# Poincaré inequalities in a metric setting

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This talk is based on joint work with:

- Estibalitz Durand-Cartagena (UNED)
- Nageswari Shanmugalingam (University of Cincinnati)

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- It has been realized that metric spaces, possibly endowed with some additional features, are a natural setting for many problems in analysis and geometry.

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    - Harmonic Analysis
    - Quasiconformal Mapping Theory
    - Nonlinear Potential Theory
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2. The interest on relevant types of spaces which are very different from the classical euclidean or riemannian cases. For example:
    - Carnot Groups
    - Fractals

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- The Heisenberg group $\mathbb{H}$ is the group of matrices of the form

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- Topologically, $\mathbb{H} \simeq \mathbb{R}^3$, so we can consider the Heisenberg group as the space $\mathbb{R}^3$ endowed with a special group operation and a special metric.

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- For $p, q \in \mathbb{R}^3$, the *Carnot-Carathéodory distance* is given by

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- A piecewise smooth path $\gamma : [a, b] \to \mathbb{R}^3$ is *admissible* if

$$\gamma'(t) \in H_{\gamma(t)}$$

for almost all $t \in [a, b]$.

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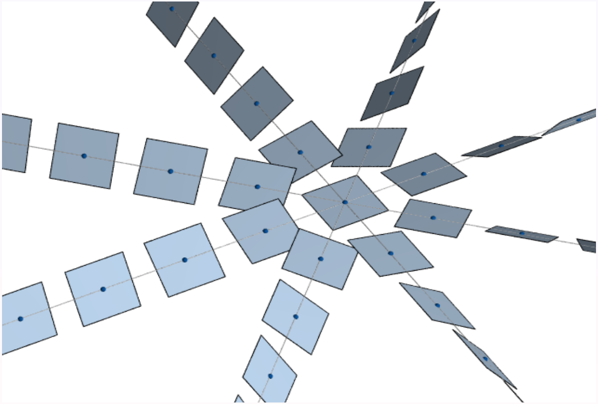

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- From Chow Theorem it follows that every pair of points can be joined by an admissible path.

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- The Hausdorff dimension of Heisenberg group is 4.

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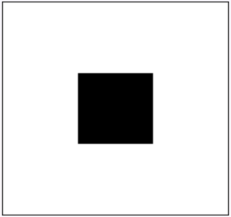

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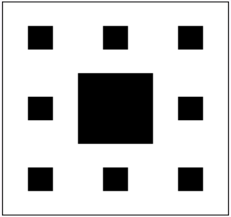

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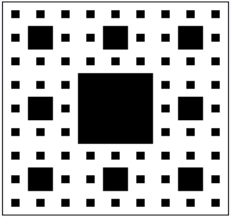

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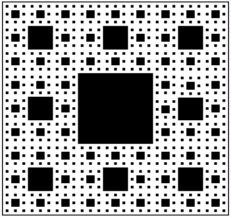

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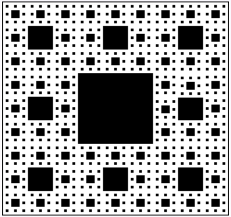
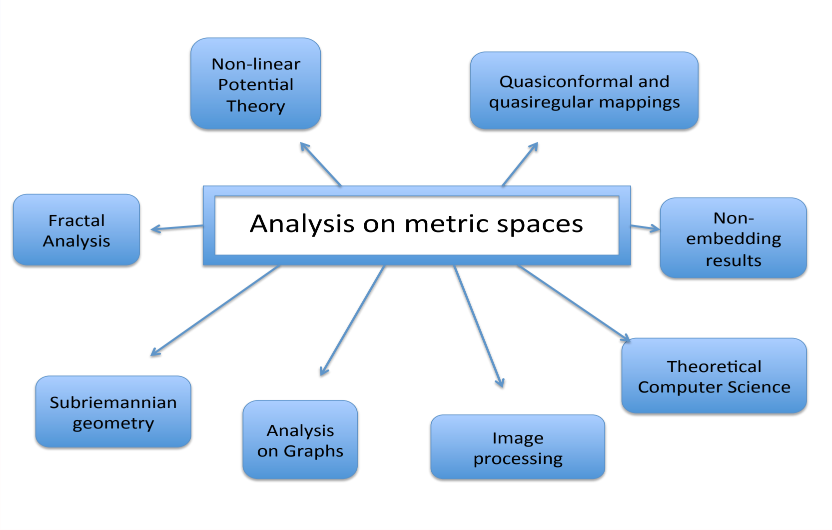


- We endow the Sierpinski Carpet with euclidean distance.
- The Hausdorff dimension is $\frac{log 8}{log 3}$.

# Motivation


Non-linear Potential Theory
Quasiconformal and quasiregular mappings
Fractal Analysis
Analysis on metric spaces
Non-embedding results
Subriemannian geometry
Analysis on Graphs
Image processing
Theoretical Computer Science

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We say that $(X, d, \mu)$ is *doubling* if there is a constant $C \geq 1$ such that, for every $x \in X$ and every $r > 0$:

$$0 < \mu(B(x, 2r)) \leq C \cdot \mu(B(x, r)) < +\infty$$

# Our setting

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- It can be shown that if a complete metric space supports a doubling measure, then it is locally compact.

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- But this class of spaces is too general to allow a 1-st order calculus.

# Poincaré inequalities

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- One needs plenty of curves, well distributed along the space.
- One way to make this idea precise is to assume that the space supports a *p-Poincaré inequality*.

# Classical Poincaré inequalities

**Classical Poincaré Inequality:** there exists a dimensional constant $C > 0$ such that, for each ball $B$ in the euclidean space $\mathbb{R}^n$ and all functions $f \in W^{1,p}(B)$, we have:

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- The path $\gamma : [a, b] \to X$ is *rectifiable* if $\ell(\gamma) < \infty$, where

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- If $f$ is a smooth function defined on $\mathbb{R}^n$ or on a riemannian manifold, then $|\nabla f|$ is an upper gradient of $f$.

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- Also the *pointwise Lipschitz constant* of $f$:

$$g(x) = \mathrm{Lip}\, f(x) := \mathrm{lím\,sup}_{y \to x} \frac{|f(x) - f(y)|}{d(x, y)}$$

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$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq C \, r \, \|g\|_{L^\infty(B(x,\lambda r))} \qquad (\text{ for } p = \infty)$$

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- Carnot groups (and, in particular, Heisenberg group) admit a 1-P.I.
- Gromov-Haussdorf limits of measured spaces with a $p$-P.I. also satisfy a $p$-P.I.

# A scale of properties

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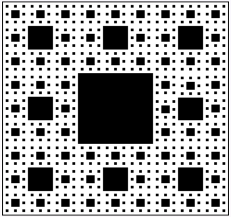
- Thus the $\infty$-Poincaré inequality is the weakest one.
- Given $1 \leq q < p$ it is possible to construct spaces supporting a $p$-Poincaré inequality, but not a $q$-Poincaré inequality.
- **Theorem** (Keith and Zhong, 2008). Let $X$ be a complete metric space equipped with a doubling measure satisfying a $p$-Poincaré inequality for some $1 < p < \infty$. Then there exists $\varepsilon > 0$ such that $X$ supports a $q$-Poincaré inequality for all $q > p - \varepsilon$.

# Examples: Sierpinski Carpet



- The Sierpinski Carpet does not admit $\infty$-Poincaré inequality.

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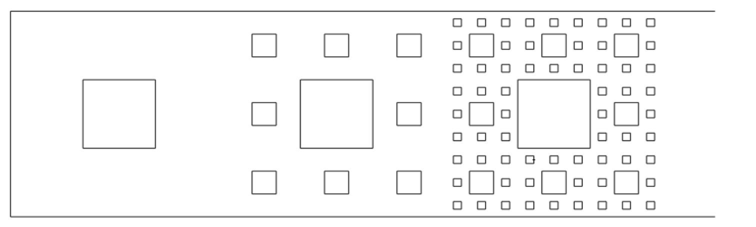

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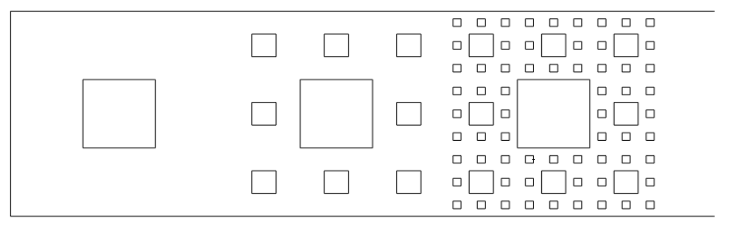

- The Sierpinski Strip admits $\infty$-Poincaré inequality, but admits no $p$-Poincaré inequality for $1 < p < \infty$.

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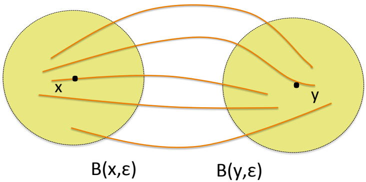
- If a metric measure space $X$ supports a $p$-Poincaré inequality ($1 \leq p \leq \infty$), then $X$ is connected.
- (Cheeger, Semmes, 1999) Every complete metric space $X$ supporting a doubling measure and a $p$-Poincaré inequality ($1 \leq p < \infty$) is quasiconvex.

# Geometric implications: quasiconvexity

- If a metric measure space $X$ supports a $p$-Poincaré inequality $(1 \leq p \leq \infty)$, then $X$ is connected.
- (Cheeger, Semmes, 1999) Every complete metric space $X$ supporting a doubling measure and a $p$-Poincaré inequality $(1 \leq p < \infty)$ is quasiconvex.
- A metric space $(X, d)$ is *quasiconvex* if there exists a constant $C \geq 1$ such that, for every $x, y \in X$, there is a path $\gamma$ in $X$ from $x$ to $y$, with length $\ell(\gamma) \leq C\, d(x, y)$.

# Thick quasiconvexity

A metric measure space will be said *thick quasiconvex* if every pair of sets of positive measure, which are a positive distance apart, can be connected by a "thick" family of quasiconvex paths.



# Modulus of paths

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- If some property holds for all such paths, except for a set $\Gamma$ with $\mathrm{Mod}_p \Gamma = 0$, we say that the property holds for *p-a.e. path*.
- If a set of paths $\Gamma$ satisfies that $\mathrm{Mod}_p \Gamma > 0$, we say that $\Gamma$ is *p*-thick.

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- A metric measure space $(x, d, \mu)$ is *p-thick quasiconvex* if there exists $C \geq 1$ such that for every $x, y \in X$, every $0 < \varepsilon < \frac{1}{4}d(x, y)$, and all measurable sets $E \subset B(x, \varepsilon)$, $F \subset B(y, \varepsilon)$ satisfying $\mu(E)\mu(F) > 0$ we have that

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$$\mathrm{Mod}_p(\Gamma(E, F, C)) > 0,$$

where $\Gamma(E, F, C)$ denotes the set of paths $\gamma_{p,q}$ connecting $p \in E$ and $q \in F$ with $\ell(\gamma_{p,q}) \leq C\, d(p, q)$.

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- A complete, $p$-thick quasiconvex, doubling metric measure space, is quasiconvex.

# Thick quasiconvexity

- (Durand-Cartagena, J., Shamungalingam, Williams, 2011-2012) In a complete metric space with a doubling measure, for $1 \leq p \leq \infty$:

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- Durand-Cartagena, Shamungalingam and Williams (2012). In a complete metric space with a doubling measure, for $1 \leq p < \infty$:

$$p\text{-thick quasiconvexity} \not\Rightarrow p\text{-Poincaré inequality}$$

# Transversal paths

A path $\gamma$ in a metric measure space $X$ is *transversal* to a subset $E \subset X$ if $\gamma$ intersects $E$ on a set of zero-length, in the sense that:

$$\mathcal{L}^1(\{t : \gamma(t) \in E\}) = 0.$$

# Geometric characterization

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**Theorem** (Durand-Cartagena, J., Shanmugalingam, 2016). Let $X$ be a locally complete, doubling metric measure space. The following conditions are equivalent:

(a) $X$ supports an $\infty$-Poincaré inequality.

(b) $X$ is $\infty$-thick quasiconvex.

(c) There is a constant $C \geq 1$ such that, for every null set $N \subset X$ and for every pair $x, y \in X$, there exists a path $\gamma$ in $X$ transversal to $N$, connecting $x$ and $y$ and such that

$$\ell(\gamma) \leq C\, d(x, y).$$

¡MUCHAS GRACIAS!

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