APLICACIONES LINEALES QUE PRESERVAN FUNCIONES ORTOGONALES.

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1. Introducción

El estudio de la continuidad de las aplicaciones lineales entre espacios normados es uno de los primeros temas que aparecen en cualquier asignatura de introducción al *Análisis Funcional*. Los espacios de operadores lineales continuos definidos entre espacios de Banach y de Hilbert conducen, de manera natural, a la definición de C*-álgebra y otras álgebras de operadores. La definición del término "Análisis Funcional" que puede encontrarse en la *Encyclopedia Britannica* es la siguiente:

> *"Functional Analysis, Branch of mathematical analysis dealing with functionals, or functions of functions. It emerged as a distinct field in the 20th century, when it was realized that diverse mathematical processes, from arithmetic to calculus procedures, exhibit very similar properties. A functional, like a function, is a relationship between objects, but the objects may be numbers, vectors, or functions."*

Esta definición está fuertemente influenciada por el hecho contrastado de que los primeros espacios de Banach de dimensión infinita que aparecen en la literatura que origina el Análisis Funcional son los espacios de funciones. Los puntos o vectores de dichos espacios son funciones. Uno de los espacios de Banach que más atención ha recibido y más estudios ha motivado es el espacio $C(K)$ de las funciones continuas $\mathbb{C}$-valuadas sobre un espacio topológico compacto y Hausdorff K. Es decir los elementos de $C(K)$ son funciones continuas $f : K \to \mathbb{C}$. Es bien conocido que el espacio $C(K)$ equipado con la norma del supremo

$$\|f\|_\infty := \sup\{|f(t)| : t \in K\} \quad (f \in C(K)),$$

es un espacio de Banach (que en la mayoría de ocasiones tiene dimensión infinita). En el trabajo que abordaremos necesitaremos considerar algunos subespacios notables de $C(K)$. Por supuesto el espacio $C(K, \mathbb{R})$ de las funciones continuas de K en el cuerpo de los números reales $\mathbb{R}$. Para dar algún otro ejemplo recordamos que, dado un espacio topológico localmente compacto Hausdorff L y una función $f : \Omega \to \mathbb{C}$, se dice que f *se anula en infinito* si para cada $\varepsilon > 0$, el conjunto

$$\{t \in L : |f(t)| \geq \varepsilon\}$$

es compacto. Notaremos mediante el símbolo $C_0(L)$ al espacio de las funciones continuas de L en $\mathbb{C}$ que se anulan en infinito, es decir, si $L \cup \{\infty\}$ denota la compactificación de L mediante un punto (∞ en este caso), consideramos las funciones continuas $f : L \cup \{\infty\} \to \mathbb{C}$ verificando $f(\infty) = 0$. De nuevo, $C_0(L)$ es un espacio de Banach (complejo) cuando lo equipamos con la norma del supremo. El espacio $C_0(L, \mathbb{R})$ se define de forma análoga.

Otra de las grandes virtudes del Análisis Funcional moderno es, si lugar a dudas, su gran versatilidad para adaptar y aplicar herramientas de otras disciplinas y ramas de la Matemática a problemas propios, junto con la capacidad de exportar resultados y herramientas al estudio de otros problemas existentes en otras ramas de la Matemática y la Física. En algunos espacios de Banach, el Análisis Matemático, el Álgebra, la Geometría y la Topología se encuentran tan mutuamente entrelazadas que ciertas propiedades analíticas están completamente determinadas mediante propiedades algebraicas y viceversa. Uno de los ejemplos donde este fenómeno se puede apreciar de manera más evidente es el problema que estudiaremos en esta Escuela Taller.

Para comenzar vamos a presentar otra parte de la estructura de los espacios $C(K)$ y $C_0(L)$ que nos hemos dejado atrás. En estos espacios es posible definir el producto de dos funciones f, g como otra función dada por la expresión $(fg)(t) := f(t)g(t)$. Este sencillo producto se conoce como producto puntual de funciones. La asociatividad y la conmutatividad del producto de $\mathbb{C}$ nos permite asegurar que el producto puntual, tanto en $C(K)$ como en $C_0(L)$ es asociativo y conmutativo. Además, la desigualdad

$$\|fg\| \leq \|f\| \ \|g\|,$$

se verifica para todo par de funciones continuas f y g en los anteriores espacios. Existe otra operación algebraica en los espacios $C(K)$ y $C_0(L)$, nos referimos a la involución natural $f \mapsto f^*$, donde $f^*(t) = \overline{f(t)}$. Es fácil comprobar que f toma valores reales si, y solo si, $f^* = f$. Por último, resaltar una identidad donde se mezclan las estructuras algebraicas y analíticas de $C(K)$

$$\|ff^*\|_\infty = \|f\|_\infty^2,$$

igualdad conocida como identidad o axioma de Gelfand-Naimark.

2. Cuando las propiedades algebraicas determinan las propiedades analítico-geométricas

En $C(K)$ tenemos dos estructuras bien diferenciadas, la estructura analítico-geométrica que proporciona su norma de espacio de Banach ($\|.\|_\infty$), y por otro lado su estructura algebraica de álgebra conmutativa con involución. Las conexiones que tienen estas dos estructuras parecen estar muy limitadas a las propiedades que hemos comentado anteriormente. Sin embargo los resultados que las conectan son sorprendentes. Veremos, por ejemplo, que, a nivel algebraico, un *homomorfismo* (es decir, una aplicación lineal que preserva los productos puntuales de funciones) $T : C(K_1) \to C(K_2)$

tiene que ser automáticamente continuo. Los homomorfismos de $C(K)$ en el cuerpo $\mathbb{C}$ tienen incluso mejores propiedades. Esto permitirá comprobar que toda identificación algebraica de dos $C(K)$-espacios (es decir, toda biyección lineal que preserva los productos y las involuciones entre dichos espacios), es una isometría sobreyectiva. En otras palabras, una identificación a nivel algebraico de dos $C(K)$-espacios nos proporciona una identificación a nivel analítico de dichos espacios.

Existe una versión con menos requerimientos. Recordamos que, dadas dos funciones continuas $f, g : K \to \mathbb{C}$, diremos que f y g son *ortogonales* o *disjuntas* o tienen soportes disjuntos (y lo notamos mediate $f \perp g$) cuando $fg = 0$. Supongamos que tenemos un homomorfismo $T : C(K_1) \to C(K_2)$. Si $f \perp g$ en $C(K_1)$, entonces $T(f)T(g) = T(fg) = 0$ en $C(K_2)$. Es decir, todo homomorfismo entre $C(K)$-espacios preserva funciones ortogonales.

Sean K_1 y K_2 dos espacios topológicos compactos Hausdorff y sea $T : C(K_1) \to C(K_2)$ una aplicación lineal. Diremos que T *preserva ortogonalidad* o es un *operador de Lamperti* si

$$f \perp g \text{ in } C(K_1) \Rightarrow T(f) \perp T(g) \text{ en } C(K_2).$$

Ahora solo suponemos que T preserva funciones ortogonales. Todo homomorfismo preserva ortogonalidad, pero existen otros casos, por ejemplo, sea h una función en $C(K_2)$ y sea $\varphi : K_2 \to K_1$ una función continua en el conjunto $\{s \in K - 2 : h(s) \neq 0\}$, entonces el operador

$$T : C(K_1) \to C(K_2)$$
$$T(f)(s) = (h.C_\varphi)(f)(s) = h(s)f(\varphi(s)) \quad (f \in C(K_1), s \in K_2),$$

es un operador lineal que preserva ortogonalidad y no es necesariamente un homomorfismo. El gran objetivo de este tema será determinar la estructura que tiene toda transformación lineal que preserva ortogonalidad entre espacios $C(K)$. Pare ello nos proponemos revisar los Teoremas de Arendt [1] y Jarosz [3] que permiten describir este tipo de operadores como generalizaciones de operadores de composición con peso.

Primero supondremos que T es continuo y determinaremos que T es exactamente un operador de composición con peso como el expuesto anteriormente. El trabajo de Jarosz permite una descripción sin asumir continuidad alguna sobre T. Entre las consecuencias de este resultado, probaremos que toda biyección lineal $T : C(K_1) \to C(K_2)$ que preserva ortogonalidad es automáticamente continua, y en tal caso K_1 y K_2 son topológicamente homeomorfos. Es decir, unas propiedades algebraicas sobre la aplicación T determinan una consecuencias analíticas y topológicas.

3. Cuando las propiedades analítico-geométricas determinan las propiedades algebraicas

La magia no termina demostrando implicaciones analíticas y topológicas desde hipótesis algebraicas, el otro camino también puede ser explorado. En este caso nos encontraremos con uno de los grandes resultados establecidos durante los primeros años de desarrollo del Análisis Funcional, nos referimos al Teorema de Banach-Stone. Este resultado establece que para toda isometría lineal y sobreyectiva $T : C(K_1) \to C(K_2)$, existen h en $C(K_2)$ y $\varphi : K_2 \to K_1$ continua y biyectiva con $|h(t)| = 1, \forall t \in K_2$ tales que

$$T(f)(s) = h(s)f(\varphi(s)),$$

para todo $f \in C(K_1)$, $s \in K_2$. En este caso, $T(f)T(g)^* \neq T(fg^*)$, pero $T(f)T(g)^*T(h) = T(fg^*h)$, para toda terna f, g y h en $C(K_1)$. Es decir, una identificación analítico-geométrica de $C(K_1)$ y $C(K_2)$ permite obtener una identificación algebraica de estos espacios, y una identificación topológica de K_1 y K_2.

Si tenemos tiempo y disponibilidad exploraremos estos resultados en el caso en que $C(K)$ es reemplazado por $C_0(L)$.

Conocimientos Previos: Conocimientos básicos de Análisis Funcional: espacios de Banach, operadores lineales continuos, isometrías, dual de un espacio de Banach. Conocimientos básicos de Topología: continuidad, compacidad, axiomas de separación (espacios de Hausdorff), Lemma de Urysohn, particiones de la unidad.

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AUTOMATIC CONTINUITY OF SEPARATING LINEAR ISOMORPHISMS

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ABSTRACT. A linear map $A : C(T) \longrightarrow C(S)$ is called separating if $f \cdot g \equiv 0$ implies $Af \cdot Ag \equiv 0$. We describe the general form of such maps and prove that any separating isomorphism is continuous.

Let T, S be compact Hausdorff spaces and let A be a linear map from the Banach space $C(T)$ into $C(S)$. The map A is said to be *separating* or *disjointness preserving* if $f \cdot g \equiv 0$ implies $Af \cdot Ag \equiv 0$ for all f, g in $C(T)$. For f in $C(T)$ or $C(S)$ we define the cozero set of f by $\mathrm{coz}(f) = \{t : f(t) \neq 0\}$. Hence A is separating if and only if it maps functions with disjoint cozero sets into functions with disjoint cozero sets.

The concept of separating maps in this context was introduced by E. Beckenstein and L. Narici [5–7]. However, disjointness preserving maps between general vector lattices and similar automatic-continuity problems were considered earlier by other authors; see e.g., [1, 2, 8] and [3, 4]. In [7] the authors prove that if A is separating and satisfies a number of additional conditions then it is automatically continuous.

In this note we describe the general form of a separating linear map $A : C(T) \longrightarrow C(S)$. Roughly speaking we can always divide S into three subsets. On the first part A is just the zero map, on the second part A is given by a composition of a continuous map from a subset of S into T and a multiplication by a continuous scalar function. The third part of S is finite, possibly empty, and A is discontinuous at every point of this part. As a consequence we prove that any separating isomorphism is automatically continuous but we also show that there is always a discontinuous separating linear map A from $C(T)$ into $C(S)$, provided T is infinite.

Our results hold both in the real and in the complex case.

THEOREM. *Let A be a linear separating map from $C(T)$ into $C(S)$. Then S is a sum of three disjoint sets S_1, S_2, S_3 where S_2 is open and S_3 is closed, there is a continuous map $\varphi : S_1 \cup S_2 \longrightarrow T$ and a continuous, bounded, non-vanishing scalar-valued function χ on S_1 such that for any $f \in C(T)$*

$$(*) \qquad \begin{aligned} A(f)(s) &= \chi(s) f \circ \varphi(s) \quad && \forall s \in S_1 \\ A(f)(s) &= 0 && \forall s \in S_3. \end{aligned}$$

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Furthermore the set $F = \varphi(S_2)$ *is finite, all functionals of the form*

$$C(T) \ni f \longmapsto A(f)(s) \quad \text{for } s \in S_2$$

are discontinuous and

$$A(f)|_{S_2} \equiv 0 \quad \text{if } \operatorname{supp} f \cap F = \emptyset.$$

PROOF. For any $s \in S$ we denote by δ_s the functional "*evaluation at the point s*". We define $S_3 = \{s \in S : \delta_s \circ A = 0\}$, $S_2 = \{s \in S : \delta_s \circ A \text{ is discontinuous}\}$ and $S_1 = S \setminus (S_2 \cup S_3)$. For any $s \in S$ we define $\operatorname{supp}(\delta_s \circ A)$ to be the set of all $t \in T$ such that for any open neighborhood U of t there is an f in $C(T)$ with $A(f)(s) \neq 0$ and $\operatorname{coz}(f) \subseteq U$. We contend that $\operatorname{supp}(\delta_s \circ A)$ contains at most one point. Assuming the contrary we get two open, disjoint sets U_1 and U_2, both having non-empty intersection with $\operatorname{supp}(\delta_s \circ A)$ and then $f_1, f_2 \in C(T)$ with $\operatorname{coz}(f_j) \subseteq U_j$, $A(f_j)(s) \neq 0$, $j = 1, 2$ which contradicts the assumption that A is separating. Assume now $\operatorname{supp}(\delta_s \circ A) = \emptyset$. Then there is an open finite cover of T, $T = U_1 \cup U_2 \cup \ldots \cup U_n$ such that $Af(s) = 0$ if $\operatorname{coz}(f) \subset U_j$, for some $j = 1, \ldots, n$. Let $\mathbf{1} = \sum_{j=1}^n f_j$ be a continuous decomposition of the identity subordinate to $\{U_j\}_{j=1}^n$. For any $f \in C(T)$ we have $Af(s) = A(\sum_{j=1}^n f_j f)(s) = \sum_{j=1}^n A(f_j f)(s) = 0$, and this means $\delta_s \circ A = 0$, so $s \in S_3$. Hence we can define a function $\varphi : S_1 \cup S_2 \longrightarrow T$ by $\{\varphi(s)\} = \operatorname{supp}(\delta_s \circ A)$.

Note that by exactly very similar arguments as above, we also get $Af(s) = 0$ for any $f \in C(T)$ such that $\varphi(s) \notin \overline{\operatorname{coz}(f)} =: \operatorname{supp} f$.

LEMMA 1. *φ is continuous.*

PROOF OF THE LEMMA. Assuming the contrary, by the compactness of T, there is a net $(s_\alpha)_{\alpha \in \Gamma}$ in $S_1 \cup S_2$ convergent to $s_0 \in S_1 \cup S_2$ such that $\varphi(s_\alpha) = t_\alpha$ converges to $t_1 \neq t_0 = \varphi(s_0)$. Let U_0, U_1 be open, disjoint neighborhoods respectively of t_0 and t_1, and let $f_0 \in C(T)$ be such that $\operatorname{coz}(f_0) \subset U_0$ and $Af_0(s_0) \neq 0$. Fix an $\alpha \in \Gamma$ such that $Af_0(s_\alpha) \neq 0$ and $t_\alpha \in U_1$. Let $f_1 \in C(T)$ be such that $\operatorname{coz}(f_1) \subset U_1$ and $Af_1(s_\alpha) \neq 0$. We get $f_0 \cdot f_1 \equiv 0$ but $Af_0 \cdot Af_1(s_\alpha) \not\equiv 0$, which contradicts the assumption that A is separating.

The definition of φ and Lemma 1 are taken from [7]; we present the above proof here for the sake of completeness.

LEMMA 2. *Let $(s_n)_{n=1}^\infty$ be a sequence in $S_1 \cup S_2$ such that $t_n = \varphi(s_n)$, $n \in \mathbf{N}$ are distinct points of T. Then*

$$\limsup \|\delta_{s_n} \circ A\| < \infty.$$

Note that the above says, in particular, that $\|\delta_{s_n} \circ A\| < \infty$ for all, but finitely many $n \in \mathbf{N}$.

PROOF OF THE LEMMA. Assume the contrary. Taking an appropriate subsequence, we can assume without loss of generality that

$$\|\delta_{s_n} \circ A\| > n^2, \quad \forall n \in \mathbf{N}, \tag{1}$$

and that there is a sequence (U_n) of pairwise disjoint open subsets of T with $t_n \in U_n$. By the definition of φ and (1), there is a sequence (f_n) in $C(T)$ such that

$$\operatorname{supp} f_n \subset U_n, \|f_n\| \leqq 1/n, \quad \text{and } |Af_n(s_n)| \geqq n.$$

Put

$$f = \sum_{n=1}^{\infty} f_n.$$

By the comment preceding Lemma 1 we have

$$|Af(s_{n_0})| = |A(f_{n_0})(s_{n_0}) + A\left(\sum_{n \neq n_0} f_n\right)(s_{n_0})| = |A(f_{n_0})(s_{n_0})| \geqq n_0.$$

Hence Af is unbounded, which is not possible. This proves the lemma.

Put

$$F = \{t \in T : \sup\{\|\delta_s \circ A\| : s \in \varphi^{-1}(t)\} = \infty\}.$$

By Lemma 2, F is a finite set. We want to show that $F = \varphi(S_2)$. The inclusion $\varphi(S_2) \subseteq F$ is obvious by the definition of S_2; to show the converse one fix a $t \in F$ and define

$$\Phi : C(T) \longrightarrow C(\varphi^{-1}(t)) \quad \text{by } \Phi(f) = Af|_{\varphi^{-1}(t)}.$$

Since $t \in F$, the map Φ is discontinuous, and by the closed graph theorem there is a sequence $(f_n)_{n=1}^{\infty}$ in $C(T)$ convergent to 0 and such that $(\Phi(f_n))_{n=1}^{\infty}$ is convergent to a non-zero function $g_0 \in C(\varphi^{-1}(t))$. Let $s \in \varphi^{-1}(t)$ be such that $g_0(s) \neq 0$. We have $f_n \xrightarrow[n=\infty]{} 0$ and $\delta_s \circ A(f_n) \longrightarrow g_0(s) \neq 0$ so $s \in S_2$ and hence $F \subseteq \varphi(S_2)$.

Fix now an $s \in S_1$ and put

$$J_s = \{f \in C(T) : \varphi(s) \notin \operatorname{supp} f\}$$
$$K_s = \{f \in C(T) : f(\varphi(s)) = 0\}.$$

Fix $g \in K_s$ and $\epsilon > 0$. Put $T_1 = \{t \in T : |g(t)| \geqq \epsilon\}$, $T_2 = \{t \in T : |g(t)| \leqq (1/2)\epsilon\}$ and let $g' \in C(T)$ be such that $\|g'\| = 1$, $g'|_{T_1} \equiv \mathbf{1}$, $g'|_{T_2} \equiv 0$. We have $g \cdot g' \in J_s$ and $\|g \cdot g' - g\| \leqq \epsilon$, so J_s is a dense subspace of K_s. Moreover, since $s \in S_1$, $\delta_s \circ A$ is a non-zero continuous functional and by the remark before Lemma 1 $J_s \subseteq \ker(\delta_s \circ A)$. Hence $K_s \subseteq \ker(\delta_s \circ A)$ and since the codimensions of these spaces are both equal to one we have $\ker(\delta_s \circ A) = K_s$ and so $\delta_s \circ A$ is of the form

$$\delta_s \circ A(f) = \chi(s) f(\varphi(s)), \quad \forall f \in C(T),$$

for some scalar $\chi(s) \neq 0$. Let $f \in C(T)$ be such that $f(\varphi(s)) \neq 0$. In some neighborhood of s, namely on $\{s \in S_1 : f(\varphi(s)) \neq 0\}$ we have $\chi = A(f)/f \circ \varphi$. Since s is an arbitrary point of S_1, by Lemma 1, χ is locally a well-defined quotient of two continuous functions and so is continuous itself on S_1. It is also a bounded function, since otherwise $A(\mathbf{1})$ would be unbounded.

It remains to prove that S_2 is open. For any $f \in C(T)$ we have

$$\sup\{|Af(s)| : s \in \overline{S_1 \cup S_3}\} = \sup\{|Af(s)| : s \in S_1 \cup S_3\} \leqq \|\chi\| \, \|f\|.$$

Hence $S_1 \cup S_3 = \{s \in S : \delta_s \circ A \text{ is continuous}\}$ is closed, and we are done.

From the theorem and the definition of φ, we can immediately deduce the following observations:

(2) A is surjective $\Rightarrow S_3 = \emptyset$ and $\varphi|_{S_1}$ is injective.
(3) $S_3 = \emptyset \Rightarrow S_1$ is a compact subset of S.
(4) F consists of non-isolated points only.
(5) A is injective $\Leftrightarrow \overline{\varphi(S_1)} = \overline{\varphi(S_1 \cup S_2)} = T$.

Statements (2) and (3) are obvious. To prove (4), assume $\varphi(s_0) = t_0$ is an isolated point of T. By the definition of φ, $A(f)(s_0) = 0$ if $f(t_0) = 0$, hence $\delta_{s_0} \circ A = \alpha\delta_{t_0}$ for some scalar α, so $\varphi(s_0) \notin F$. Implication "$\Leftarrow$" of (5) is obvious; to get "$\Rightarrow$" assume $\overline{\varphi(S_1)} \subsetneq T$. By (4) and since F is finite, we get $\overline{\varphi(S_1)} \cup F \subsetneq T$, so there is an $f \in C(T)$ such that $f \neq 0$ and $\operatorname{supp} f \cap (\overline{\varphi(S_1)} \cup F) = \emptyset$. By Theorem, $Af = 0$ and A is not injective.

Corollary. *Assume A is a linear, separating isomorphism from $C(T)$ onto $C(S)$. Then A is continuous and S and T are homeomorphic.*

Proof. By (2), (3), and (5), since φ is continuous we get $\varphi(S_1) = T$. For any $f \in C(T)$ we have

$$Af|_{S_1} \equiv 0 \Rightarrow f \equiv 0 \Rightarrow Af|_{S_2} \equiv 0.$$

Hence, since A is surjective, we get $S_2 = \emptyset$, and by (2) φ is a homeomorphism from S onto T.

Example. Let T be an infinite compact set, S a compact set, and let E be a linear subspace of $C(S)$ with $\dim E \leqq \mathfrak{c} :=$ *continuum*. We show that there is a discontinuous, linear separating map A from $C(T)$ onto E. Observe that the cardinality of any separable metric space is at most $\mathfrak{c}$, so E may be any separable linear subspace of $C(S)$. There are also many non-separable Banach spaces E with $\dim E \leqq \mathfrak{c}$, e.g., $E = l^\infty$. Hence, in particular we have an example of a discontinuous, linear, separating map from $c =$ *Banach space of all convergent sequences* onto l^∞.

Let $(U_n)_{n=1}^\infty$ be a sequence of pairwise disjoint, non-empty, open subsets of T, and let $t_n \in U_n$, for $n \in \mathbf{N}$. Fix an $x_0 \in \beta\mathbf{N} - \mathbf{N}$, where $\beta\mathbf{N}$ is the Stone-Čech compactification of the set of positive integers. Any sequence $(a_n)_{n=1}^\infty$ of non-negative real numbers can be extended to a continuous function from $\beta\mathbf{N}$ into $\mathbf{R}^+ \cup \{+\infty\}$, which we denote by $[(a_n)_{n=1}^\infty]$. We define two vector spaces

$$V = \{(f(t_n))_{n=1}^\infty \in l^\infty : f \in C(T)\}$$

$$V_0 = \{(a_n)_{n=1}^\infty \in V : x_0 \notin \operatorname{supp}[(a_n)_{n=1}^\infty]\}.$$

LEMMA. $\dim(V/V_0) = \mathfrak{c}$.

The equation $\dim(V/V_0) \leqq \mathfrak{c}$ is obvious since $\dim(V/V_0) \leqq \dim V \leqq \dim l^\infty = \mathfrak{c}$. The converse equation can be proven in several ways. Probably the shortest one is to observe that V/V_0 can be seen as a subset of the non-standard model $^*\mathbf{C}(^*\mathbf{R})$ of the set of all complex (real) numbers, that V/V_0 contains the monad M_0 of 0, and that M_0 is a $\mathfrak{c}$-dimensional linear space over $\mathbf{C}(\mathbf{R})$ [9, 10]. To get a more elementary and self-contained proof let $f_n \in C(T)$ be such that $\operatorname{supp} f_n \subset U_n$ and $\|f_n\| = 1 = f(t_n)$. Let $\mathcal{A}$ be the set of all infinite subsets of $\mathbf{N}$. Clearly $\operatorname{card}(\mathcal{A}) = \mathfrak{c}$. Let $(\alpha_n)_{n=1}^\infty$ be any decreasing sequence of positive numbers tending to zero and such that

$$\lim_n \frac{\alpha_{n+1}}{\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_n} = 0.$$

For any $A \in \mathcal{A}$ we define a sequence $(a_n^A)_{n=1}^\infty$ by

$$a_n^A = \prod_{k \in A(n)} \alpha_k \quad \text{for } n \in \mathbf{N},$$

where $A(n) = \{k \in \mathbf{N} : n - k \in A\}$; if $A(n) = \emptyset$ then we understand that $a_n^A = 1$. Let A, B be distinct subsets of $\mathbf{N}$ and let k_0 be the smallest integer which is contained in exactly one of these sets, say $k_0 \in A$. Then

$$(6) \qquad 0 < \frac{a_n^A}{a_n^B} \leq \frac{\alpha_{n-k_0}}{\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_{n-k_0-1}} \xrightarrow[n\to\infty]{} 0.$$

For any $A \in \mathcal{A}$ we now define $f_A \in C(T)$ by

$$f_A = \sum_{n=1}^\infty a_n^A f_n.$$

Let $A_1, \ldots, A_2$ be distinct subsets of $\mathbf{N}$. By (6) sequences $(f_{A_j}(t_n))_{n=1}^\infty$ tend to zero with quite "different speed", this means in particular that there is one set among $A_1, \ldots, A_k$, say A_1, such that

$$\lim_n \frac{f_{A_j}(t_n)}{f_{A_1}(t_n)} = +\infty \quad \text{for } j = 2, \ldots, k.$$

Hence a non-trivial linear combination of $(f_{A_1}(t_n))_{n=1}^\infty, \ldots, (f_{A_k}(t_n))_{n=1}^\infty$ is distinct from zero for all, except possibly finitely many, indices; hence the set

$$\{(f_A(t_n))_{n=1}^\infty + V_0 \in (V/V_0) : A \in \mathcal{A}\}$$

is linearly independent, so $\dim(V/V_0) = \mathfrak{c}$.

Let Φ be any linear map from V onto E such that $V_0 \subseteq \ker\Phi$ and

$$\Phi\left(\left(\frac{1}{n}\right)_{n=1}^\infty\right) \neq 0.$$

We define $A : C(T) \longrightarrow C(S)$ by

$$Af(s) = \Phi((f(t_n))_{n=1}^{\infty}) \quad \text{for } s \in S.$$

The map A is evidently discontinuous. We prove that it is separating.

Let $h_1, h_2 \in C(T)$ be such that

$$\{t \in T : h_1(t) \neq 0\} \cap \{t \in T : h_2(t) \neq 0\} = \emptyset.$$

Put

$$N_i = \{n \in \mathbf{N} : h_i(t_n) \neq 0\}, \quad \text{for } i = 1, 2.$$

We have $N_1 \cap N_2 = \emptyset$ and hence the closures of these sets in $\beta\mathbf{N}$ are also disjoint. This means that at most one of the sets $\overline{N_1}$ or $\overline{N_2}$ contains x_0. Assume that $x_0 \notin N_1$. Then $(h_1(t_n))_{n=1}^{\infty} \in V_0$ and $Ah_1 \equiv 0$.

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Weighted Composition Operators of $C_0(X)$'s*

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In this paper, we prove that into isometries and disjointness preserving linear maps from $C_0(X)$ into $C_0(Y)$ are essentially weighted composition operators $Tf = h \cdot f \circ \varphi$ for some continuous map φ and some continuous scalar-valued function h.

1. INTRODUCTION

Let X and Y be locally compact Hausdorff spaces. Let $C_0(X)$ (resp. $C_0(Y)$) be the Banach space of continuous scalar-valued (i.e., real- or complex-valued) functions defined on X (resp. Y) vanishing at infinity and equipped with the supremum norm. The classical Banach–Stone theorem gives a description of surjective isometries from $C_0(X)$ onto $C_0(Y)$. They are all *weighted composition operators* $Tf = h \cdot f \circ \varphi$ (i.e., $Tf(y) = h(y)f(\varphi(y)), \forall y \in Y$) for some homeomorphism φ from Y onto X and some continuous scalar-valued function h on Y with $|h(y)| \equiv 1$, $\forall y \in Y$. Different generalizations (see, e.g., [1, 2, 4, 5, 7]) of the Banach–Stone Theorem have been studied for many years. Some of them discuss the structure of *into* isometries and disjointness preserving linear maps (see, e.g., [3, 6]). A linear map from $C_0(X)$ into $C_0(Y)$ is said to be *disjointness preserving* if $f \cdot g = 0$ in $C_0(X)$ implies $Tf \cdot Tg = 0$ in $C_0(Y)$. In this paper, we shall discuss the structure of weighted composition operators from $C_0(X)$ into $C_0(Y)$. We prove that every into isometry and every disjoint-

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ness preserving linear map from $C_0(X)$ into $C_0(Y)$ is essentially a weighted composition operator.

THEOREM 1. *Let X and Y be locally compact Hausdorff spaces and T a linear isometry from $C_0(X)$ into $C_0(Y)$. Then there exist a locally compact subset Y_1 (i.e., Y_1 is locally compact in the subspace topology) and a weighted composition operator T_1 from $C_0(X)$ into $C_0(Y_1)$ such that for all f in $C_0(X)$,*

$$Tf_{|Y_1} = T_1 f = h \cdot f \circ \varphi,$$

for some quotient map φ from Y_1 onto X and some continuous scalar-valued function h defined on Y_1 with $|h(y)| \equiv 1$, $\forall y \in Y_1$.

THEOREM 2. *Let X and Y be locally compact Hausdorff spaces and T a bounded disjointness preserving linear map from $C_0(X)$ into $C_0(Y)$. Then there exist an open subset Y_1 of Y and a weighted composition operator T_1 from $C_0(X)$ into $C_0(Y_1)$ such that for all f in $C_0(X)$, Tf vanishes outside Y_1 and*

$$Tf_{|Y_1} = T_1 f = h \cdot f \circ \varphi,$$

for some continuous map φ from Y_1 into X and some continuous scalar-valued function h defined on Y_1 with $h(y) \neq 0$, $\forall y \in Y_1$.

Since weighted composition operators from $C_0(X)$ into $C_0(Y)$ are disjointness preserving, Theorem 2 gives a complete description of all such maps. When X and Y are both compact, Theorems 1 and 2 reduce to the results of W. Holsztynski [3] and K. Jarosz [6], respectively. It is plausible to think that Theorems 1 and 2 could be easily obtained from their compact space versions by simply extending an into isometry (or a bounded disjointness preserving linear map) $T: C_0(X) \to C_0(Y)$ to a bounded linear map $T_\infty : C(X_\infty) \to C(Y_\infty)$ of the same type, where $X_\infty = X \cup \{\infty\}$ and $Y_\infty = Y \cup \{\infty\}$ are the one-point compactifications of the locally compact Hausdorff spaces X and Y, respectively. However, the example given in Section 4 will show that this idea is sometimes fruitless because T can have *no* such extensions at all. We thus have to modify, and in some cases give new arguments to, the proofs of W. Holsztynski [3] and K. Jarosz [6] to fit into our more general settings in this paper.

Recall that for f in $C_0(X)$, the *cozero* of f is $\text{coz}(f) = \{x \in X : f(x) \neq 0\}$ and the *support* $\text{supp}(f)$ of f is the closure of $\text{coz}(f)$ in X_∞. A linear map $T: C_0(X) \to C_0(Y)$ is disjointness preserving if T maps functions with disjoint cozeros to functions with disjoint cozeros. For x in X, δ_x denotes the point evaluation at x, that is, δ_x is the linear functional on $C_0(X)$ defined by $\delta_x(f) = f(x)$. For y in Y, let $\text{supp}(\delta_y \circ T)$ be the set of all x in

X_∞ such that for any open neighborhood U of x in X_∞ there is an f in $C_0(X)$ with $Tf(y) \neq 0$ and $\operatorname{coz}(f) \subset U$. The kernel of a function f is denoted by ker f.

2. ISOMETRIES FROM $C_0(X)$ INTO $C_0(Y)$

DEFINITION. Let X and Y be locally compact Hausdorff spaces. A map φ from Y into X is said to be *proper* if preimages of compact subsets of X under φ are compact in Y.

It is obvious that a continuous map φ is proper if and only if $\lim_{y\to\infty}\varphi(y) = \infty$. As a consequence, a proper continuous map φ from a locally compact Hausdorff space Y onto a locally compact Hausdorff space X is a quotient map, i.e., $\varphi^{-1}(O)$ is open in Y if and only if O is open in X. A quotient map from a locally compact space onto another is, however, not necessarily proper. For example, the quotient map φ from $(-\infty, +\infty)$ onto $[0, +\infty)$ defined by

$$\varphi(y) = \begin{cases} y, & y > 0, \\ 0, & y \leq 0 \end{cases}$$

is not proper.

LEMMA 3. *Let X and Y be locally compact Hausdorff spaces, φ a map from Y into X, and h a continuous scalar-valued function defined on Y with bounds $M, m > 0$ such that $m \leq |h(y)| \leq M$, $\forall y \in Y$. Then the weighted composition $Tf = h \cdot f \circ \varphi$ defines a (necessarily bounded) linear map from $C_0(X)$ into $C_0(Y)$ if and only if φ is continuous and proper.*

Proof. For the sufficiency, we need to verify that $h \cdot f \circ \varphi$ vanishes at ∞ for all f in $C_0(X)$. For any $\epsilon > 0$, $|f(x)| < \epsilon/M$ outside some compact subset K of X. Since φ is proper, $\varphi^{-1}(K)$ is compact in Y. Now the fact that $|h(y) \cdot f(\varphi(y))| \leq M|f(\varphi(y))| < \epsilon$ outside $\varphi^{-1}(K)$ indicates that $h \cdot f \circ \varphi \in C_0(Y)$. The boundedness of T is trivial in this case.

For the necessity, we first check the continuity of φ. Suppose $y_\lambda \to y$ in Y. We want to show that $x_\lambda = \varphi(y_\lambda) \to \varphi(y)$ in X. Suppose not, by passing to a subnet if necessary, we can assume that x_λ either converges to some $x \neq \varphi(y)$ in X or ∞. If $x_\lambda \to x$ in X then for all f in $C_0(X)$,

$$\begin{aligned} h(y)f(x) &= \lim h(y_\lambda)f(x_\lambda) = \lim h(y_\lambda)f(\varphi(y_\lambda)) \\ &= \lim Tf(y_\lambda) = Tf(y) = h(y)f(\varphi(y)). \end{aligned}$$

As $h(y) \neq 0$, $f(x) = f(\varphi(y))$, $\forall f \in C_0(X)$. Consequently, we obtain a contradiction $x = \varphi(y)$. If $x_\lambda \to \infty$ then a similar argument gives $f(\varphi(y))$

$= 0$ for all f in $C_0(X)$. Hence $\varphi(y) = \infty$, a contradiction again. Therefore, φ is continuous from Y into X. Finally, let K be a compact subset of X and we are going to see that $\varphi^{-1}(K)$ is compact in Y, or equivalently, closed in $Y_\infty = Y \cup \{\infty\}$, the one-point compactification of Y. To see this, suppose $y_\lambda \to y$ in Y_∞ and $x_\lambda = \varphi(y_\lambda) \in K$. We want $y \in \varphi^{-1}(K)$, i.e., $y \neq \infty$ and $\varphi(y) \in K$. Without loss of generality, we can assume that $x_\lambda \to x$ for some x in K. Now,

$$\lim|Tf(y_\lambda)| = \lim|h(y_\lambda)f(\varphi(y_\lambda))| \geq m \lim|f(x_\lambda)| = m|f(x)|$$

for all f in $C_0(X)$. This implies that $y \neq \infty$ and then a similar argument gives $\varphi(y) = x \in K$. ∎

The assumption on the bounds of f in Lemma 3 is significant. For example, let $X = Y = \mathbb{R} = (-\infty, +\infty)$ and define

$$h(y) = \begin{cases} e^y, & y < 0, \\ 1, & y \geq 0, \end{cases} \quad \text{and} \quad \varphi(y) = \begin{cases} \sin y, & y < 0, \\ y, & y \geq 0. \end{cases}$$

Then the weighted composition operator $Tf = h \cdot f \circ \varphi$ from $C_0(\mathbb{R})$ into $C_0(\mathbb{R})$ is well-defined. It is not difficult to see that $\varphi^{-1}([-\frac{1}{2}, \frac{1}{2}])$ is not compact in $\mathbb{R}$. On the other hand, if we redefine $h(y) = e^y$ and $\varphi(y) = y$ for all y in $\mathbb{R}$ then the weighted composition operator T is not well-defined from $C_0(\mathbb{R})$ into $C_0(\mathbb{R})$, even though φ is proper and continuous in this case.

Recall that a bounded linear map T from a Banach space E into a Banach space F is called an *injection* if there is an $m > 0$ such that $\|Tx\| \geq m\|x\|$, $\forall x \in E$. It follows from the open mapping theorem that T is an injection if and only if T is one-to-one and has closed range.

PROPOSITION 4. *Let X and Y be locally compact Hausdorff spaces, φ a map from Y into X, and h a continuous scalar-valued function defined on Y. The weighted composition operator $Tf = h \cdot f \circ \varphi$ from $C_0(X)$ into $C_0(Y)$ is an injection if and only if φ is continuous, proper, and onto and h has bounds $M, m > 0$ such that $m \leq |h(y)| \leq M$, $\forall y \in Y$. In this case, φ is a quotient map and thus X is a quotient space of Y.*

Proof. The sufficiency follows easily from Lemma 3 and the observation that $\|Tf\| = \|h \cdot f \circ \varphi\| \geq m\|f\|$, $\forall f \in C_0(X)$. For the necessity, we first note that there are constants $M, m > 0$ such that $m\|f\| \leq \|Tf\| \leq M\|f\|$ for all f in $C_0(X)$. It is then obvious that $m \leq |h(y)| \leq M$, $\forall y \in Y$. By Lemma 3, φ is continuous and proper. Finally, we check that φ is onto. It is not difficult to see that φ has dense range. In fact, if $\varphi(Y)$ were not dense in X, then there would be an x in X and a neighborhood U of x in X such that $U \cap \varphi(Y) = \varnothing$. Choose an f in $C_0(X)$ such that $f(x) = 1$

and f vanishes outside U. Then $Tf(y) = h(y)f(\varphi(y)) = 0$ for all y in Y, i.e., $Tf = 0$. Since T is an injection, we get a contradiction that $f = 0$. We now show that $\varphi(Y) = X$. Let $x \in X$ and K a compact neighborhood of x in X. By the density of $\varphi(Y)$ in X, there is a net $\{y_\lambda\}$ in Y such that $x_\lambda = \varphi(y_\lambda) \to x$ in X. Without loss of generality, we can assume that x_λ belongs to K for all λ. Since $\varphi^{-1}(K)$ is compact in Y, $\varphi(\varphi^{-1}(K))$ is a compact subset of X containing the net $\{x_\lambda\}$. Consequently, $x = \lim x_\lambda$ belongs to $\varphi(\varphi^{-1}(K)) \subset \varphi(Y)$.

Proof of Theorem 1. We adopt some notations from W. Holsztynski [3] and K. Jarosz [6]. Let $X_\infty = X \cup \{\infty\}$ and $Y_\infty = Y \cup \{\infty\}$ be the one-point compactifications of X and Y, respectively. For each x in X and y in Y, put

$$S_x = \{f \in C_0(X) : |f(x)| = \|f\| = 1\},$$
$$R_y = \{g \in C_0(Y) : |g(y)| = \|g\| = 1\},$$

and

$$Q_x = \{y \in Y : T(S_x) \subset R_y\}.$$

We first claim that $\{Q_x\}_{x \in X}$ is a disjoint family of non-empty subsets of Y. In fact, for $f_1, f_2, \ldots, f_n$ in S_x, let $h = \sum_{i=1}^n \overline{f_i(x)} f_i$. Then $\|h\| = n$ and thus $\|Th\| = n$. Hence there is a y in Y such that $|\sum_{i=1}^n \overline{f_i(x)} Tf_i(y)| = |Th(y)| = n$. This implies $|Tf_i(y)| = 1$ for all $i = 1, 2, \ldots, n$. In other words, $y \in \bigcap_{i=1}^n (Tf_i)^{-1}(\Gamma)$, where $\Gamma = \{z : |z| = 1\}$. We have just proved that the family $\{(Tf)^{-1}(\Gamma) : f \in S_x\}$ of closed subsets of the compact space Y_∞ has finite intersection property. It is plain that $\infty \notin (Tf)^{-1}(\Gamma)$ for all f in S_x. Hence $Q_x = \bigcap_{f \in S_x} (Tf)^{-1}(\Gamma)$ is non-empty for all x in X. Moreover, $Q_{x_1} \cap Q_{x_2} = \varnothing$ if $x_1 \neq x_2$ in X. In fact, f_1 in S_{x_1} and f_2 in S_{x_2} exist such that $\operatorname{coz}(f_1) \cap \operatorname{coz}(f_2) = \varnothing$. If there is a y in $Q_{x_1} \cap Q_{x_2}$ then it follows from $Tf_1 \in R_y$ and $Tf_2 \in R_y$ that $1 = \|f_1 + f_2\| = \|T(f_1 + f_2)\| = |T(f_1 + f_2)(y)| = 2$, a contradiction.

Let $Y_1 = \bigcup_{x \in X} Q_x$. It is not difficult to see that $\operatorname{supp}(\delta_y \circ T) = \{x\}$ whenever $y \in Q_x$. So we can define a surjective map $\varphi : Y_1 \to X$ by

$$\{\varphi(y)\} = \operatorname{supp}(\delta_y \circ T).$$

Note that for all f in $C_0(X)$ and for all y in Y_1,

$$\varphi(y) \notin \operatorname{supp}(f) \Rightarrow T(f)(y) = 0. \tag{1}$$

In fact, if $Tf(y) \neq 0$, without loss of generality, we can assume $Tf(y) = r > 0$ and $\|f\| = 1$. Since $\varphi(y) \notin \operatorname{supp}(f)$, there is a g in $C_0(X)$ such that

$\operatorname{coz}(f) \cap \operatorname{coz}(g) = \varnothing$ and $Tg(y) = \|g\| = 1$. Hence $1 + r = T(f+g)(y) > \|f + g\| = 1$, a contradiction.

Now, we want to show that φ is continuous. Suppose φ were not continuous at some y in Y_1, without loss of generality, let $\{y_\lambda\}$ be a net converging to y in Y_1 such that $\varphi(y_\lambda) \to x \neq \varphi(y)$ in X_∞. Then there exist disjoint neighborhoods U_1 and U_2 of x and $\varphi(y)$ in X_∞, respectively, and a λ_0 such that $\varphi(y_\lambda) \in U_1$, $\forall \lambda \geq \lambda_0$. Let $f \in C_0(X)$ such that $\operatorname{coz}(f) \subseteq U_2$ and $T(f)(y) = \|f\| = 1$. As $\operatorname{supp}(f) \cap U_1 = \varnothing$, we have $\varphi(y_\lambda) \notin \operatorname{supp}(f)$, $\forall \lambda \geq \lambda_0$. By (1), $T(f)(y_\lambda) = 0$, $\forall \lambda \geq \lambda_0$. This implies $T(f)$ is not continuous at y, a contradiction.

For each y in Y_1, put

$$J_y = \{f \in C_0(X) : \varphi(y) \notin \operatorname{supp}(f)\},$$

and

$$K_y = \{f \in C_0(X) : f(\varphi(y)) = 0\}.$$

For f in K_y and $\varepsilon > 0$, let $X_1 = \{x \in X : |f(x)| \geq \varepsilon\}$ and $X_2 = \{x \in X : |f(x)| \leq \varepsilon/2\}$. Let g be a continuous function defined on X such that $0 \leq g(x) \leq 1$, $\forall x \in X$, $g(x) = 1$, $\forall x \in X_1$, and $g(x) = 0$, $\forall x \in X_2$. Let $f_\varepsilon = g \cdot f$. Then $f_\varepsilon \in J_y$ and $\|f_\varepsilon - f\| \leq 2\varepsilon$. One thus can show that J_y is a dense subset of K_y. By (1), $J_y \subset \ker(\delta_y \circ T)$, and hence $\ker(\delta_{\varphi(y)}) = K_y \subset \ker(\delta_y \circ T)$. Consequently, there exists a scalar $h(y)$ such that $\delta_y \circ T = h(y) \cdot \delta_{\varphi(y)}$, i.e.,

$$T(f)(y) = h(y) \cdot f(\varphi(y)), \qquad \forall f \in C_0(X).$$

It follows from the definition of Y_1 that h is continuous on Y_1 and $|h(y)| = 1$, $\forall y \in Y_1$.

It is the time to see that Y_1 is locally compact. For each y_1 in Y_1 and a neighborhood U_1 of y_1 in Y_1, we want to find a compact neighborhood K_1 of y_1 in Y_1 such that $y_1 \in K_1 \subset U_1$. Let $x_1 = \varphi(y_1)$ in X. Then

$$|Tf(y_1)| = |f(x_1)|, \qquad \forall f \in C_0(X).$$

Fix f_1 in S_{x_1}. Then $V_1 = \varphi^{-1}(\{x \in X : |f_1(x)| > \frac{1}{2}\}) \cap U_1$ is an open neighborhood of y_1 in Y_1 and contained in U_1. Since $V_1 = W \cap Y_1$ for some neighborhood W of y_1 in Y, there exists a compact neighborhood K of y_1 in Y such that $y_1 \in K \subset W$. We are going to verify that $K_1 = K \cap Y_1$ is a compact neighborhood of y_1 in Y_1. Let $\{y_\lambda\}$ be a net in $K_1 \subset V_1$. By passing to a subnet, we can assume that y_λ converges to y in K and we want to show $y \in Y_1$. Let $x_\lambda = \varphi(y_\lambda)$ in X. Since X_∞ is compact, by passing to a subnet again, we can assume that x_λ converges to x in X or $x_\lambda \to \infty$. If $x_\lambda \to x$ in X, $|Tf(y)| = \lim|Tf(y_\lambda)| = \lim|h(y_\lambda)f(\varphi(y_\lambda))| =$

$\lim|f(x_\lambda)| = |f(x)|$, for all f in $C_0(X)$. Hence $y \in Q_x$, and thus $y \in Y_1$. If $x_\lambda \to \infty$, $|Tf_1(y)| = \lim|Tf_1(y_\lambda)| = \lim|h(y_\lambda)f_1(\varphi(y_\lambda))| = \lim|f_1(x_\lambda)| = 0$. However, the fact that $y_\lambda \in V_1$ ensures $|Tf_1(y_\lambda)| = |f_1(x_\lambda)| > 1/2$ for all λ, a contradiction. Hence Y_1 is locally compact.

Let $T_1 : C_0(X) \to C_0(Y_1)$ be defined by $T_1 f = h \cdot f \circ \varphi$. It is clear that T_1 is a linear isometry and $Tf_{|Y_1} = T_1 f$. By Proposition 4, the surjective continuous map φ is proper and thus a quotient map. The proof is complete. ∎

In Theorem 1, Y_1 can be neither open nor closed in Y and φ may not be an open map. See the following examples.

EXAMPLE 5. Let $X = [\,0, +\infty)$ and $Y = [-\infty, +\infty]$. Let T be a linear isometry from $C_0(X)$ into $C_0(Y)$ defined for all f in $C_0(X)$ by

$$Tf(y) = \begin{cases} f(y), & 0 \le y < +\infty, \\ \dfrac{e^y}{2}(f(-y) + f(0)), & -\infty < y \le 0, \\ 0, & y = \pm\infty. \end{cases}$$

Then in the notation of Theorem 1, $Y_1 = [\,0, +\infty)$ is neither closed nor open in Y. In this case, $\varphi(y) = y$ for all y in $[0, +\infty)$, and X and Y_1 are homeomorphic.

EXAMPLE 6. Let $X = \mathbb{R}$ and $Y = \{(x, y) \in \mathbb{R}^2 : y = 0\} \cup \{(x, y) \in \mathbb{R}^2 : 0 \le x, 0 \le y \le 1\}$. Let $\varphi : Y \to X$ be defined by $\varphi(u_1, u_2) = u_1$. Then φ is continuous, onto, and proper, and thus a quotient map. Moreover, $T : C_0(X) \to C_0(Y)$ defined by $Tf = f \circ \varphi$ is a linear isometry. Note that $O = \{(x, y) \in \mathbb{R}^2 : 0 \le x < 1, 0 < y \le 1\}$ is open in Y, but $\varphi(O) = [\,0, 1)$ is not open in X. Hence φ is not an open map.

3. DISJOINTNESS PRESERVING LINEAR MAPS FROM $C_0(X)$ INTO $C_0(Y)$

It is clear that Theorem 2 follows from the following more general result in which discontinuity of the linear disjointness preserving map T is allowed. The payoff of the discontinuity is a finite subset F of X at which the behaviour of T is not under control.

THEOREM 7. *Let X and Y be locally compact Hausdorff spaces and T a disjointness preserving linear map from $C_0(X)$ into $C_0(Y)$. Then Y can be written as a disjoint union $Y = Y_1 \cup Y_2 \cup Y_3$, in which Y_2 is open and Y_3 is* *closed. A continuous map φ from $Y_1 \cup Y_2$ into X_∞ exists such that for every f in $C_0(X)$,*

$$\varphi(y) \notin \operatorname{supp}(f) \Rightarrow T(f)(y) = 0. \tag{2}$$

Moreover, a continuous bounded non-vanishing scalar-valued function h on Y_1 exists such that

$$Tf_{|Y_1} = h \cdot f \circ \varphi,$$

and

$$Tf_{|Y_3} = 0.$$

Furthermore, $F = \varphi(Y_2)$ is a finite set and the functionals $\delta_y \circ T$ are discontinuous on $C_0(X)$ for all y in Y_2.

Proof. We shall follow the plan of K. Jarosz in his compact space version [6]. Set

$$Y_3 = \{y \in Y \mid \delta_y \circ T \equiv 0\},$$

$$Y_2 = \{y \in Y \mid \delta_y \circ T \text{ is discontinuous}\},$$

and

$$Y_1 = Y \setminus (Y_2 \cup Y_3).$$

First, we claim that $\operatorname{supp}(\delta_y \circ T)$ contains exactly one point for every y in $Y_1 \cup Y_2$. Suppose on the contrary that $\operatorname{supp}(\delta_y \circ T)$ contains two distinct points x_1 and x_2 in X_∞. Let U_1 and U_2 be neighborhoods of x_1 and x_2 in X_∞, respectively, such that $U_1 \cap U_2 = \varnothing$. Let f_1 and f_2 in $C_0(X)$ with $\operatorname{coz}(f_1) \subset U_1$ and $\operatorname{coz}(f_2) \subset U_2$ be such that $Tf_1(y) \neq 0$ and $Tf_2(y) \neq 0$. However, $f_1 f_2 = 0$ implies $Tf_1 Tf_2 = 0$, a contradiction. Suppose $\operatorname{supp}(\delta_y \circ T)$ is empty. Then we can write the compact Hausdorff space X_∞ as a finite union of open sets $X_\infty = \bigcup_{i=1}^n U_i$ such that $Tf(y) = 0$ whenever $\operatorname{coz}(f) \subset U_i$ for some $i = 1, 2, \ldots, n$. Let $\mathbf{1} = \sum_{i=1}^n f_i$ be a continuous decomposition of the identity subordinate to $\{U_i\}_{i=1}^n$. Then for all f in $C_0(X)$, $Tf(y) = \sum_{i=1}^n T(ff_i)(y) = 0$. This says $\delta_y \circ T \equiv 0$ and thus $y \in Y_3$.

Next we define a map φ from $Y_1 \cup Y_2$ into X_∞ by

$$\{\varphi(y)\} = \operatorname{supp}(\delta_y \circ T).$$

We now prove (2). Assume $\varphi(y) \notin \operatorname{supp}(f)$. Then there is an open neighborhood U of $\varphi(y)$ disjoint from $\operatorname{coz}(f)$. Let $g \in C_0(X)$ such that

$\mathrm{coz}(g) \subset U$ and $Tg(y) \neq 0$. Since $fg = 0$ and T is disjointness preserving, $Tf(y) = 0$ as asserted.

It then follows from (2) the continuity of φ as one can easily modify an argument of the proof of Theorem 1 for this goal. Similarly, it also follows from (2) the desired representation

$$Tf(y) = h(y)f(\varphi(y)), \qquad \forall f \in C_0(X), \forall y \in Y_1, \tag{3}$$

where h is a continuous non-vanishing scalar-valued function defined on Y_1.

Claim. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in $Y_1 \cup Y_2$ such that the $x_n = \varphi(y_n)$'s are distinct points of X. Then

$$\limsup \|\delta_{y_n} \circ T\| < \infty.$$

In particular, only finitely many $\delta_y \circ T$ can have infinite norms.

Assume the contrary and, by passing to a subsequence if necessary, we have

$$\|\delta_{y_n} \circ T\| > n^4, \qquad n = 1, 2, \ldots.$$

Let $f_n \in C_0(X)$ with $\|f_n\| \leq 1$ such that

$$|Tf_n(y_n)| \geq n^3, \qquad n = 1, 2, \ldots.$$

Let V_n, W_n, and U_n be open subsets of X such that $x_n \in V_n \subseteq \overline{V_n} \subseteq W_n \subseteq \overline{W_n} \subseteq U_n$ and $U_n \cap U_m = \varnothing$ if $n \neq m$, $n, m = 1, 2, \ldots,$ and let $g_n \in C(X_\infty)$ such that $0 \leq g_n \leq 1$, $g_{n|V_n} \equiv 1$ and $g_{n|X_\infty \setminus W_n} \equiv 0$, $n = 1, 2, \ldots.$ Then (2) implies

$$\begin{aligned} Tf_n(y_n) &= T(f_n g_n)(y_n) + T(f_n(1 - g_n))(y_n) \\ &= T(f_n g_n)(y_n), \qquad n = 1, 2, \ldots. \end{aligned}$$

Therefore, we can assume $\operatorname{supp} f_n \subset U_n$. Let $f = \sum_{n=1}^{\infty}(1/n^2)f_n$ in $C_0(X)$. By (2) again, $|Tf(y_n)| = |(1/n^2)Tf_n(y_n)| \geq n$ for $n = 1, 2, \ldots.$ This conflicts with the boundedness of Tf in $C_0(Y)$, and the claim is thus verified.

The assertion that $F = \varphi(Y_2)$ is a finite subset of X is clearly a consequence of the claim while the boundedness of h follows from the claim and (3). It is also plain that $Y_3 = \cap\{\ker Tf : f \in C_0(X)\}$ is closed in Y. Finally, to see that Y_2 is open, we consider for every f in $C_0(X)$,

$$\begin{aligned} \sup\{|Tf(y)| : y \in \overline{Y_1 \cup Y_3}\} &= \sup\{|Tf(y)| : y \in Y_1 \cup Y_3\} \\ &= \sup\{|Tf(y)| : y \in Y_1\} \\ &= \sup\{|h(y)f(\varphi(y))| : y \in Y_1\} \leq M\|f\|, \end{aligned}$$

where $M > 0$ is a bound of h on Y_1. It follows that the linear functional $\delta_y \circ T$ is bounded for all y in $\overline{Y_1 \cup Y_3}$, and thus $Y_2 \cap \overline{Y_1 \cup Y_3} = \varnothing$. Hence, $Y_1 \cup Y_3 = \overline{Y_1 \cup Y_3}$ is closed. In other words, Y_2 is open. ∎

THEOREM 8. *Let X and Y be locally compact Hausdorff spaces and T a bijective disjointness preserving linear map from $C_0(X)$ onto $C_0(Y)$. Then T is a bounded weighted composition operator, and X and Y are homeomorphic.*

Proof. We adopt the notations used in Theorem 7. Since T is surjective, $Y_3 = \varnothing$. We are going to verify that $Y_2 = \varnothing$, too. First, we note that the finite set $F \setminus \{\infty\}$ consists of non-isolated points in X. In fact, if $y \in Y_2$ such that $x = \varphi(y)$ is an isolated point in X then it follows from (2) that for every f in $C_0(X)$, $f(x) = 0$ implies $\varphi(y) = x \notin \operatorname{supp} f$ and thus $Tf(y) = 0$. Hence, $\delta_y \circ T = \lambda \delta_x$ for some scalar λ. Therefore, $\delta_y \circ T$ is continuous, a contradiction to the assumption that $y \in Y_2$. We then claim that $\varphi(Y) = \varphi(Y_1 \cup Y_2)$ is dense in X. In fact, if a nonzero f in $C_0(X)$ exists such that $\operatorname{supp} f \cap \varphi(Y) = \varnothing$ then $Tf = 0$ by (2), conflicting with the injectivity of T. Since

$$X = \overline{\varphi(Y)} = \overline{\varphi(Y_1) \cup \varphi(Y_2)} = \overline{\varphi(Y_1) \cup F} = \overline{\varphi(Y_1)} \quad \text{or} \quad \overline{\varphi(Y_1)} \cup \{\infty\},$$

for every f in $C_0(X)$,

$$Tf_{|Y_1} = 0 \Rightarrow f_{|\varphi(Y_1)} = 0 \Rightarrow f = 0 \Rightarrow Tf_{|Y_2} = 0.$$

Therefore, the open set $Y_2 = \varnothing$ by the surjectivity of T. Theorem 7 then gives

$$Tf = h \cdot (f \circ \varphi), \qquad \forall f \in C_0(X).$$

This representation implies that T^{-1} is also a bijective disjointness preserving linear map from $C_0(Y)$ onto $C_0(X)$. The above discussion provides that

$$T^{-1}g = h_1 \cdot g \circ \varphi_1, \qquad \forall g \in C_0(Y),$$

for some continuous non-vanishing scalar-valued function h_1 on X and continuous function φ_1 from X into Y. It is plain that $\varphi_1 = \varphi^{-1}$ and thus X and Y are homeomorphic. ∎

4. A COUNTER EXAMPLE

The following example shows that not every into isometry or bounded disjointness preserving linear map from $C_0(X)$ into $C_0(Y)$ can be extended to a bounded linear map from $C(X_\infty)$ into $C(Y_\infty)$ of the same type. Here X and Y are locally compact Hausdorff spaces with one-point compactifications X_∞ and Y_∞, respectively.

EXAMPLE 9. Let $X = [0, +\infty)$, $Y = (-\infty, +\infty)$ and the underlying scalar field is the field $\mathbb{R}$ of real numbers. Let

$$h(y) = \begin{cases} 1, & y > 2, \\ y - 1, & 0 \le y \le 2, \\ -1, & y < 0, \end{cases}$$

and

$$\varphi(y) = \begin{cases} y, & y \ge 0, \\ -y, & y < 0. \end{cases}$$

Then the weighted composition operator $Tf = h \cdot f \circ \varphi$ is simultaneously an into isometry and a bounded disjointness preserving linear map from $C_0([0, +\infty))$ into $C_0((-\infty, +\infty))$. However, no bounded linear extension T_∞ from $C([0, \infty])$ into $C((-\infty, +\infty) \cup \{\infty\})$ of T can be an into isometry or a disjointness preserving linear map.

Suppose, on the contrary, T_∞ were an into isometry. Consider f_n in $C_0([0, +\infty))$ defined by

$$f_n(x) = \begin{cases} 1, & 0 \le x \le n, \\ \dfrac{2n - x}{n}, & n < x < 2n, \\ 0, & 2n \le x \le +\infty, \end{cases} \qquad n = 1, 2, \ldots.$$

Note that $\delta_y \circ T_\infty$ can be considered as a bounded Borel measure m_y on $[0, +\infty]$ for all point evaluation δ_y at y in $(-\infty, +\infty) \cup \{\infty\}$ with total variation $\|m_y\| = \|\delta_y \circ T_\infty\| \le 1$. Let $\mathbf{1}$ be the constant function $\mathbf{1}(x) \equiv 1$ in $C([0, +\infty])$. For all y in $(-\infty, +\infty)$,

$$\begin{aligned} T_\infty \mathbf{1}(y) &= \delta_y \circ T_\infty(\mathbf{1}) = \int_{[0, +\infty]} \mathbf{1}\, dm_y \\ &= \lim_{n \to \infty} \int_{[0, +\infty]} f_n \, dm_y + m_y(\{\infty\}) = \lim_{n \to \infty} \delta_y \circ T_\infty(f_n) + m_y(\{\infty\}) \\ &= \lim_{n \to \infty} Tf_n(y) + m_y(\{\infty\}) = \lim_{n \to \infty} h(y) \cdot f_n(\varphi(y)) + m_y(\{\infty\}) \\ &= h(y) + m_y(\{\infty\}). \end{aligned}$$

Let $g(y) = m_y(\{\infty\})$ for all y in $(-\infty, +\infty)$. Then $g(y) = T_\infty \mathbf{1}(y) - h(y)$ is continuous on $(-\infty, +\infty)$ and $|g(y)| = |m_y(\{\infty\})| \le \|m_y\| \le 1$, $\forall y \in$

$(-\infty, +\infty)$. Note that $\|T_\infty \mathbf{1}\| = 1$. Therefore, $g(y) = T_\infty \mathbf{1}(y) - 1 \leq 0$ when $y > 2$, and $g(y) = T_\infty \mathbf{1}(y) + 1 \geq 0$ when $y < -2$. We claim that $g(y)g(-y) = 0$ whenever $|y| > 2$. In fact, if for example $g(y_0) < -\delta$ for some $y_0 > 2$ and some $\delta > 0$, then for each small $\epsilon > 0$, $0 \leq T_\infty \mathbf{1}(y) < 1 - \delta$ for all y in $(y_0 - \epsilon, y_0 + \epsilon)$. We can choose an f in $C_0([0, +\infty))$ satisfying that $f(y_0) = \|f\| = 1$ and f vanishes outside $(y_0 - \epsilon, y_0 + \epsilon) \subset (2, +\infty)$. Now,

$$
\begin{aligned}
T_\infty(\mathbf{1} + \delta f)(y) &= T_\infty(\mathbf{1})(y) + \delta T_\infty(f)(y) \\
&= T_\infty(\mathbf{1})(y) + \delta T(f)(y) \\
&= h(y) + g(y) + \delta h(y) f(\varphi(y)) \\
&= \begin{cases} 1 + g(y) + \delta f(y), & y > 2, \\ T_\infty \mathbf{1}(y), & -2 \leq y \leq 2, \\ -1 + g(y) - \delta f(-y), & y < -2. \end{cases}
\end{aligned}
$$

Since $\|T_\infty(\mathbf{1} + \delta f)\| = \|1 + \delta f\| = 1 + \delta$ and $|T_\infty(\mathbf{1} + \delta f)(y)| \leq 1$ unless $-y \in (y_0 - \epsilon, y_0 + \epsilon)$, there is a y_1 in $(y_0 - \epsilon, y_0 + \epsilon)$ such that $|-1 + g(-y_1) - \delta f(y_1)| = 1 + \delta$. It forces that $g(-y_1) = 0$. Since ϵ can be arbitrary small, we have $g(-y_0) = 0$ and our claim that $g(y)g(-y) = 0$ whenever $|y| > 2$ has thus been verified. As $T_\infty \mathbf{1}$ is continuous on $(-\infty, +\infty) \cup \{\infty\}$, we must have

$$\lim_{y \to +\infty} T_\infty \mathbf{1}(y) = \lim_{y \to -\infty} T_\infty \mathbf{1}(y),$$

that is,

$$\lim_{y \to +\infty} -1 + g(y) = \lim_{y \to -\infty} 1 + g(y).$$

Let L be their common (finite) limit. Then

$$\lim_{y \to +\infty} g(y) = L + 1, \qquad \lim_{y \to -\infty} g(y) = L - 1.$$

Consequently,

$$0 = \lim_{y \to +\infty} g(y) g(-y) = L^2 - 1.$$

It follows that $L = \pm 1$, and thus either $\lim_{y \to +\infty} g(y) = 2$ or $\lim_{y \to -\infty} g(y) = -2$. Both of them contradicts the fact that $|g(y)| \leq 1$, $\forall y \in (-\infty, +\infty)$.

On the other hand, suppose T_∞ were disjointness preserving. Since $f_n(\mathbf{1} - f_{2n}) = 0$, we have $T_\infty f_n \cdot T_\infty(\mathbf{1} - f_{2n}) = 0$. That is,

$$T_\infty f_n(y) \cdot T_\infty(\mathbf{1} - f_{2n})(y) = 0, \qquad \forall y \in (-\infty, +\infty) \cup \{\infty\}.$$

When $|y| < n$ and $y \neq 1$, $T_\infty f_n(y) = Tf_n(y) = h(y) \neq 0$ and hence $T_\infty(\mathbf{1})(y) = T_\infty(f_{2n})(y) = T(f_{2n})(y) = h(y)$. Since $T_\infty \mathbf{1}$ is continuous on $(-\infty, +\infty) \cup \{\infty\}$, we must have

$$+1 = \lim_{y \to +\infty} h(y) = \lim_{y \to -\infty} h(y) = -1,$$

a contradiction again.

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