Almost everywhere convergence of Fourier Series and Kalton's lattice constants

María J. Carro

University of Barcelona

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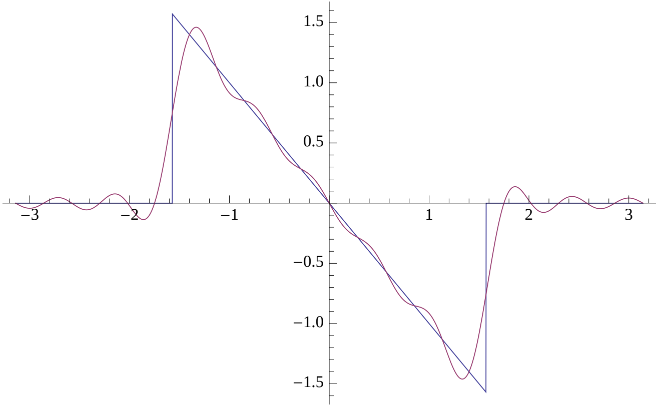
Abstract

We shall review some historical results on the problem of almost everywhere convergence of Fourier series and present some recent developments on this topic using the so called Kalton lattice constants.



N. Kalton

Introduced his lattice constants in order to study when a space contains uniformly complemented ℓ_2^n.

Kalton's constant

Quasi-Banach constant

Let $(X, \|\cdot\|_X)$ be a quasi-Banach function space. Then:

$$\Big\| \sum_{i=1}^{n} f_i \Big\|_X \le c_n(X) \sum_{i=1}^{n} \|f_i\|_X .$$

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Examples

1. If $1 < p \leq \infty$, $c_n(L^p) = 1$, for every n.
2. If $p \leq 1$, $c_n(L^p) = n^{\frac{1}{p}-1}$
3. If

$$L^{1,\infty} = \Big\{f : \|f\|_{L^{1,\infty}} = \sup_{y>0} y|\{x : |f(x)| > y| < \infty\Big\},$$

then $c_n(L^{1,\infty}) = 1 + \log n$.

Kalton's constant

Lattice constants

In 1993, N. Kalton defined $e_n(X)$ to be the least constant such that,

$$\Big\| \sum_{i=1}^{n} f_i \Big\|_X \leq e_n(X) \max_{1 \leq i \leq n} \|f_i\|_X,$$

and $d_n(X)$ the least constant so that

$$\sum_{i=1}^{n} \Big\| f_i \Big\|_X \leq d_n(X) \Big\| \sum_{i=1}^{n} f_i \Big\|_X$$

for every collection of pairwise disjoint functions.

N.J. Kalton, Lattice Structures on Banach Spaces, Memoirs Amer. Math. Soc. Vol. 103, No 493, 1993.

Properties of Kalton's constant

Under certain conditions on X,

1. $d_n(X) = e_n(X^*)$.
2. $d_n(X^*) = e_n(X)$.
3. $$\underline{\lim} \frac{d_n}{n^\alpha} = 0 \implies X \text{ is } \frac{1}{1-\alpha} \text{ concave.}$$
4. If $\underline{\lim} \frac{d_n}{\sqrt{\log n}} = 0$, then X does not contain uniformly complemented copies of ℓ_n^2.
5. etc.

Some similar constants

On many interesting examples, $e_n(X)$ coincide with the least constant satysfying

$$\Big\| \sum_{i=1}^{n} f_i \Big\|_X \leq e_n(X),$$

for every collection of pairwise disjoint functions so that $\|f_i\|_X = 1$, and in fact, if we define $f_n(x)$ as the least constant satisfying

$$\Big\| \sum_{i=1}^{n} \frac{\chi_{A_i}}{\|\chi_{A_i}\|_X} \Big\|_X \leq f_n(X),$$

we also have that $e_n(X) = f_n(X)$.

Mastylo, Mieczysaw, *Lattice structures on some Banach spaces*. Proc. Amer. Math. Soc. 140 (2012), no. 4, 1413–1422.

Embedding of spaces

A very easy and useful remark

By definition of $f_n(X)$, given n, there exists $\{A_1^n, A_2^n, \cdots, A_n^n\}$ a collection of pairwise disjoint sets so that

$$\frac{f_n(X)}{2} \leq \Big\| \sum_{i=1}^{n} \frac{\chi_{A_i^n}}{\|\chi_{A_i^n}\|_X} \Big\|_X$$

Embedding of spaces

Theorem

Let Y be a Banach space. Then

$$Y \subset X \quad \Longrightarrow \quad \frac{f_n(X)}{n} \lesssim \sup_i \frac{\|\chi_{A_i^n}\|_Y}{\|\chi_{A_i^n}\|_X}, \ \forall n.$$

Proof

$$\begin{aligned} \frac{f_n(X)}{2} &\leq \Big\| \sum_{i=1}^n \frac{\chi_{A_i^n}}{\|\chi_{A_i^n}\|_X} \Big\|_X \lesssim \Big\| \sum_{i=1}^n \frac{\chi_{A_i^n}}{\|\chi_{A_i^n}\|_X} \Big\|_Y \\ &\lesssim n \sup_i \frac{\|\chi_{A_i^n}\|_Y}{\|\chi_{A_i^n}\|_X}. \quad \square \end{aligned}$$

Connection with Fourier series

Jean Baptiste Joseph Fourier (1768–1830).

Fourier introduced this series for the purpose of solving the heat equation on a metal plate and his initial results were published in 1807.

The almost everywhere convergence of the Fourier series

Let $\mathbb{T} \equiv [0,1)$ and let $f \in L^1(\mathbb{T})$. Then, the Fourier coefficients are defined as

$$\widehat{f}(k) = \int_0^1 f(x) e^{-2\pi ikx}\, dx, \qquad k \in \mathbb{Z},$$

The almost everywhere convergence of the Fourier series

Let $\mathbb{T} \equiv [0, 1)$ and let $f \in L^1(\mathbb{T})$. Then, the Fourier coefficients are defined as

$$\widehat{f}(k) = \int_0^1 f(x) e^{-2\pi i k x}\, dx, \qquad k \in \mathbb{Z},$$

and the Fourier series of f at $x \in \mathbb{T}$ is given by

$$S[f](x) = \sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{2\pi i k x}.$$

The problem of the almost everywhere convergence of the Fourier series consists in studying when:

$$\lim_{n\to\infty} S_n f(x) = f(x), \qquad \text{a.e. } x \in \mathbb{T}$$

where

$$S_n f(x) = \sum_{|k|\le n} \widehat{f}(k) e^{2\pi i k x}.$$

Paul du Bois-Reymond (1831–1889).

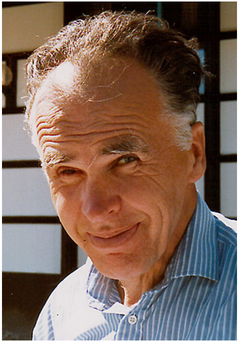
A Negative Result

In 1873, du Bois-Reymond constructed a continuous function whose Fourier series is not convergent at a point.

Andrey Kolmogorov (1903 –1987).

Another Negative Result

In 1922, Andrey Kolmogorov gave an example of a function $f \in L^1(\mathbb{T})$ whose Fourier series diverges almost everywhere.



Lennart Carleson (1928–).

The Positive Result

In 1966, L. Carleson proved that if $f \in L^2(\mathbb{T})$, then

$$\lim_{n\to\infty} S_n f(x) = f(x), \qquad \text{a.e. } x \in \mathbb{T}.$$

In fact:

Carleson-Hunt, 1967

If $f \in L^p(\mathbb{T})$, for some $p > 1$, then

$$\lim_{n\to\infty} S_n f(x) = f(x), \qquad \text{a.e. } x \in \mathbb{T}.$$

In fact:

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If $f \in L^p(\mathbb{T})$, for some $p > 1$, then

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Open question today:

To characterize the space of all integrable functions for which the almost everywhere convergence of the Fourier series holds.

Carleson maximal operator

Let S be the Carleson maximal operator,

$$Sf(x) = \sup_{n \in \mathbb{N}} |S_n f(x)|.$$

Essentially, to solve the problem of the almost everywhere convergence in a space X, we have to prove that

$$S : X \longrightarrow L^0$$

is bounded, where L^0 is the set of measurable functions with the convergence in measure topology.

Carleson-Hunt main estimate

For every measurable set $E \subset \mathbb{T}$ and every $y > 0$,

$$\Big|\{x \in [0,1] : |S\chi_E(x)| > y\}\Big|^{1/p} \leq \frac{C}{y(p-1)}|E|^{1/p}, \qquad p > 1,$$

Consecuencia:

For every $p > 1$,

$$S : L^p \longrightarrow L^p, \qquad \frac{C}{p-1}.$$

The situation up to now

For every $1 < p < 2 < q$,

$$L^q(\mathbb{T}) \subset L^2(\mathbb{T}) \subset L^p(\mathbb{T}) \subset \boxed{??????} \subset L^1(\mathbb{T})$$

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All the above spaces are of the same form:

$$\int_0^1 |f(x)| D(|f(x)|) dx < \infty.$$

Open question:

Find the best D such that the almost everywhere convergence of the Fourier series holds true for every f satisfying:

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Notation:

$$\log_{(1)} y = \log y, \quad \log_{(n)} y = \log_{(n-1)} \log y, \qquad y >> 1.$$

Observe that

$$\log_{(n)} y \leq \log_{(n-1)} y.$$

Possible and non possible functions D

Possible and non possible functions D

State of the art:

$$1 \leq \sqrt{\frac{\log y}{\log_{(2)} y}}$$

$$\leq \boxed{\log y} \leq \log y \log_{(3)} y$$

$$\leq \log y \log_{(2)} y \leq (\log y)^2 \leq y^{\varepsilon}$$

Possible and non possible functions D

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Y. Antonov, 1996:

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S. V. Konyagin, 2004:

$$D(y) = \sqrt{\frac{\log y}{\log_{(2)} y}}.$$

Yano's contribution, 1951

$L(\log L)^2$

If T is a sublinear operator such that

$$T : L^p \longrightarrow L^p, \qquad \frac{C}{(p-1)^m},$$

then

$$T : L(\log L)^m \longrightarrow L^1.$$

Corollary

$$S : L(\log L)^2 \longrightarrow L^1.$$

Yano, Shigeki Notes on Fourier analysis. An extrapolation theorem. J. Math. Soc. Japan 3, (1951). 296–305.

Antonov's contribution

Carleson-Hunt's estimate, 1967

For every measurable set $E \subset \mathbb{T}$ and every $y > 0$,

$$\|S\chi_E\|_{L^{1,\infty}} \lesssim |E|\left(1 + \log \frac{1}{|E|}\right) := D(|E|)$$

1996

For every bounded function by 1 and every $y > 0$,

$$\|Sf\|_{L^{1,\infty}} \lesssim D(\|f\|_1)$$

Antonov's contribution

For every measurable function f,

$$f = \sum_{n\in\mathbb{Z}} 2^{2^{n+1}} f_n(x)$$

where

$$f_n(x) = \frac{f(x)\chi_{\{2^{2^n}\leq|f|<2^{2^{n+1}}\}}(x)}{2^{2^{n+1}}}$$

Corollary

$$S : L\log L\log_3 L \longrightarrow L^{1,\infty}.$$

Antonov, N. Yu. Convergence of Fourier series. East J. Approx. 2 (1996), no. 2, 187–196

Arias de Reyna's contribution, 2004

Let $f \in L^1(\mathbb{T})$ and let us write

$$f = \sum_n f_n, \qquad f_n \in L^\infty.$$

Then:

$$|Sf| \le \sum_n |Sf_n| = \sum_n \|f_n\|_\infty \left| S\Big(\frac{f_n}{\|f_n\|_\infty}\Big)\right|$$

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Using Stein-Weiss Lemma

$$\begin{aligned} ||Sf||_{1,\infty} &= \sum_n (1+\log n)||f_n||_\infty \Big\| S\Big(\frac{f_n}{||f_n||_\infty}\Big)\Big\|_{1,\infty} \\ &\lesssim \sum_n (1+\log n)||f_n||_\infty D\Big(\frac{||f_n||_1}{||f_n||_\infty}\Big). \end{aligned}$$

Definition

A measurable function $f \in L^0$ belongs to QA if there exists a sequence $(f_n)_{n=1}^{\infty}$, with $f_n \in L^\infty$, such that

$$f = \sum_{n=1}^{\infty} f_n, \quad a.e.$$

and

$$\sum_{n=1}^{\infty} (1 + \log n) \|f_n\|_\infty D\left(\frac{\|f_n\|_1}{\|f_n\|_\infty} \right) < \infty.$$

We endow this space with the quasi-norm

$$\|f\|_{QA} = \inf \left\{ \sum_{n=1}^{\infty} (1 + \log n) \|f_n\|_\infty D\left(\frac{\|f_n\|_1}{\|f_n\|_\infty} \right) \right\}.$$

The best result nowadays:

Theorem

For every function $f \in QA$,

$$\lim_{n\to\infty} S_n f(x) = f(x),$$

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Recovering Antonov's result:

$$L \log L \log_{(3)} L(\mathbb{T}) \subset QA$$

Arias-de-Reyna, J. Pointwise convergence of Fourier series. J. London Math. Soc. (2) 65 (2002), no. 1, 139–153.

Questions in the air:

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General question

Find the best Lorentz space contained in QA.

Kalton's constants

Functional properties of QA

Kalton's constants

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- QA is a quasi-Banach r.i. space.

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Kalton's constants

Functional properties of QA

- QA is a quasi-Banach r.i. space.
- $QA \hookrightarrow L\log L(\mathbb{T})$.
- The Banach envelope of QA is $L\log L(\mathbb{T})$.
- QA is logconvex in the sense of Kalton; that is:

$$\|x_1 + \ldots + x_n\| \leq C \sum_{j=1}^{n} (1 + \log j)\|x_j\| .$$

C., M.J.; Mastylo, M.; Rodríguez-Piazza, L. *Almost everywhere convergent Fourier series.* J. Fourier Anal. Appl. 18 (2012), no. 2, 266–286.

Kalton's constants

Theorem

For each positive integer n,

$$c\, n(1+\log n) \leq e_n(QA) \approx f_n(QA) \leq n(1+\log n)\,.$$

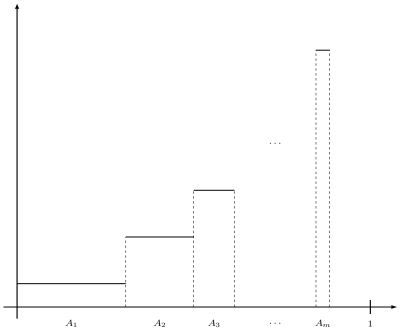
Moreover:

For each positive integer n,

$$c\, n(1+\log n) \leq \Big\| \sum_{i=1}^{n} \frac{\chi_{A_i^n}}{\|\chi_{A_i^n}\|_X} \Big\|_{QA}$$

with A_i^n as follows:

Kalton's constants



where, if $\mu_j = |A_j|$, then

$$\mu_{j+1} = \left(\frac{\mu_j}{e}\right)^{n^3(1+\log n)}, \qquad j = 1, \cdots, n.$$

$L\log L\log_{(4)} L(\mathbb{T}) \subset QA$?

Theorem

Let Y be a Banach space. Then

$$Y \subset X \quad \Longrightarrow \quad \frac{f_n(X)}{n} \lesssim \sup_i \frac{\|\chi_{A_i^n}\|_Y}{\|\chi_{A_i^n}\|_X}, \; \forall n.$$

Is it true that?

$$(1 + \log n) \lesssim \sup_i \frac{\|\chi_{A_i^n}\|_Y}{\|\chi_{A_i^n}\|_X}, \; \forall n.$$

Answers on the earth:

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$$L \log L \log_{(4)} L(\mathbb{T}) \not\subseteq QA.$$

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The conjecture is false:

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is not the best Lorentz space embedded in QA.

In fact, more can be said:

Answers on the earth:

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The conjecture is false:

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In fact, more can be said:

Answer to the general question:

There is not the best Lorentz space contained in QA.

Carleson maximal operator

Let S be the Carleson maximal operator,

$$Sf(x) = \sup_{n \in \mathbb{N}} |S_n f(x)|.$$

Essentially, to solve the problem of the almost everywhere convergence in a space X, we have to prove that

$$S : X \longrightarrow L^0$$

is bounded, where L^0 is the set of measurable functions with the convergence in measure topology.

Carleson maximal operator

Let S be the Carleson maximal operator,

$$Sf(x) = \sup_{n\in\mathbb{N}} |S_n f(x)|.$$

Essentially, to solve the problem of the almost everywhere convergence in a space $X \subset L^1$, we have to prove that

$$S : X \longrightarrow L^{1,\infty}$$

is bounded.

A bigger space

Let
$$S(L^{1,\infty}) = \{f : Sf \in L^{1,\infty}\}$$
with the norm
$$\|f\|_{S(L^{1,\infty})} = \|Sf\|_{L^{1,\infty}}.$$

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Corollary

i) If $f \in S(L^{1,\infty})$,

$$\lim_n S_n f(x) = f(x), a.e.x$$

ii)

$$QA \subset S(L^{1,\infty})$$

Our next goal: to study the space $S(L^{1,\infty})$

Properties

i) It is a quasi-Banach space.
ii)
$$f_n(S(L^{1,\infty})) \leq e_n(S(L^{1,\infty})) \lesssim n(1+\log n)$$

Open questions

i) $n(1+\log n) \lesssim f_n(S(L^{1,\infty}))$?

ii) How to construct A_i^n ?

The end

THANKS FOR YOUR ATTENTION