } i\text{th position}\}$ belongs to $\mathcal{H}$.

The assumption that if all the switches are in the ith position, then the light is also in the state L_i can always be achieved by appropriate labeling of the states of the light. As we have already mentioned in the previous section, the original switch-lamp problem follows, since on a finite set any ultrafilter is principal (i.e., it contains a one-element set, and the switch in this set determines the state of the light).

In the solution of the general switch-lamp problem we consider functions $f : X \to \{0, 1, 2\}$ ($f(S)$ is the position of the switch S) and an operator Φ that assigns to each such f a value in $\{0, 1, 2\}$ (the state of the lamp). The assumption in the problem takes the following form: if f_0 and f_1 differ everywhere, then $\Phi(f_0) \neq \Phi(f_1)$. We use this property over and over again in the solution. We have also assumed that if g_i is the identically constant function satisfying $g_i(S) = i$ ($S \in X$), then $\Phi(g_i) = i$.

First, assume that A is a subset of X and that $B = X \setminus A$. If $f : A \to \{0, 1, 2\}$ and $g : B \to \{0, 1, 2\}$ are functions defined on A and B, respectively, then we simply write $f * g$ for the function on $X = A \cup B$ that coincides with f on A and with g on B, and we also use the notation $(c)_A$ for the function that is identically c on A. Then $\Phi((0)_A * (0)_B) = 0$ and $\Phi((1)_A * (1)_B) = 1$, hence we must have

$$\Phi((1)_A * (0)_B) \neq \Phi((0)_A * (0)_B)$$

or

$$\Phi((1)_A * (0)_B) \neq \Phi((1)_A * (1)_B).$$

We show that if the first of these holds, then the state of the lamp depends only on the state of the switches contained in A. The other assumption leads in an identical fashion to the conclusion that the state of the lamp then depends only on the state of the switches in B.

Suppose that $\Phi((1)_A * (0)_B) \neq 0$. Since

$$\Phi((1)_A * (0)_B) \neq \Phi((2)_A * (2)_B) = 2,$$

we must have $\Phi((1)_A * (0)_B) = 1$. If $\overline{g} : B \to \{1, 2\}$, then $(2)_A * \overline{g}$ is pointwise different from both $(0)_A * (0)_B$ and $(1)_A * (0)_B$, so necessarily $\Phi((2)_A * \overline{g}) = 2$. We denote this fact by $\Phi((2)_A * (1, 2)_B) = 2$ (i.e., henceforth $(1, 2)_B$ denotes an arbitrary function $B \to \{1, 2\}$). If now $g : B \to \{1, 2\}$ and $\overline{g}$ is defined by $\overline{g}(i) = 3 - g(i)$ for each i in B, then the functions $(0)_A * (0)_B$, $(1)_A * g$, and $(2)_A * \overline{g}$ assume three different values everywhere, which implies that $\Phi((1)_A * g) = 1$ (i.e., $\Phi((1)_A * (1, 2)_B) = 1$). Thus, $\Phi((2)_A * \overline{g}) = 2$ and $\Phi((1)_A * \overline{g}) = 1$, so $\Phi((0)_A * g) = 0$ (i.e., it also follows that $\Phi((0)_A * (1, 2)_B) = 0$). So far we have verified that $\Phi((i)_A * (1, 2)_B) = i$ for $i = 1, 2, 3$.

Next let $g : B \to \{0, 1, 2\}$ be arbitrary, and let $\overline{g} : B \to \{1, 2\}$ be a function that is everywhere different from g. Since $\Phi((i)_A * g)$ is different from any of the two values $\Phi((j)_A * \overline{g}) = j$ ($j = 1, 2, 3$, $j \neq i$), we must have $\Phi((i)_A * g) = i$ for $i = 1, 2$, and 3. Accordingly, we have verified that $\Phi((i)_A * g) = i$ for all i, whence

$$\Phi((i_0, i_1)_A * g) \in \{i_0, i_1\} \qquad (i_0, i_1 \in \{0, 1, 2\}), \tag{2}$$

where $g : B \to \{0, 1, 2\}$ is arbitrary.

Assume, finally, that two functions agree on A but that Φ assigns different values to them. Thus, for some function $f : A \to \{0, 1, 2\}$ and functions $g_0, g_1 : B \to \{0, 1, 2\}$ we have $\Phi(f * g_0) \neq \Phi(f * g_1)$—say, $\Phi(f * g_0) = i_0$ and $\Phi(f * g_1) = i_1$. Then there is a function $\overline{f} : A \to \{i_0, i_1\}$ that is everywhere different from f and a function $g : B \to \{0, 1, 2\}$ that is everywhere different from both g_0 and g_1. For these we have $\Phi(\overline{f} * g) \neq i_0, i_1$. On the other hand, the value $\Phi(\overline{f} * g)$ must be in $\{i_0, i_1\}$ by (2). This contradiction establishes that if two functions coincide on A, then the associated Φ-values are the same. In other words, the state of the lamp depends only on the switches in A.

What we have just shown is that if $X = A \cup B$ is a disjoint decomposition, then one and only one of $Y = A$ or $Y = B$ has the property that $\Phi(f)$ depends on the restriction of f to Y. Let $\mathcal{H}$ be the collection of those subsets Y of X with this property. As we have just observed, for an arbitrary subset A of X exactly one of A or $X \setminus A$ is in $\mathcal{H}$. Clearly, $\emptyset \notin \mathcal{H}$. Moreover, if $A \in \mathcal{H}$ and $A \subset B$, then $B \in \mathcal{H}$. It is also immediate that $\mathcal{H}$ is closed under finite intersection (Exercise 4). Thus, $\mathcal{H}$ is an ultrafilter.

To conclude the proof, for an arbitrary $f : X \to \{1, 2, 3\}$ let $C_i(f) = \{S \in X : f(S) = i\}$ $(i = 0, 1, 2)$. These are disjoint sets with union X, hence exactly one of them, say C_{i_0}, belongs to $\mathcal{H}$. Since on C_{i_0} the function f coincides with the constant function g_{i_0} and $\Phi(f)$ depends only on the restriction of f to C_0, we have $\Phi(f) = \Phi(g_{i_0}) = i_0$, as claimed.

4. AN ARBITRARY NUMBER OF VOTERS. For arbitrarily many voters Problem 3 takes the following form (see, for example, Kirman and Sondermann [**7**]):

> Suppose that in an election there are finitely many $n(\geq 3)$ candidates $\{c_1, \dots, c_n\}$ and a set X of voters. Each voter makes a ranking of the candidates, and the outcome of the election is determined by two rules:
>
> - if all the voters enter the same ranking, then this is the outcome;
> - whether a candidate a precedes candidate b in the outcome depends only on their order on the different ranking lists of the individual voters (and it does not depend on where a and b are on those lists; i.e., on how the voters ranked other candidates).
>
> Show that there is an ultrafilter $\mathcal{H}$ on X such that the outcome is an ordering π of $\{c_1, \dots, c_n\}$ if and only if the set F_π of those voters whose ranking is π belongs to $\mathcal{H}$.

We point out that the two conditions imply that if all voters rank a ahead of b, then a is also ahead of b in the outcome.

The outline of the proof is this: we prove the theorem for up to four voters. This gives the desired conclusion when the voters vote in blocks and there are at most four such blocks. Finally, we show that this is already sufficient to obtain the full conclusion for an arbitrary set of voters.

We work with blocks of voters who rank the candidates the same way. However, in block-voting each block is like an individual voter (in other words, each block can be replaced with any one of its members), hence it is convenient to consider first the case when there are only finitely many voters (X is finite). We call a voter *decisive* if the outcome of the vote always agrees with her list.

First we show that if there are only two voters A and B, then one of them is decisive. We establish the following notation: we signify the fact that candidates a, b, and c are listed on A's ballot in some order (say $\dots, a, \dots, c, \dots, b, \dots$), on B's ballot in some (possibly different) order (say $\ldots, c, \ldots, a, \ldots, b, \ldots$), and in the outcome in a (potentially third) order (say $\ldots, b, \ldots, c, \ldots, a, \ldots$), by writing

$$\begin{array}{rl} \text{A :} & acb \\ \text{B :} & cab \\ & \text{---} \\ \text{Outcome :} & bca \end{array}$$

Suppose now that A is not decisive. Then there is an election with some candidates a and b in the order ab on A's ballot such that in the outcome their order is ba. Then on B's ballot their order is necessarily ba (otherwise a and b would be listed on both lists in the order ab, which would be the outcome, as well). We show that B is decisive. Since the order in the outcome is the result of the order of the pairs of candidates, it is sufficient to show that B is decisive for each pair of candidates. Let c be a third candidate. Each column in the following table implies the next one (by this we mean the following: if we assume, for example, that, as indicated in column one, the outcome is $\ldots b \ldots a \ldots$ if A's list has $\ldots a \ldots b \ldots$ and B's has $\ldots b \ldots a \ldots$, then the outcome will be $\ldots c \ldots b \ldots a \ldots$ when A's list reads $\ldots a \ldots c \ldots b \ldots$ and B's list reads $\ldots c \ldots b \ldots a \ldots$, and this is what is stated in column two):

$$\begin{array}{rccccccc} \text{A :} & ab & acb & ac & abc & bc & bac & ba \\ \text{B :} & ba & cba & ca & cab & cb & acb & ab \\ & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \\ \text{Outcome :} & ba & cba & ca & cab & cb & acb & ab \end{array}$$

This proves that B is decisive for the pair a and b.

Now we repeat the same argument with column 1 replaced with column 3 (respectively, column 5)—in other words, we exchange a and b with a and c (respectively, with b and c)—to conclude that B is also decisive for the pairs a and c (respectively, b and c). Thus, the decisiveness of B for the pair a and b has been established, and here a and b can be replaced with any c different from a and b. By means of at most two such replacements we can get to any pair of candidates, so the decisiveness of B has been confirmed.

Next we show that if there are four voters A, B, C, and D, then one of them is decisive. In fact, suppose first that A and B form a block (i.e., they always vote the same way) and C and D also form a block. Then we have two block voters, hence one of them is decisive, say the AB block. We claim that if A and B vote the same way, then they are decisive even if C and D do not vote in a block. If this is not the case, then there are candidates p and q and a vote tally in which A and B rank them in the order pq, whereas in the outcome the order is qp. In the following table $p'q'$ and $p''q''$ denote permutations of the "phrase" pq, and again each column implies the next one (a is a candidate different from p and q):

$$\begin{array}{rccc} \text{A :} & pq & paq & aq \\ \text{B :} & pq & paq & aq \\ \text{C :} & p'q' & p'q'a & qa \\ \text{D :} & p''q'' & p''q''a & qa \\ & \text{---} & \text{---} & \text{---} \\ \text{Outcome :} & qp & qpa & qa \end{array} \tag{3}$$

This contradicts the decisiveness of the block AB over CD, and this contradiction proves the decisiveness of the block AB. Now *fix* the rankings of C and D in some order $\pi(c_1), \ldots, \pi(c_n)$ (for both of them), where $\pi : \{c_1, \ldots, c_n\} \to \{c_1, \ldots, c_n\}$ is some permutation of the candidates, allowing A and B to vote as they wish. In this way we get a two-member voting scheme, hence either A or B is decisive in it; for definiteness let it be A. We claim that A is decisive in the original four-voter scheme. Suppose that this is not the case. Then there are candidates p and q and an election in which A ranks them in the order pq, whereas their order in the outcome is qp. Since the block AB was decisive, this is possible only if B ranks them in the order qp. Then, if the last element of the fixed order is $b = \pi(c_n)$, each column in the following table implies the next one:

$$\begin{array}{rccc}
\mathrm{A}: & pq & pbq & bq \\
\mathrm{B}: & qp & qpb & qb \\
\mathrm{C}: & p'q' & p'q'b & qb \\
\mathrm{D}: & p''q'' & p''q''b & qb \\
 & \text{---} & \text{---} & \text{---} \\
\text{Outcome}: & qp & qpb & qb
\end{array}$$

This contradicts the decisiveness of A when C and D vote in the fixed order $\pi(c_1)$, $\ldots, \pi(c_n)$ (if one of p or q is the last element $b = \pi(c_n)$, then work symmetrically with the first element in the fixed order; if p and q coincide with the last and first elements, then first replace one of them in the indicated manner with a third element; after these steps we are back to previously considered cases). With this, the claim that in a four-member voting scheme there is always a decisive voter has been verified. The same argument establishes the same claim if there are three voters.

Having dealt with these cases, we turn to an arbitrary set X of voters. We call a subset F of X *decisive* if it is true that when all members of F vote the same way, this is always the outcome. An argument similar to that in (3) demonstrates the following: *if F is decisive in the two-block voting scheme consisting of the blocks F and $X \setminus F$, then F is decisive*. In fact, in that case (3) should be changed to (4), where V_1 and V_2 are the set of voters in $X \setminus F$ who rank p and q in the orders pq and qp, respectively, and a is again a candidate different from p and q. Indeed, if F were not decisive, then there would be an election (indicated in the first column in (4)) whose outcome would lead to a contradiction:

$$\begin{array}{rccc}
\mathrm{F}: & pq & paq & aq \\
\mathrm{V}_1: & pq & pqa & qa \\
\mathrm{V}_2: & qp & qpa & qa \\
 & \text{---} & \text{---} & \text{---} \\
\text{Outcome}: & qp & qpa & qa
\end{array} \tag{4}$$

Here the last column would contradict the decisiveness of F in the two-block voting consisting of F and $X \setminus F$.

Thus, we have shown that if F is decisive in the two-block voting scheme consisting of the blocks F and $X \setminus F$, then F is decisive. This already implies something apparently stronger: *if F is decisive in any finite block-voting scheme in which F is one of the blocks, then F is decisive* (note that F is then automatically decisive in the two-block voting consisting of F and $X \setminus F$).

Let $\mathcal{H}$ be the set of decisive subsets of X. We show that $\mathcal{H}$ is an ultrafilter on X. It is clear that $\emptyset \notin \mathcal{H}$ (the presence of $\emptyset$ in $\mathcal{H}$ would mean a fixed outcome irrespective of the votes). If $F \in \mathcal{H}$ and $F \subset F'$, then $F' \in \mathcal{H}$. Also, at most one of F or $X \setminus F$ can belong to $\mathcal{H}$. That one of them is actually in $\mathcal{H}$ follows from the decisiveness in the two-member voting schemes. Therefore, to show that $\mathcal{H}$ is an ultrafilter, it suffices to verify that if F_1 and F_2 belong to $\mathcal{H}$, then so is $F_1 \cap F_2$. Consider the four-member block voting scheme when the blocks are $F_1 \cap F_2$, $X \setminus (F_1 \cup F_2)$, $F_1 \setminus F_2$ and $F_2 \setminus F_1$ (i.e., the voters in each block vote the same way, and if one of these sets is empty, then the relevant block-voter is missing). We know that one of them is decisive (we have verified decisiveness if there are at most four voters). Since both F_1 and F_2 are decisive, this decisive block cannot be any of the last three, so it must be $F_1 \cap F_2$, placing $F_1 \cap F_2$ in $\mathcal{H}$.

Now consider an arbitrary election. For a permutation π of the candidates consider the set F_π of those voters v in X whose ballots rank candidates in the order given by π. Since $X = \cup_\pi F_\pi$ is a finite disjoint decomposition, exactly one of the F_π belongs to $\mathcal{H}$, say F_{π_0} has this property. Then F_{π_0} is decisive, so the outcome of the vote must be π_0.

5. BANACH-LIMITS. Ever since the rigorous notion of sequential convergence took hold in mathematics, there has been a desire to extend it to more general sequences. Even before that time many great mathematicians routinely associated limits with sequences and series that are not considered to be convergent these days, but those limits can be justified by modern summation methods. Several generalizations of convergence were introduced in the nineteenth and early twentieth centuries, but it was Stefan Banach [**2**, chap. 2, sec. 3] who established in 1932 that a generalized limit, now called a *Banach-limit*, can be associated with every bounded sequence. Furthermore, this generalized limit retains many of the desirable features of standard convergence. In this section we show that ultrafilters provide an elegant way to define Banach-limits.

Let $\{x_n\}$ be a bounded real sequence. If it happens to be convergent (with respect to standard topology of the real line), then denote its limit by $\lim x_n$. Our aim is to assign a value (generalized limit) $\lim^* x_n$ to every bounded sequence $\{x_n\}$, and we want to do so in such a way that certain properties of convergence are preserved. In particular, for starters we require the following properties:

(A) if $\{x_n\}$ is convergent, then $\lim^* x_n = \lim x_n$;

(B) $\lim^*(x_n + y_n) = \lim^* x_n + \lim^* y_n$;

(C) $\lim^*(cx_n) = c \lim^* x_n$.

In (B), $\{x_n + y_n\}$ is the termwise sum of the sequences $\{x_n\}$ and $\{y_n\}$, and in (C), c is any real constant. Property (A) expresses the fact that convergent sequences should have the same generalized limits as their conventional limits, while (B) and (C) require that taking generalized limit should be a linear operation. There is one further useful property of ordinary limits that we would like to preserve: the limit is never larger (in absolute value) than any bound for the absolute value of the individual terms in the sequence. Therefore, we stipulate:

(D) $|\lim^* x_n| \leq A$ if $|x_m| \leq A$ for $m = 1, 2, \ldots$.

If, as usual, $\sup_m |x_m|$ denotes the least upper bound of the numbers $|x_m|$ $(m = 1, 2, \ldots)$, then (D) is clearly the same as

(E) $|\lim^* x_n| \leq \sup_n |x_n|$.

One can also prove (Exercise 8) that, assuming (A)–(C), property (E) is equivalent to positivity:

(F) if $\{x_n\}$ is a nonnegative sequence, then $\lim^* x_n \geq 0$.

For those readers who are familiar with the basic notions of functional analysis, we point out the following. If we denote by $\mathcal{B}$ the set of all bounded sequences, then this is a linear space and what we are actually looking for is a linear functional from $\mathcal{B}$ into $\mathbf{R}$ that preserves convergence for convergent sequences. The usual norm in $\mathcal{B}$ is the so-called supremum norm: $\|x_n\| = \sup_n |x_n|$. Property (E) demands nothing other than that this linear functional have norm at most 1. The standard construction of Banach-limits is via the Hahn-Banach extension theorem, according to which the linear functional $\{x_n\} \to \lim x_n$ can be extended from the space of convergent sequences to the whole of $\mathcal{B}$ while preserving its norm. Here we furnish an alternative approach using ultrafilters (we could not trace the origin of this approach; it may be folklore).

Let $\mathcal{H}$ be a nonprincipal ultrafilter on $\mathbf{N}$, the set of natural numbers. As we have discussed in section 2, the fact that $\mathcal{H}$ is nonprincipal means that it contains all cofinite sets in $\mathbf{N}$ (i.e., all subsets of $\mathbf{N}$ whose complements are finite sets). Recall the definition of standard limit: $\{x_n\}$ has limit s if for every $\varepsilon > 0$ we have $|x_n - s| < \varepsilon$ for all but finitely many n. Note that this is the same as saying that all the sets $U(s, \varepsilon) = \{n \ : \ |x_n - s| < \varepsilon\}$ are cofinite. With this in mind we make the following definition:

Definition. For a bounded real sequence $\{x_n\}$ let $\lim^* x_n = s$ if all the sets $U(s, \varepsilon) = \{n \ : \ |x_n - s| < \varepsilon\}$ $(\varepsilon > 0)$ belong to $\mathcal{H}$.

We have to show that this is a good definition (i.e., $\lim^* x_n$ exists for each bounded sequence $\{x_n\}$ in $\mathbf{R}$ and it is unique). Indeed, let $L = \sup_n |x_n|$. Then $-L \le x_n \le L$ for all n. Given a positive integer m, we divide the interval $[-L, L]$ into 2^m pairwise disjoint intervals

$$I_{m,j} = [-L + (j-1)2L/2^m, -L + j2L/2^m) \qquad (j = 1, \ldots, 2^m - 1)$$

and $I_{m,2^m} = [L - 2L/2^m, L]$ of equal length, and we consider the sets $J_{m,j} = \{n \ : \ x_n \in I_{m,j}\}$. These are pairwise disjoint, and their union is $\mathbf{N}$. Consequently, one and only one of them, say J_{m,j_m}, belongs to $\mathcal{H}$. Note that on the next level every interval $I_{m+1,k}$ is a subset of one of the $I_{m,j}$, and then $I_{m+1,j_{m+1}}$ must be part of I_{m,j_m} (otherwise $\mathcal{H}$ would contain two disjoint sets). Hence, the closures of the intervals I_{m,j_m} form a nested sequence of closed intervals with diameter tending to 0, so they have a common point that we call s. If $\varepsilon > 0$ is arbitrary, then for large m the interval I_{m,j_m} is a subset of $(s - \varepsilon, s + \varepsilon)$, hence the index set $U(s, \varepsilon) = \{n \ : \ |x_n - s| < \varepsilon\}$ contains J_{m,j_m}. Since this latter set belongs to $\mathcal{H}$, $U(s, \varepsilon)$ is likewise in $\mathcal{H}$. Because this is true for each $\varepsilon > 0$, by definition $\lim^* x_n = s$. Furthermore, it is not possible to have $\lim^* x_n = s$ and $\lim^* x_n = s'$ for different s and s', for $U(s, \varepsilon)$ and $U(s', \varepsilon)$ are disjoint for small ε, whence at most one of them can belong to $\mathcal{H}$. Thus, the generalized limit exists, and it is unique. Incidentally, we have also established that $\lim^* x_n$ lies in $[-L, L]$, therefore property (E) also holds. Verifying properties (A)–(C) is quite straightforward; we leave it to the reader (Exercise 7).

There is one additional property that it is customary to include in this theory, namely, translation invariance. By this we mean that if $y_n = x_{n+1}$ for all n, then $\lim^* y_n = \lim^* x_n$. In particular, translation invariance implies that if we alter finitely many terms in a sequence, then the generalized limit remains the same (i.e., the limit does not depend on how the sequence begins). Standard convergence has this property, but unfortunately $\lim^*$ as defined does not. Indeed, if we consider the sequence $\{x_n\} = \{0, 1, 0, 1, 0, 1, \ldots\}$, then the definition implies immediately that $\lim^* x_n$ is either 0 or 1. Now with $\{y_n\} = \{1, 0, 1, 0, \ldots\}$ the sum sequence $\{x_n + y_n\}$ is the sequence $\{1, 1, \ldots\}$, hence $\lim^* y_n = 1 - \lim^* x_n$, so $\lim^* y_n$ is definitely different from $\lim^* x_n$. But $y_n = x_{n+1}$ for all n, ruling out translation invariance. However, from $\lim^*$ it is easy to construct a translation-invariant generalized limit with properties (A)–(F). Indeed, let

$$z_n = \frac{x_0 + \cdots + x_n}{n+1},$$

and set

$$\lim{}^{**} x_n = \lim{}^{*} z_n. \tag{5}$$

It is easy to see that $\lim^{**}$ also has properties (A)–(F). If $y_n = x_{n+1}$ and

$$z'_n = \frac{y_0 + \cdots + y_n}{n+1},$$

then it is clear that $\lim(z_n - z'_n) = 0$. Hence, the generalized limits of $\{z_n\}$ and $\{z'_n\}$ are the same, which shows that $\lim^{**} x_n = \lim^{**} y_n$.

As a final note for readers who are familiar with the notion of topological spaces, we add the following. The Banach-limit $\lim^* x_n$ is a special case of the more general notion of an ultrafilter limit. Given a set X equipped with an ultrafilter $\mathcal{H}$, a topological space Y, and a function $f : X \to Y$, we say that a in Y is the *limit of f determined by $\mathcal{H}$* if for every neighborhood U of a the set $\{x \in X : f(x) \in U\}$ belongs to $\mathcal{H}$. Such a limit always exists if Y is compact, and it is unique if the topology on Y is Hausdorff (i.e., any two points have disjoint neighborhoods). This latter fact is clear from the properties of ultrafilters and this definition of ultrafilter limit. To prove the existence of the limit when Y is compact, assume to the contrary that no limit existed. Then for each a in Y we would have a neighborhood U_a such that $\{x \in X : f(x) \in U_a\} \notin \mathcal{H}$. For some finite cover $Y = \cup_{k=1}^m U_{a_k}$ of Y by these U_a (guaranteed by the compactness of Y) we would get $\cup_{k=1}^m \{x \in X : f(x) \in U_{a_k}\} \notin \mathcal{H}$, which is absurd, since the set on the left-hand side is

$$\{x \in X : f(x) \in \cup_{k=1}^m U_{a_k}\} = \{x \in X : f(x) \in Y\} = X,$$

which must be in $\mathcal{H}$. This contradiction shows that the limit of f relative to $\mathcal{H}$ exists.

6. EXERCISES. The best way to grasp concepts and ideas in mathematics is to work with them. We also share Paul Halmos's view that the heart of mathematics is problem solving. Therefore, we urge the reader to verify the following statements as exercises related to this article. Further problems on ultrafilters can be found in [**8**, chap. 17].

1. A filter $\mathcal{H}$ on the set X is an ultrafilter if and only if it is a maximal filter, in the sense that on X there is no filter that strictly includes $\mathcal{H}$.
2. Under the correspondence (1) nontrivial measures correspond to nonprincipal ultrafilters.
3. On a finite set X every ultrafilter $\mathcal{H}$ is principal (i.e., there is an a in X such that $\mathcal{H} = \{Y \subset X : a \in Y\}$).
4. With the notations of section 3, let $\mathcal{H}$ be the system of those subsets Y of X with the property that $\Phi(f)$ depends only on the restriction of f to Y. Then $\mathcal{H}$ is closed under finite intersection.

5. The statement in the switch-lamp problem is not true for the standard two-way lamp.
6. What would $\lim^* x_n$ be if $\mathcal{H}$ were a principal ultrafilter?
7. Properties (A)–(C) hold for the generalized limit defined in section 5.
8. Assuming (A)–(C), properties (E) and (F) are equivalent.
9. Show that for the generalized limit defined in section 5 it is also true that $\lim^* x_n$ is always in the cluster set of $\{x_n\}$ (i.e., coincides with the (ordinary) limit of some subsequence of $\{x_n\}$).
10. Conversely, if s is the limit of some subsequence of $\{x_n\}$, then there is a nonprincipal ultrafilter $\mathcal{H}$ on $\mathbf{N}$ such that for the generalized limit that it generates we have $\lim^* x_n = s$.
11. For an arbitrary real sequence $\{x_n\}$ set $\lim^* x_n = \infty$ if and only if $\{n : p < x_n\} \in \mathcal{H}$ holds whenever $p < \infty$, and define $\lim_D^* x_n = -\infty$ analogously. Then every real sequence has a (possibly infinite) generalized limit (in this case linearity is not required, for we do not define $\infty - \infty$).
12. Associate with any subset Y of $\mathbf{N}$ its density $\sigma(Y)$ as follows: let

$$s_n = \frac{\#\{j \ : \ j \le n, \ j \in Y\}}{n+1}$$

be the relative density of Y in $\{0, 1, \ldots, n\}$, and set $\sigma(Y) = \lim^* s_n$. Show that $\sigma(Y)$ lies between 0 and 1 and that the correspondence $Y \mapsto \sigma(Y)$ is finitely additive (i.e., $\sigma(Y_1 \cup Y_2) = \sigma(Y_1) + \sigma(Y_2)$ when Y_1 and Y_2 are disjoint).
13. Show that there is an ultrafilter $\mathcal{H}$ on the set of natural numbers such that if A belongs to $\mathcal{H}$, then $\sum_{n\in A} 1/n = \infty$.
14. Let $\mathcal{H}$ be a nonprincipal ultrafilter on $\mathbf{N}$. Two players consecutively pick natural numbers $0 < n_0 < n_1 < \cdots$, with player I beginning. Player I wins if the set $[0, n_0) \cup [n_1, n_2) \cup \cdots$ is in $\mathcal{H}$, otherwise player II wins. Show that neither player has a winning strategy (a winning strategy is a description of how to react in all possible situations so that if the player follows it, then she wins no matter what the opponent does; such a strategy must be shared with all parties).

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Cassius J. Keyser, *The Human Worth of Rigorous Thinking: Essays and Addresses*, Columbia University Press, New York, 1925, p. 59.

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Doron Zeilberger, Closed form (pun intended!), *Contemporary Mathematics* **143** (1993) 579.

"*Mathematics is the science of the infinite*, its goal the symbolic comprehension of the infinite with human, that is finite, means."

Hermann Weyl, *The Open World: Three Lectures on the Metaphysical Implications of Science*, Yale University Press, New Haven, 1932, p. 7.

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