Aplicaciones lineales que preservan órdenes parciales

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Introduction

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Linear preserver problems: the study of linear maps between Banach algebras that leave invariant certain function, subset o property.

Let $T : A \to B$ be a linear map between Banach algebras.

- **(I) Function preservers**: $T(F(a)) = F(T(a))$ for a scalar, set or vector valued function F.

Example (Frobenius '1897)

Every linear map $T : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ with $det(T(R)) = det(R)$ for all $R \in M_n(\mathbb{C})$ has the form

$$T(R) = MRN, \qquad \text{for every } R \in M_n(\mathbb{C}) \quad \text{or}$$

$$T(R) = MR^t N, \qquad \text{for every } R \in M_n(\mathbb{C})$$

with $det(MN) = 1$.

Further examples: spectrum, spectral radius, norm, reduced minimum modulus...

- **(II) Subset preservers**: $T(S_A) \subset S_B$ or $T(S_A) = S_B$.

Example (Kaplansky's problem)

Let A, B be unital semisimple Banach algebras and $T : A \longrightarrow B$ a linear unital surjective map such that $T(A^{-1}) = B^{-1}$.

Then T is a Jordan homomorphism.

- Many positive partial results.
- **Still unsolved in general C*-algebras!**

Further examples: idempotent elements, regular elements, **extreme points of the unit ball**...

- **(III) Relation preservers**: $a \sim b \Longrightarrow T(a) \sim T(b)$.

Example (Burgos, Fernández-Polo, Garcés, Martínez, Peralta '08)

Let A, B be C*-algebras, $T : A \to B$ a bounded linear map, $h = T^{**}(1)$ and $r = r(h)$.

Then T preserves orthogonality if, and only if, there exists a triple homomorphism $S : A \to B^{**}$ such that $h^* S(z) = S(z^*)^* h$, $S(z)h^* = hS(z^*)^*$ and

$$T(z) = \frac{hr^* S(z) + S(z) r^* h}{2} = hr^* S(z) = S(z) r^* h$$

for all $z \in A$.

Further examples: commutativity, zero product, **partial order relations**...

- **(IV) Strongly preservers**: $T(F(a)) = F(T(a))$, for a map F defined from the algebra into itself.

Example (Boudi, Mbekhta '10)

Let A, B be Banach algebras with A unital and $T : A \to B$ an additive map.

Then $T(a^{-1}) = T(a)^{-1}$ for every $a \in A^{-1}$ if, and only if, $T(1)T$ is a Jordan homomorphism and $T(1)$ commutes with $T(A)$.

Further examples: maps strongly preserving **Drazin invertibility**, **group invertibility**, **Moore–Penrose invertibility**...

- **(V) Approximate preservers**: characterization of linear maps that "almost" preserve a function, subset or property.

Example (Alaminos, Extremera, Villena)

Let $K, \varepsilon > 0$.

Then there is $\delta > 0$ such that for every linear map $T : B(X) \to B(Y)$ with $\|T\| < K$ and $q(T) > K^{-1}$, the condition

$$\mathrm{dist}_H(\sigma(\phi(a)), \sigma(a)) < \delta$$

implies

$$\mathrm{dist}(T, JIsom) < \varepsilon.$$

Our work mainly contains contributions to the lines:

- **(II) Subset preservers**
- **(III) Relation preservers**
- **(IV) Strongly preservers**
- **(V) Approximate preservers**

Some definitions

Definition (Generalized inverse)

b is a generalized inverse of a

$$aba = a \quad \text{and} \quad bab = b.$$

It is **not unique** in general.

$A^{\wedge}$ denotes the set of all elements with generalized inverse in A.

Definition (Group inverse)

b is the group inverse of a if

$$aba = a, \quad bab = b \quad \text{and} \quad ab = ba.$$

Notation:

- $b = a^{\sharp}$,
- $A^{\sharp}$ denotes the set of all group invertible elements in A.

Definitions

- ***Prime algebra***: *$aAb = \{0\}$ implies $a = 0$ or $b = 0$.*
- ***Socle of an algebra***:

$$\mathrm{soc}(A) = \{x \in A : xAx \text{ is finite-dimensional}\} = \\ = \mathrm{span}\{p = p^2 : pAp \text{ is one-dimensional}\}.$$

*The socle is **essential** if it has non-trivial intersection with any other non zero ideal.*

- ***Real rank zero C*-algebra***: *every selfadjoint element can be approximated by real linear combination of orthogonal projections.*

Let A, B be Banach algebras and $T : A \to B$ a linear map.

Definition (Homomorphism)

$$T(ab) = T(a)T(b)$$

Definition (Anti-homomorphism)

$$T(ab) = T(b)T(a)$$

Definition (Jordan homomorphism)

$$T(a^2) = T(a)^2$$

Let A, B be C*-algebras and $T : A \to B$ a linear map.

Definition (Homomorphism)

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Definition (Jordan homomorphism)

$$T(a^2) = T(a)^2$$

Definition (Selfadjoint)

$$T(a^*) = T(a)^*$$

Linear preservers of partial orders

Preservers of the sharp partial order

Definition (Mitra '87)

$$M \leq_\sharp N \quad \Leftrightarrow \quad MM^\sharp = NM^\sharp = M^\sharp N \quad \text{for} \quad M, N \in M_n(\mathbb{C})$$

Definition (Sharp partial order)

$$a \leq_\sharp b \quad \Leftrightarrow \quad aa^\sharp = ba^\sharp = a^\sharp b$$

Definitions

T ***preserves the sharp partial order*** *if $a \leq_\sharp b \Rightarrow T(a) \leq_\sharp T(b)$,*
T ***preserves the sharp partial order in both directions*** *if*
$a \leq_\sharp b \Leftrightarrow T(a) \leq_\sharp T(b)$.

Theorem (Burgos, Márquez, Patricio '15)

Let A be a unital real rank zero C-algebra, B a Banach algebra and $T : A \to B$ a bounded linear map.*

Then T preserves "$\leq_\sharp$" if, and only if, $T = T(1)S$ where S is a Jordan homomorphism and $T(1) \in A^\sharp$ commutes with $S(A)$.

Remark: the previous result **does not hold** in general unital C*-algebras.

Theorem (Burgos, Márquez, Patricio '15)

Let A, B be unital Banach algebras, $soc(A)$ essential and$T : A \to B$ a bijective linear map.

Then T preserves "$\leq_\sharp$" if, and only if, T is a Jordan isomorphism multiplied by a central invertible element.

Preservers of the star partial order

Definition (Drazin '78)

$$M \leq_* N \quad \Leftrightarrow \quad MM^* = NM^* \quad \text{and} \quad M^*M = M^*N \quad \text{for} \quad M, N \in M_n(\mathbb{C})$$

Definition (Star partial order)

$$a \leq_* b \quad \Leftrightarrow \quad aa^* = ba^* \quad \text{and} \quad a^*a = a^*b$$

Definitions

T ***preserves the star partial order*** *if* $a \leq_* b \Rightarrow T(a) \leq_* T(b)$,
T ***preserves the star partial order in both directions*** *if*
$a \leq_* b \Leftrightarrow T(a) \leq_* T(b)$.

Theorem (Guterman '01)

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $T : M_n(\mathbb{K}) \to M_n(\mathbb{K})$ a bijective linear map such that $A \leq_* B \Rightarrow T(A) \leq_* T(B)$.

Then $T(A) = \alpha UAV$ for every $A \in M_n(\mathbb{K})$ or $T(A) = \alpha UA^t V$ for every $A \in M_n(\mathbb{K})$, where $\alpha \in \mathbb{K}$ and U, V are unitary matrices.

(Star partial order vs. Orthogonality)

$$a \perp b \quad \Leftrightarrow \quad a \leq_* (a+b)$$

Consequence: if T is linear, then T preserves the star partial order $\Leftrightarrow$ T preserves orthogonality.

Definition (Restriction of the star partial order)

$$a \leq b \quad \Leftrightarrow \quad a = pb = bq \quad \text{for some} \quad p, q \in Proj(A)$$

- $a \leq b \Rightarrow a \leq_* b$,
- $a \leq_* b \Rightarrow a \leq b$ whenever $a \in A^\wedge$.

Let A, B be C*-algebras with A unital and $T : A \to B$ a linear map preserving "$\leq$".

Theorem (Burgos, Márquez, Patricio '15)

If A linearly spanned by its projections then T preserves orthogonality.

Theorem (Burgos, Márquez, Patricio '15)

If A is of real rank zero and T is bounded then T preserves orthogonality.

Theorem (Burgos, Márquez, Patricio '15)

*If A has essential socle and T is bijective then T is a Jordan *-homomorphism multiplied by an invertible element.*

Preservers of the minus partial order

Definition (Hartwig '80)

$$A \leq^- B \quad \Leftrightarrow \quad AA^- = BA^- \quad \text{and} \quad A^-A = A^-B$$

for $A, B \in M_n(\mathbb{C})$ and $AA^-A = A$.

Definition (Šemrl '10)

$$A \leq^- B \quad \Leftrightarrow \quad R(P) = \overline{R(A)}, \quad N(A) = N(Q), \quad PA = PB \quad \text{and} \quad AQ = BQ$$

for $A, B \in B(H)$ and $P, Q \in B(H)^\bullet$.

Theorem (Šemrl '10)

Let H be an infinite dimensional Hilbert space and $T : B(H) \to B(H)$ a bijective map with $a \leq^- b \Leftrightarrow T(a) \leq^- T(b)$.

Then $T(A) = RAS$ for every $A \in B(H)$ or $T(A) = RA^*S$ for every $A \in B(H)$, where R, S are bounded bijective both linear or both conjugate linear maps on H.

Definition (Minus partial order, Djordjevic, Rakic, Marovt '13)

$$a \leq^{-} b \quad \Leftrightarrow \quad a = pb = bq$$

for some $p, q \in A^{\bullet}$ with $ann_l(p) = ann_l(a)$ and $ann_r(q) = ann_r(a)$.

Let A, B be unital semisimple Banach algebras with essential socle and $T : A \to B$ a surjective linear map with $a \leq^{-} b \Leftrightarrow T(a) \leq^{-} T(b)$.

Theorem (Burgos, Márquez, Morales '15)

If $T(A^{\wedge}) = B^{\wedge}$ then T is Jordan isomorphism multiplied by an invertible element.

Theorem (Burgos, Márquez, Morales '15)

If $B = B(X)$ or B is a prime C-algebra then T is an isomorphism or an anti-isomorphism multiplied by an invertible element.*

Preservers of the diamond partial order

Definition (Baksalary, Hauke '90)

$$M \leq_\diamond N \quad \Leftrightarrow \quad R(M) \subset R(N), \quad R(M^*) \subset R(N^*) \quad \text{and} \quad MM^*M = MN^*M$$

Definition (Diamond partial order)

$$a \leq_\diamond b \quad \Leftrightarrow \quad aA \subset bA, \quad Aa \subset Ab \quad \text{and} \quad aa^*a = ab^*a$$

Theorem (Burgos, Márquez, Morales)

Let A, B be unital C-algebras with essential socle, with B prime, and $T : A \to B$ a surjective linear map.*

*Then preserves "$\leq_\diamond$" in both directions if and only if $T(1)T(1)^* = T(1)^*T(1) \in \mathbb{C}1$ and $T = T(1)S$ where S is a *-homomorphism or a *-anti-homomorphism.*

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- M. Burgos, A. C. Márquez-García, A. Morales-Campoy, *Maps preserving the diamond partial order*, submitted.

Thanks for your attention!