The Cesàro operator on power series spaces

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Aim of the lecture

AIM

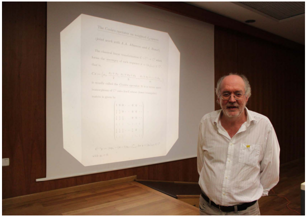
Investigate the continuity, the compactness, the mean ergodicity and determine the spectrum of the Cesàro operator C acting on power series spaces and their duals, with applications to spaces of analytic functions on the disc.

We report on joint work in progress with Angela A. Albanese (Univ. Lecce, Italy) and Werner J. Ricker (Univ. Eichstaett, Germany).

Ernesto Cesàro (1859-1906)

Albanese and Ricker

Angela Albanese



Werner Ricker

The discrete Cesàro operator

The *Cesàro operator* C is defined for a sequence $x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}$ of complex numbers by

$$\mathrm{C}(x) = \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)_n, \quad x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}.$$

Proposition.

The operator $C\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is a bicontinuous isomorphism of $\mathbb{C}^{\mathbb{N}}$ onto itself with

$$C^{-1}(y) = (ny_n - (n-1)y_{n-1})_n, \quad y = (y_n)_n \in \mathbb{C}^{\mathbb{N}}, \tag{1}$$

where we set $y_{-1} := 0$.

Recall that $\mathbb{C}^{\mathbb{N}}$ is a Fréchet space for the topology of coordinatewise convergence.

Theorem. Hardy. 1920.

Let $1 < p < \infty$. The Cesàro operator maps the Banach space ℓ^p continuously into itself, which we denote by $C^{(p)}: \ell^p \to \ell^p$, and $\|(C^{(p)})\| = p'$, where $\frac{1}{p} + \frac{1}{p'} = 1$, for all $n \in \mathbb{N}$.

In particular, **Hardy's inequality** holds:

$$\|(C^{(p)})\|_p \leq p' \|x\|_p, \quad x \in \ell^p.$$

Clearly C is not continuous on ℓ_1, since $C(e_1) = (1, 1/2, 1/3, ...)$.

Proposition.

The Cesàro operators $\mathsf{C}^{(\infty)} \colon \ell^\infty \to \ell^\infty$, $\mathsf{C}^{(c)} \colon c \to c$ and $\mathsf{C}^{(0)} \colon c_0 \to c_0$ are continuous, and $\|\mathsf{C}^{(\infty)}\| = \|\mathsf{C}^{(c)}\| = \|\mathsf{C}^{(0)}\| = 1$.

Moreover, $\lim \mathsf{C}x = \lim x$ for each $x \in c$.

Spectrum and point spectrum

X is a Hausdorff locally convex space (lcs).

$\mathcal{L}(X)$ (resp. $\mathcal{K}(X)$) is the space of all continuous (resp. compact) linear operators on X.

The **resolvent set** $\rho(T,X)$ of $T \in \mathcal{L}(X)$ consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$.

The **spectrum** of T is the set $\sigma(T,X) := \mathbb{C} \setminus \rho(T,X)$. The **point spectrum** is the set $\sigma_{pt}(T,X)$ of those $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective. The elements of $\sigma_{pt}(T,X)$ are called eigenvalues of T.

Spectrum and point spectrum

Notation:

$\Sigma := \{\frac{1}{m} \colon m \in \mathbb{N}\}$ and $\Sigma_0 := \Sigma \cup \{0\}$.

Proposition.

(i) $\sigma(\mathsf{C}; \mathbb{C}^{\mathbb{N}}) = \sigma_{pt}(\mathsf{C}; \mathbb{C}^{\mathbb{N}}) = \Sigma$.

(ii) Fix $m \in \mathbb{N}$. Let $x^{(m)} := (x_n^{(m)})_n \in \mathbb{C}^{\mathbb{N}}$ where $x_n^{(m)} := 0$ for $n \in \{1, \ldots, m-1\}$, $x_m^{(m)} := 1$ and $x_n^{(m)} := \frac{(n-1)!}{(m-1)!(n-m)!}$ for $n > m$. Then the eigenspace

$$\mathrm{Ker}\left(\frac{1}{m}I - \mathsf{C}\right) = \mathrm{span}\{x^{(m)}\} \subseteq \mathbb{C}^{\mathbb{N}}$$

is 1-dimensional.

Spectrum and point spectrum

Theorem. Leibowitz. 1972.

(i) $\sigma(\mathsf{C}; \ell^\infty) = \sigma(\mathsf{C}; c_0) = \{\lambda \in \mathbb{C} \mid \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\}$.

(ii) $\sigma_{pt}(\mathsf{C}; \ell^\infty) = \{(1, 1, 1, ...)\}$.

(iii) $\sigma_{pt}(\mathsf{C}; c_0) = \emptyset$.

Spectrum and point spectrum

Theorem. Leibowitz. 1972.

Let $1 < p < \infty$ and $1/p + 1/p' = 1$.

(i) $\sigma(\mathsf{C}; \ell^p) = \{\lambda \in \mathbb{C} \mid \left|\lambda - \frac{p'}{2}\right| \leq \frac{p'}{2}\}$.

(ii) $\sigma_{pt}(\mathsf{C}; \ell^p) = \emptyset$.

In particular, C is not compact in the spaces $\ell^p, 1 < p \leq \infty$, or in the space c_0.

The Cesàro operator for analytic functions

The Cesàro operator is defined for analytic functions on the disc $\mathbb{D}$ by

$$Cf = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_n \right) z^n, \quad f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}).$$

The Cesàro operator acts continuously and has the integral representation

$$Cf(z) = \frac{1}{z} \int_0^z \frac{f(\rho)}{1-\rho}\, d\rho, \quad f \in H(\mathbb{D}),\ z \in \mathbb{D}.$$

The Cesàro operator for analytic functions

Indeed, for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$, we have

$$
\begin{aligned}
Cf(z) &= \frac{1}{z}\int_0^z \frac{f(\rho)}{1-\rho}\, d\rho = \frac{1}{z}\int_0^z \left(\sum_{n=0}^{\infty} a_n \rho^n\right)\left(\sum_{m=0}^{\infty} \rho^m\right) \\
&= \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} \frac{1}{z}\int_0^z \rho^{n+m}\, d\rho = \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} \frac{z^{n+m}}{n+m+1} \\
&= \sum_{n=0}^{\infty} a_n \sum_{k=n} \frac{z^k}{k+1} = \sum_{k=0}^{\infty} \left(\frac{1}{k+1}\sum_{n=0}^{k} a_n\right) z^k.
\end{aligned}
$$

$H(\mathbb{D})$ as a power series space

The map

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}) \to (a_n)_{n=0}^{\infty}$$

defines an isomorphism between the Fréchet space $H(\mathbb{D})$ endowed with the topology of uniform convergence on the compact sets and the sequence space

$$\Lambda_0((n)_n) := \bigcap_{k \in \mathbb{N}} c_0(w_k),$$

$H(\mathbb{D})$ as a power series space

In the Fréchet space

$$\Lambda_0((n)_n) := \bigcap_{k\in\mathbb{N}} c_0(w_k),$$

we take $w_k(n) := (r_k)^n, k \in \mathbb{N}, n = 0, 1, 2, \ldots$ and $r_k = 1 - (1/k), k \in \mathbb{N}$, an increasing sequence tending to 1. Moreover,

$$c_0(w_k) := \left\{x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \lim_{n\to\infty} w_k(n)|x_n| = 0\right\},$$

equipped with the norm $\|x\|_{0,w_k} := \sup_{n\in\mathbb{N}} w_k(n)|x_n|$ for $x \in c_0(w_k)$.

The Korenblum space

For $\gamma > 0$ the growth classes $A^{-\gamma} := A^{-\gamma}(\mathbb{D})$ are defined by

$$A^{-\gamma} = \{f \in H(\mathbb{D}) \colon \|f\|_{-\gamma} := \sup_{z\in\mathbb{D}} (1-|z|)^{\gamma}|f(z)| < \infty\}.$$

The Cesàro operator acts continuously on $A^{-\gamma}$. Its spectrum on these (and many other spaces of analytic functions on the disc) has been studied by Aleman and Persson 2008-2010.
The **Korenblum space** $A^{-\infty} := A^{-\infty}(\mathbb{D})$ is defined as

$$A^{-\infty} = \cup_{0<\gamma<\infty} A^{-\gamma} = \cup_{n\in\mathbb{N}} A^{-n},$$

and it is endowed with the finest locally convex topology such that all the inclusion $A^{-n} \hookrightarrow A^{-\infty}$ are continuous, that is, $A^{-\infty} = \mathrm{ind}_n A^{-n}$.

$A^{-\infty}$ as a dual power series space

The map

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \to (a_n)_{n=0}^{\infty}$$

defines an isomorphism between $A^{-\infty}$ and the countable inductive limit $E_\alpha := \cup_{k\in\mathbb{N}} c_0(v_k)$ of weighted c_0 spaces defined for the weight sequence

$$v_k(n) = (n+e)^{-k}, k \in \mathbb{N}, n = 0, 1, 2, \ldots$$

New aim

In order to investigate the Cesàro operator on the spaces $H(\mathbb{D})$ and $A^{-\infty}$, we have to **study the behaviour of the discrete Cesàro operator** C **on weighted Banach** c_0 **spaces.**

We will also need **abstract tools to deduce properties of an operator acting on intersections or unions of Banach spaces** from properties of the behavior of the operator acting between the steps.

The space $c_0(w)$

- Let $w = (w(n))_{n=1}^{\infty}$ be a bounded, strictly positive sequence. Define

$$c_0(w) := \left\{ x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \colon \lim_{n\to\infty} w(n)|x_n| = 0 \right\},$$

  equipped with the norm $\|x\|_{0,w} := \sup_{n\in\mathbb{N}} w(n)|x_n|$ for $x \in c_0(w)$.

- $c_0(w)$ is isometrically isomorphic to c_0 via the linear multiplication operator $\Phi_w \colon c_0(w) \to c_0$ given by

$$x = (x_n)_{n\in\mathbb{N}} \to \Phi_w(x) := (w(n)x_n)_{n\in\mathbb{N}}. \tag{2}$$

- We are interested in the case when $\inf_{n\in\mathbb{N}} w(n) = 0$. Otherwise $c_0(w) = c_0$ with equivalent norms.

Theorem.

Let w be a bounded, strictly positive sequence.
The Cesàro operator $\mathrm{C}^{(0,w)} \in \mathcal{L}(c_0(w))$ if and only if

$$\left\{ \frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} \right\}_{n\in\mathbb{N}} \in \ell_\infty. \tag{3}$$

Moreover, $\|\mathrm{C}^{(0,w)}\| \geq 1$.

If w is decreasing, then (3) is satisfied and $\|\mathrm{C}^{(0,w)}\| = 1$.

Compactness of C on $c_0(w)$

Theorem.

Let w be a bounded, strictly positive sequence.
The following conditions are equivalent.

(a) $C^{(0,w)}$ is weakly compact.

(b) $C^{(0,w)}$ is compact.

(c) The sequence

$$\left\{ \frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} \right\}_{n\in\mathbb{N}} \in c_0. \tag{4}$$

Let $w = (w(n))_{n=1}^{\infty}$ be two strictly positive sequences. Let $T_w\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ denote the linear operator given by

$$T_w x := \left(\frac{w(n)}{n} \sum_{k=1}^{n} \frac{x_k}{w(k)} \right)_{n\in\mathbb{N}}, \quad x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}. \qquad (5)$$

Then $\Phi_w C = T_w \Phi_v$. Therefore, the Cesàro operator C maps $c_0(w)$ continuously (resp., compactly) into $c_0(w)$ if and only if the operator $T_w \in \mathcal{L}(c_0)$ (resp., $T_w \in \mathcal{K}(c_0)$).

Continuity of C on $c_0(w)$. A classical lemma

Lemma. Banach's Book.

Let $A = (a_{nm})_{n,m\in\mathbb{N}}$ be a matrix with entries from $\mathbb{C}$ and $T\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ be the linear operator defined by

$$Tx := \left(\sum_{m=1}^{\infty} a_{nm}x_m\right)_{n\in\mathbb{N}}, \quad x = (x_n)_{n\in\mathbb{N}}, \tag{6}$$

interpreted to mean that Tx exists in $\mathbb{C}^{\mathbb{N}}$ for every $x \in \mathbb{C}^{\mathbb{N}}$.
Then $T \in \mathcal{L}(c_0)$ if and only if the following two conditions are satisfied:

(i) $\lim_{n\to\infty} a_{nm} = 0$ for each fixed $m \in \mathbb{N}$;

(ii) $\sup_{n\in\mathbb{N}} \sum_{m=1}^{\infty} |a_{nm}| < \infty$.

In this case, $\|T\| = \sup_{n\in\mathbb{N}} \sum_{m=1}^{\infty} |a_{nm}|$.

Examples

- Let $w(2n+1) = \frac{1}{n+1}$ for $n \geq 0$ and $w(2n) = 2^{-n}$ for $n \geq 1$. Clearly $\lim_{n\to\infty} w(n) = 0$, but C does not act continuously in $c_0(w)$.

- Let $\alpha > 0$ and $w(n) := \frac{1}{n^\alpha}$ for all $n \in \mathbb{N}$. Since w is decreasing, $C^{(0,w)} \in \mathcal{L}(c_0(w))$. But $C^{(0,w)}$ is not compact, since

$$\begin{aligned}\frac{w(n)}{n}\sum_{k=1}^{n}\frac{1}{w(k)} &= \frac{1}{n^{\alpha+1}}\sum_{k=1}^{n}k^\alpha \geq \frac{1}{n^{\alpha+1}}\sum_{k=1}^{n}\int_{k-1}^{k}x^\alpha\,dx \\ &= \frac{1}{n^{\alpha+1}}\int_0^n x^\alpha\,dx = \frac{1}{\alpha+1}.\end{aligned}$$

Examples of compact operators Cesàro operators on $C^{(0,w)}$

Proposition.

Let w be bounded, strictly positive and satisfy

$$\limsup_{n\to\infty} \frac{w(n+1)}{w(n)} \in [0,1),$$

then $C^{(0,w)} \in \mathcal{K}(c_0(w))$.

Moreover, $\sigma_{pt}(C^{(0,w)}) = \Sigma; \quad \sigma(C^{(0,w)}) = \Sigma_0$.

One checks that that the condition (4) is valid to prove compactness.

- $C^{(0,w)} \in \mathcal{K}(c_0(w))$ for the following sequences:

  (1) $w(n) := a^{-\alpha_n}$, $n \in \mathbb{N}$, with $a > 1$, $\alpha_n \uparrow \infty$ and $\lim_{n\to\infty}(\alpha_n - \alpha_{n-1}) = \infty$.

  (2) $w(n) := \frac{n^\alpha}{a^n}$ for $n \in \mathbb{N}$, where $a > 1$ and $\alpha \in \mathbb{R}$.

  (3) $w(n) := \frac{a^n}{n!}$ for $n \in \mathbb{N}$, where $a \geq 1$.

  (4) $w(n) := n^{-n}$ for $n \in \mathbb{N}$.

- Let $w(n) := e^{-\sqrt{n}}$ or $w(n) := e^{-(\log n)^\beta}$, $\beta > 1$, for $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \frac{w(n+1)}{w(n)} = 1$, but $C \in \mathcal{K}(c_0(w))$.

Spectrum of $C^{(0,w)}$

Given a bounded, strictly positive sequence w, let

$$S_w := \{s \in \mathbb{R} \colon \sum_{n=1}^{\infty} \frac{1}{n^s w(n)} < \infty\}.$$

In case $S_w \neq \emptyset$ we define $s_0 := \inf S_w$.

Moreover, let

$$R_w := \{t \in \mathbb{R} \colon \lim_{n\to\infty} n^t w(n) = 0\}.$$

In case $R_w \neq \mathbb{R}$ we define $t_0 := \sup R_w$. If $R_w = \mathbb{R}$ we set $t_0 = \infty$.

Recall $\Sigma := \{\frac{1}{m} \colon m \in \mathbb{N}\}$ and $\Sigma_0 := \Sigma \cup \{0\}$.

Spectrum of $C^{(0,w)}$. Main result

Theorem.

Let w be a bounded, strictly positive sequence such that $C^{(0,w)} \in \mathcal{L}(c_0(w))$.

(1) The following inclusion holds:

$$\Sigma_0 \subseteq \sigma(C^{(0,w)}).$$

(2) Let $\lambda \notin \Sigma_0$. Then $\lambda \in \rho(C^{(0,w)})$ if and only if both of the conditions

(i) $\lim_{n\to\infty} \frac{w(n)}{n^{1-\alpha}} = 0$, and

(ii) $\sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{m=1}^{n-1} \frac{w(n)n^{\alpha}}{w(m)m^{\alpha}} < \infty$,

are satisfied, where $\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right)$.

Spectrum of $C^{(0,w)}$. Main result continued

Theorem continued.

(3) Suppose that $R_w \neq \mathbb{R}$, i.e., $t_0 < \infty$. Then we have the inclusions

$$\left\{ \frac{1}{m} : m \in \mathbb{N},\ 1 \leq m < t_0 + 1 \right\} \subseteq \sigma_{pt}(C^{(0,w)}) \subseteq$$

$$\subseteq \left\{ \frac{1}{m} : m \in \mathbb{N},\ 1 \leq m \leq t_0 + 1 \right\}.$$

In particular, $\sigma_{pt}(C^{(0,w)})$ is a proper subset of Σ.
If $R_w = \mathbb{R}$, then

$$\sigma_{pt}(C^{(0,w)}) = \Sigma.$$

Spectrum of $C^{(0,w)}$

Proposition.

Let w be a strictly positive, decreasing sequence.

(i)

$$\sigma(C^{(0,w)}) \subseteq \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}. \tag{7}$$

(ii) If $S_w \neq \emptyset$, then

$$\left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2s_0}\right| \leq \frac{1}{2s_0}\right\} \cup \Sigma \subseteq \sigma(C^{(0,w)}). \tag{8}$$

Spectrum of $\mathrm{C}^{(0,w)}$

A sequence $u = (u_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ is called **rapidly decreasing** if $(n^m u_n)_{n\in\mathbb{N}} \in c_0$ for every $m \in \mathbb{N}$. The space of all rapidly decreasing, $\mathbb{C}$-valued sequences is denoted by s.

Proposition.

Let w be a bounded, strictly positive sequence. If $\mathrm{C}^{(0,w)} \in \mathcal{K}(c_0(w))$, then

$$\sigma_{pt}(\mathrm{C}^{(0,w)}) = \Sigma \quad \text{and} \quad \sigma(\mathrm{C}^{(0,w)}) = \Sigma_0.$$

Moreover, $w \in s$ and $S_w = \emptyset$.

There exist weights $w \in s$ such that $\mathrm{C}^{(0,w)} \notin \mathcal{K}(c_0(w))$: Define w via $w(1) := 1$ and $w(n) := \frac{1}{j^j}$ if $n \in \{2^{j-1}+1, \ldots, 2^j\}$ for $j \in \mathbb{N}$.

(1) $w(n) = \frac{1}{(\log(n+1))^\gamma}$ for $n \in \mathbb{N}$ with $\gamma \geq 0$. Then $s_0 = 1$ and $t_0 = 0$. We have

$$\sigma(C^{(0,w)}) = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}, \quad \text{and}$$

$$\sigma_{pt}(C^{(0,w)}) = \emptyset \text{ if } \gamma = 0; \quad \sigma_{pt}(C^{(0,w)}) = \{1\} \text{ if } \gamma > 0.$$

Spectrum of $C^{(0,w)}$. Relevant examples

(2) $w(n) = \frac{1}{n^\beta}$ for $n \in \mathbb{N}$ with $\beta > 0$. Then $t_0 = \beta$ and

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2(\beta+1)}\right| \leq \frac{1}{2(\beta+1)}\right\} \cup \Sigma = \sigma(C^{(0,w)}), \quad \text{and}$$

$$\left\{\frac{1}{m}\colon m \in \mathbb{N},\ 1 \leq m < \beta + 1\right\} = \sigma_{pt}(C^{(0,w)}).$$

Back to spaces of analytic functions.

Theorem

The Cesàro operator satisfies

$$\sigma(\mathsf{C}, H(\mathbb{D})) = \sigma_{pt}(\mathsf{C}, H(\mathbb{D})) = \{\frac{1}{m} : m \in \mathbb{N}\}.$$

Persson showed in 2008 the following facts:

For every $m \in \mathbb{N}$ the operator $(\mathsf{C} - \frac{1}{m}) \colon H(\mathbb{D}) \to H(\mathbb{D})$ is not injective because $\mathrm{Ker}(\mathsf{C} - \frac{1}{m}) = \mathrm{span}\{e_m\}$, where $e_m(z) = z^{m-1}(1-z)^{-m}$, $z \in \mathbb{D}$, and it is not surjective because the function $f_m(z) := z^{m-1}$, $z \in \mathbb{D}$, does not belong to the range of $(\mathsf{C} - \frac{1}{m})$.

Back to spaces of analytic functions.

Theorem

The Cesàro operator $\mathsf{C}\colon A^{-\infty} \to A^{-\infty}$ is continuous and

$$\sigma(\mathsf{C}, A^{-\infty}) = \sigma_{pt}(\mathsf{C}, A^{-\infty}) = \{\frac{1}{m}\colon m \in \mathbb{N}\}.$$

The proof of the last two results requires an analysis of the spectrum of the discrete Cesàro operator on power series spaces or duals of power series spaces, that can be described as intersections or unions of weighted spaces $c_0(w)$ for decreasing weights $w = (w_n)_n$ of a special form.

Mean ergodic properties. Definitions

Power bounded operators

An operator $T \in \mathcal{L}(X)$ is said to be *power bounded* if $\{T^m\}_{m=1}^{\infty}$ is an equicontinuous subset of $\mathcal{L}(X)$.

If X is a Banach space, an operator T is power bounded if and only if $\sup_n ||T^n|| < \infty$.

If X is a Fréchet space, an operator T is power bounded if and only if the orbits $\{T^m(x)\}_{m=1}^{\infty}$ of all the elements $x \in X$ under T are bounded. This is a consequence of the uniform boundedness principle.

Mean ergodic properties. Definitions

For $T \in \mathcal{L}(X)$**, we set** $T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m$.

Mean ergodic operators

An operator $T \in \mathcal{L}(X)$ is said to be *mean ergodic* if the limits

$$Px := \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} T^m x, \quad x \in X, \tag{9}$$

exist in X.

If T is mean ergodic, then one then has the direct decomposition

$$X = \mathrm{Ker}(I - T) \oplus \overline{(I - T)(X)}.$$

Mean ergodic properties. Definitions

Uniformly mean ergodic operators

If $\{T_{[n]}\}_{n=1}^{\infty}$ happens to be convergent in $\mathcal{L}_b(X)$ to $P \in \mathcal{L}(X)$, then T is called *uniformly mean ergodic*.

Theorem. Lin. 1974.

Let T a (continuous) operator on a Banach space X which satisfies $\lim_{n\to\infty} ||T^n/n|| = 0$. The following conditions are equivalent:

(1) T is uniformly mean ergodic.

(4) $(I - T)(X)$ is closed.

Ergodic properties of C on classical spaces

Proposition.

- The Cesàro operator $C\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is power bounded and uniformly mean ergodic.
- The Cesàro operator $\mathrm{C}^{(p)}\colon \ell^p \to \ell^p$, $1 < p < \infty$, is not power bounded and not mean ergodic.
- The Cesàro operator $\mathrm{C}^{(0)}\colon c_0 \to c_0$ is power bounded, not mean ergodic.

Mean ergodicity of C on $c_0(w)$

Proposition.

Let w be a decreasing, strictly positive sequence. Then $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$ is power bounded.

Moreover, the following assertions are equivalent:

(i) $\mathsf{C}^{(0,w)}$ is mean ergodic.

(iii) The weight w satisfies $\lim_{n\to\infty} w(n) = 0$.

Uniform mean ergodicity of C on $c_0(w)$

Proposition.

Let w be a decreasing, strictly positive sequence. Then $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$ is uniformly mean ergodic if and only if w satisfies both of the conditions

(i) $\lim_{n\to\infty} w(n) = 0$, and

(ii) $\sup_{n\in\mathbb{N}} w(n+1) \sum_{m=1}^{n-1} \frac{1}{mw(m+1)} < \infty$.

Uniform mean ergodicity of C on $c_0(w)$

Proposition.

If w is a decreasing, strictly positive sequence such that $C^{(0,w)} \in \mathcal{K}(c_0(w))$, then $C^{(0,w)}$ is uniformly mean ergodic.

Examples.

(i) For $w(n) = \frac{1}{(\log(n+1))^\gamma}$ for $n \in \mathbb{N}$ with $\gamma \geq 1$, the operator $C^{(0,w)}$ is not compact, mean ergodic and not uniformly mean ergodic.

(ii) For $w(n) = \frac{1}{n^\beta}$ for $n \in \mathbb{N}$ with $\beta \geq 1$, the operator $C^{(0,w)}$ is uniformly mean ergodic but not compact

Ergodicity of C on $H(\mathbb{D})$ and $A^{-\infty}$

Theorem.

The Cesàro operator C acting on $H(\mathbb{D})$ and on $A^{-\infty}$ is power bounded and uniformly mean ergodic.

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