Weighted composition operators on the Dirichlet space: boundedness and spectral properties

Cáceres, Marzo 2016

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Joint work with Isabelle Chalendar (Lyon, France) and Jonathan R. Partington (Leeds, U.K.)

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Weighted composition operator

Why studying weighted composition operators?

Introduction

- **1930, Banach**

If X is a compact metric space and

$$\begin{array}{rccc} T: & \mathcal{C}(X) & \to & \mathcal{C}(X) \\ & f & \to & Tf \end{array}$$

is a surjective linear isometry, then

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where $|h(t)| = 1$ and φ is a homeomorphism of X onto itself.

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where $|h(t)| = 1$ and φ is a homeomorphism of X onto itself. That is, T is the **weighted composition operator** $W_{h,\varphi}$.

Introduction

- **1950, Bishop**

Let $\alpha \in (0,1)$ be an irrational number and $T_\alpha : L^2[0,1) \to L^2[0,1)$ defined by

$$(T_\alpha h)(x) = xh(\{x+\alpha\}); \qquad x \in [0;1)$$

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Open question: Does T_α have non-trivial invariant subspaces for every $\alpha \in [0,1)$?

Introduction

- $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$

Given h and φ analytic functions in $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$, we may consider the linear map

$$W_{h,\varphi} : \quad f \in \mathcal{H}(\mathbb{D}) \quad \to \quad h\,(f \circ \varphi) \in \mathcal{H}(\mathbb{D})$$

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- **1964, Forelli**

Characterization of the isometries of $\mathcal{H}^p$, $1 < p < \infty$ and $p \neq 2$.

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$$\|f\|_p = \left(\sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} < \infty.$$

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$$f \in \mathcal{H}^\infty \iff f \text{ is a bounded analytic function on } \mathbb{D}$$

Boundedness of weighted composition operators

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- **A necessary condition:** $h \in \mathcal{H}^p$.

Boundedness of weighted composition operators on Hardy spaces

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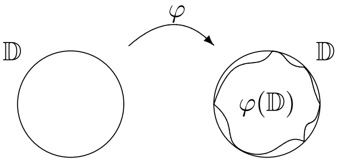
If $\varphi \in \mathcal{H}(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$, then C_φ is bounded on $\mathcal{H}^p$.



Boundedness of weighted composition operators on Hardy spaces

- **2003, Contreras and Hernández-Díaz**

Necessary and sufficient condition for boundedness of $W_{h,\varphi}$ in terms of Carleson measures.

Boundedness of weighted composition operators on Hardy spaces

- **2003, Contreras and Hernández-Díaz**

$W_{h,\varphi}$ is bounded in $\mathcal{H}^p \Leftrightarrow \mu_{h,\varphi}$ is a Carleson measure on $\overline{\mathbb{D}}$, where

$$\mu_{h,\varphi}(E) = \int_{\varphi^{-1}(E)\cap\mathbb{D}} |h|^p dm,$$

for measurable subsets $E \subseteq \mathbb{D}$.

Boundedness of weighted composition operators on Hardy spaces

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$W_{h,\varphi}$ is bounded in $\mathcal{H}^2 \Leftrightarrow$

$$\sup_{|w|<1} \left\| \frac{(1-|w|^2)^{1/2} h}{1-\overline{w}\varphi} \right\|_2 < \infty.$$

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- **2006, Harper**

$W_{h,\varphi}$ is bounded in $\mathcal{H}^2 \Leftrightarrow$

$$\sup_{|w|<1} \|W_{h,\varphi} k_w\|_2 < \infty,$$

where k_w is the normalized reproducing kernel at w in $\mathcal{H}^2$.

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- **2007, Cučković and Zhao**

Generalizations to mappings between $\mathcal{H}^p$ and $\mathcal{H}^q$ spaces.

Boundedness of weighted composition operators on Hardy spaces

- **2007, Jury**

Let $H(\varphi)$ denote the de Branges–Rovnyak space, that is, the reproducing kernel Hilbert space on $\mathbb{D}$ with reproducing kernel at w

$$k_w^{\varphi}(z) = \frac{1 - \overline{\varphi(w)}\varphi(z)}{1 - \overline{w}z}.$$

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- **Theorem.** Let φ be an analytic self-map of $\mathbb{D}$ and $h \in H(\varphi)$. Then $W_{h,\varphi}$ is bounded in $\mathcal{H}^2$ and

$$\|W_{h,\varphi}\|_2 \leq \|h\|_{H(\varphi)}.$$

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$$\|W_{h,\varphi}\|_2 \le \|h\|_{H(\varphi)}.$$

- **Remark.** This is not a necessary condition for boundedness.

Boundedness of weighted composition operators on Hardy spaces

- **2010, Kumar, GG and Partington**

- **“Inner-Outer” factorization of Hardy functions.**
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  Given a sequence of (not necessarily distinct) points $\{z_k\}$ in $\mathbb{D} \setminus \{0\}$ satisfying the *Blaschke condition*

  $$\sum_{k=1}^{\infty}(1-|z_k|) < \infty,$$

  the infinite product

  $$B(z) = \prod_{k=1}^{\infty} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \overline{z_k} z},$$

  converges uniformly on compact subsets of $\mathbb{D}$ to a holomorphic function B called the *Blaschke product with zero sequence* $\{z_k\}$.

- **"Inner-Outer" factorization of Hardy functions.**

Let $f \in \mathcal{H}^p$, then $f = BS_\sigma F$ where

- B is a Blaschke product.

  A general expression for a Blaschke product is given by

$$e^{i\theta}\, z^N \prod_{k=1}^{\infty} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \overline{z_k} z}$$

  where $N \geq 0$ is an integer.

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$$S_\sigma(z) = \exp\left(-\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\sigma(\theta)\right)$$

where σ is a positive singular measure in $\partial\mathbb{D}$.

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- B is a Blaschke product.
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$$F(z) = \lambda \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log|f(e^{i\theta})|\,d\theta\right).$$

Boundedness of weighted composition operators on Hardy spaces

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If φ is inner, then

$$H(\varphi) = K_\varphi := \mathcal{H}^2 \ominus \varphi\mathcal{H}^2,$$

and

$$P_{K_\varphi} h = \varphi P_-(\overline{\varphi} h),$$

where P_- is the orthogonal projection onto $L^2 \ominus \mathcal{H}^2$.

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where P_- is the orthogonal projection onto $L^2 \ominus \mathcal{H}^2$.
In this case, if $h \in K_\varphi$,

$$\left\| \sum_{n=0}^{\infty} a_n \varphi^n h \right\|_2^2 = \|h\|_2^2 \sum_{n=0}^{\infty} |a_n|^2,$$

since

$$\langle \varphi^n h, \varphi^m h \rangle = \begin{cases} 0 & \text{for } n \neq m, \\ \|h\|_2^2 & \text{for } n = m, \end{cases}$$

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So, if φ is inner and $h \in K_\varphi$

$$\|W_{h,\varphi} f\|_2 = \|h\|_2 \|f\|_2.$$

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So, if φ is inner and $h \in K_\varphi$

$$\|W_{h,\varphi} f\|_2 = \|h\|_2 \|f\|_2.$$

Thus the condition in Jury's Theorem holds for $h \in K_\varphi$, although not in general. Indeed, if $\varphi(z) = z$, then $\|W_{h,\varphi}\| = \|T_h\| = \|h\|_\infty$.

Boundedness of weighted composition operators on Hardy spaces

• **Definition.** For $\varphi : \mathbb{D} \to \mathbb{D}$ analytic, the **multiplier space** of φ is defined by

$$\mathcal{M}(\varphi) = \{h \in \mathcal{H}^2 : \; W_{h,\varphi} := T_h C_\varphi \text{ is bounded}\}.$$

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• **Question.** Determine $\mathcal{M}(\varphi)$.

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- **Theorem** (2010, Kumar, GG, Partington) $\mathcal{M}(\varphi) = \mathcal{H}^2$ if and only if $\|\varphi\|_\infty < 1$.

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- **Theorem** (2010, Kumar, GG, Partington) $\mathcal{M}(\varphi) = \mathcal{H}^2$ if and only if $\|\varphi\|_\infty < 1$.

- **Theorem** (2003, Contreras and Hernández-Díaz, 2008 Matache) $\mathcal{M}(\varphi) = \mathcal{H}^\infty$ if and only if φ is a finite Blaschke product.

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Let φ be an analytic self-map with $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\alpha \in \partial\mathbb{D}$.

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Let φ be an analytic self-map with $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\alpha \in \partial\mathbb{D}$.

φ has (finite) angular derivative at α, denoted by $\varphi'(\alpha)$, whenever the non-tangential limit

$$\angle \lim_{z\to\alpha} \frac{\varphi(z)-\eta}{z-\alpha} \qquad (z \in \mathbb{D}),$$

exists and is finite for some $\eta \in \partial\mathbb{D}$.

- **Julia-Carathéodory Theorem:**

Let φ be an analytic self-map of $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\alpha \in \partial\mathbb{D}$. The following conditions are equivalent

1. φ has finite angular derivative at α.
2. Both radial limits $\varphi(\alpha)$ and $\varphi'(\alpha)$ exist and are finite.
3. $\liminf_{z\to\alpha} \frac{1-|\varphi(z)|}{1-|z|} < \infty$, where the lím inf is calculated as z approaches α within $\mathbb{D}$.

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Moreover, under the above conditions it holds that $\varphi(\alpha) = \eta$,

$$|\varphi'(\alpha)| = \liminf_{z\to\alpha} \frac{1-|\varphi(z)|}{1-|z|}$$

• **Theorem** (2010, Kumar, GG, Partington) Let φ be an analytic self-map of $\mathbb{D}$. Let

$$E_\varphi = \{\zeta \in \mathbb{T} : \ \varphi \text{ has finite angular derivative at } \zeta\}.$$

If $W_{h,\varphi}$ is bounded on $\mathcal{H}^p$ for some $1 \leq p < \infty$, then h is pointwise bounded on every Stolz domain whose vertex is a point of E_φ.

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• **Remark.** The converse does not hold.

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- **Remark.** The converse does not hold.

- **Corollary** Let φ be an inner function . Then any function $h \in \mathcal{M}(\varphi)$ is essentially bounded on all relatively compact subsets of $\mathbb{T} \setminus \sigma(\varphi)$, where $\sigma(\varphi)$ denotes the *spectrum* of φ, namely, $\sigma(\varphi) = \overline{\{a_n\}_n} \cup \ \text{supp } \mu$.

Boundedness of weighted composition operators

The Dirichlet space $\mathcal{D}$

Boundedness of weighted composition operators

The Dirichlet space $\mathcal{D}$

$f \in \mathcal{D} \iff f \in \mathcal{H}(\mathbb{D})$ and

$$\|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z)$$

Boundedness of weighted composition operators

The Dirichlet space $\mathcal{D}$

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- **Question:** When does $W_{h,\varphi}$ take $\mathcal{D}$ boundedly into itself?

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- **Remark.** Not every composition operator C_φ takes $\mathcal{D}$ into itself!
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$$\begin{array}{cccc} C_\varphi : & \mathcal{D} & \to & \mathcal{D} \\ & f & \to & f \circ \varphi \end{array}$$

$n_\varphi(w) \equiv$ multiplicity of φ at w

$$S(\xi, \delta) = \{z \in \mathbb{D} : |z - \xi| < \delta\}$$

the Carleson disk centered at $\xi \in \partial\mathbb{D}$ of radius $0 < \delta < 1$.

$$C_\varphi \text{ is bounded on } \mathcal{D} \iff \int_{S(\xi,\delta)} n_\varphi(w) dA(w) \sim \mathbf{O}\ (\delta^2).$$

Boundedness of weighted composition operators on the Dirichlet space

It is clear that if C_φ is a bounded operator on $\mathcal{D}$ and u is a **multiplier** of $\mathcal{D}$, that is, the Toeplitz operator $T_u : f \mapsto uf$ is defined everywhere on $\mathcal{D}$ and hence bounded, the weighted composition operator $W_{u,\varphi}$ on $\mathcal{D}$ is obviously bounded.

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• **Multipliers of $\mathcal{D}$.** Not easy to describe.

• **1980, Stegenga** Characterization in terms of a condition involving the logarithmic capacity of their boundary values. In particular, the strict inclusion

$$\mathcal{M}(\mathcal{D}) \subset \mathcal{D} \cap \mathcal{H}^\infty$$

holds.

• **Multipliers of $\mathcal{D}$**. Not easy to describe.

• **1980, Stegenga** Characterization in terms of a condition involving the logarithmic capacity of their boundary values. In particular, the strict inclusion

$$\mathcal{M}(\mathcal{D}) \subset \mathcal{D} \cap \mathcal{H}^\infty$$

holds.

• **1999, Wu** An equivalent condition in terms of Carleson measures for $\mathcal{D}$ (that is, there is a continuous injection from $\mathcal{D}$ into $L^2(\mathbb{D}, \mu)$),

$$u \in \mathcal{M}(\mathcal{D}) \iff u \in \mathcal{H}^\infty \text{ and } d\mu(z) = |u'(z)|^2\, dA(z)$$
$$\text{is a Carleson measure for } \mathcal{D}.$$

Boundedness of weighted composition operators on the Dirichlet space

An example

An example Let $u(z) = (1-z)^2$ and let φ be the infinite Blaschke product with zeroes

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Conclusion One may construct self-maps of the unit disc φ such that $\varphi \notin \mathcal{D}$ and multipliers $u \in \mathcal{M}(\mathcal{D})$ such that $W_{u,\varphi}$ is bounded in the Dirichlet space.Therefore, facing the problem of describing the weighted composition operators taking $\mathcal{D}$ boundedly into itself deals not only with the multipliers of $\mathcal{D}$ but also with those self-maps of the unit disc that may induce unbounded composition operators in $\mathcal{D}$.

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An open question. Characterization of $\mathcal{M}(\varphi)$. Extreme cases?

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Decomposition Theorem. (2015, Chalendar, G-G, Partington) Let B be a finite Blaschke product and write K_B for the model space $K_B = \mathcal{H}^2 \ominus B\mathcal{H}^2$. Assume $B(0) = 0$. Then

1. $f \in \mathcal{H}^2$ if and only if $f = \sum_{k=0}^{\infty} g_k B^k$ (convergence in $\mathcal{H}^2$ norm) with $g_k \in K_B$ and $\sum_{k=0}^{\infty} \|g_k\|^2 < \infty$.
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Recall that if C_φ is bounded then $\mathcal{M}(\mathcal{D}) \subseteq \mathcal{M}(\varphi) \subseteq \mathcal{D}$. For φ a finite Blaschke product the space of weighted composition operators is as small as possible:

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Remark. The assumption about φ being inner cannot be relaxed; even if $\|\varphi\|_\infty = 1$ and C_φ is bounded in $\mathcal{D}$. We can have $\mathcal{M}(\varphi) \neq \mathcal{M}(\mathcal{D})$ even if $\|\varphi\|_\infty = 1$ and C_φ is bounded in $\mathcal{D}$. Let us consider

$$\varphi(z) = \frac{1-z}{2} \qquad \text{and} \qquad h(z) = \sum_{k=2}^{\infty} \frac{z^k}{k(\log k)^{3/4}}.$$

One has $h \in \mathcal{D} \setminus \mathcal{M}(\mathcal{D})$. Nonetheless, $W_{h,\varphi}$ is bounded; that is, $h \in \mathcal{M}(\varphi)$.

Boundedness of weighted composition operators on the Dirichlet space

Theorem (2015, Chalendar, G-G, Partington)
Let φ be an analytic self-map of $\mathbb{D}$. Then $\mathcal{M}(\varphi) = \mathcal{D}$ if and only if

1. $\|\varphi\|_\infty < 1$, and
2. $\varphi \in \mathcal{M}(\mathcal{D})$.

Spectral properties of weighted composition operators

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- **Open question:** Determine the spectrum of composition operators in $\mathcal{H}^p$.
- **2005, Highdon** Spectrum of composition operators induced by linear fractional self-maps of $\mathbb{D}$ acting on $\mathcal{D}$.
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Spectral properties of invertible weighted composition operators on $\mathcal{D}$

Proposition. $W_{h,\varphi}$ is invertible in $\mathcal{D}$ if and only if $h \in \mathcal{M}(D)$, bounded away from zero in $\mathbb{D}$ and φ is an automorphism of $\mathbb{D}$. In such a case, the inverse operator of $W_{h,\varphi} : \mathcal{D} \to \mathcal{D}$ is also a weighted composition operator and

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where $p \in \mathbb{D}$ and $-\pi < \theta \leq \pi$.

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Spectral properties of invertible weighted composition operators on $\mathcal{D}$

Elliptic case

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Elliptic case

Theorem (Chalendar, G-G, Partington)
Suppose that φ is an elliptic automorphism of $\mathbb{D}$ with fixed point $a \in \mathbb{D}$ and $W_{h,\varphi}$ a weighted composition operator on $\mathcal{D}$. Let $h_{(n)} = \prod_{k=0}^{n-1} h \circ \varphi_k$. Then

1. either there exists a positive integer j such that $\varphi_j(z) = z$ for all $z \in \mathbb{D}$, in which case, if m is the smallest such integer, then

$$\sigma(W_{h,\varphi}) = \overline{\{\lambda : \ \lambda^m = h_{(m)}(z), z \in \mathbb{D}\}},$$

2. or $\varphi_{\mathrm{n}} \neq \mathrm{Id}$ for every n and, if $\mathrm{W}_{\mathrm{h},\varphi}$ is invertible, then

$$\sigma(\mathrm{W}_{\mathrm{h},\varphi}) = \{\lambda : \ |\lambda| = |\mathrm{h}(\mathrm{a})|\}.$$

Parabolic case

Spectral properties of invertible weighted composition operators on $\mathcal{D}$

Parabolic case

Theorem (Chalendar, G-G, Partington)
Suppose that φ is a parabolic automorphism of $\mathbb{D}$ with fixed point $\mathrm{a} \in \mathbb{T}$ and $\mathrm{W}_{\mathrm{h},\varphi}$ a weighted composition operator on $\mathcal{D}$, determined by an $\mathrm{h} \in \mathcal{M}(\mathcal{D})$ that is continuous at a. If $\mathrm{W}_{\mathrm{h},\varphi}$ is invertible, then

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Key idea. Causal operators

Spectral properties of invertible weighted composition operators on $\mathcal{D}$

Hyperbolic case

Spectral properties of invertible weighted composition operators on $\mathcal{D}$

Hyperbolic case

Theorem (Chalendar, G-G, Partington)
Suppose that φ is a hyperbolic automorphism of $\mathbb{D}$ with attractive fixed point $a \in \mathbb{T}$ and repelling fixed point $b \in \mathbb{T}$. Let $W_{h,\varphi}$ be a weighted composition operator on $\mathcal{D}$, determined by an $h \in \mathcal{M}(\mathcal{D})$ that is continuous at a and b. If $W_{h,\varphi}$ is invertible, then

$$\rho(W_{h,\varphi}) \leq \text{máx}\{|h(a)|, |h(b)|\}/\mu,$$

where ϕ is conjugate to the automorphism

$$\psi(z) = \frac{(1+\mu)z + (1-\mu)}{(1-\mu)z + (1+\mu)},$$

with $0 < \mu < 1$. Hence $\sigma(W_{h,\varphi})$ is contained in the annulus with radii $\text{máx}\{|h(a)|, |h(b)|\}/\mu$ and $\text{mín}\{|h(a)|, |h(b)|\}\mu$.

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