Spaces contained in c_0 and spaces not containing c_0

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Motivation

A Banach space X has the *point of continuity property* (PCP) if for every $\varepsilon > 0$ and every bounded nonempty set $A \subset X$ there is a weak open set U such that $A \cap U \neq \emptyset$ and $\text{diam}(A \cap U) < \varepsilon$.

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Note that if X has the PCP, then $(A)'_\varepsilon \subsetneq A$ where

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and thus iterating this *set derivation* we may arrive to the empty set, after an ordinal number of steps called Szlenk index.

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To contain a copy of c_0 is an strong form of fail the RNP or the PCP.

Problem: *look for a set derivation that recognizes the presence of copies of c_0.*

An ordinal index

Let X be an infinite dimensional Banach space and $A \subset X$.

Essential inner radius

$$\varrho(A) = \sup\{r \geq 0 : \exists x \in A, \exists Y \subset X \text{ fin. codim. with } x + rB_Y \subset A\}$$

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If A is convex closed, then $[A]'_\varepsilon$ is again convex closed.

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The index Gz

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Finally, we take $Gz(B_X) = \sup_{\varepsilon > 0} Gz(B_X, \varepsilon)$.

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- If $Gz(B_X, \varepsilon_0) > \varepsilon_0^{-1} + 1$ for some $\varepsilon_0 \in (0, 1)$, then

$$Gz(B_X, \varepsilon) \geq c\varepsilon^{-p}$$

  for some $c > 0$, $p > 1$ and every $\varepsilon \in (0, 1)$.

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The last property is related to the construction of asymptotically uniformly smooth equivalent norms with associated modulus of power type (Knaust, Odell and Schlumprecht 1999).

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If T is the Tsirelson space, then $Gz(B_{T^*}) = \omega^\omega$.

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Assume that X is separable. Then $Gz(B_X) = \infty$ if and only if X is isomorphic to a subspace of c_0.

Therefore, we will give up on finding a set derivation to recognize copies of c_0 (we have achieved exactly the opposite!)

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The comparison with the weak* dentability, which characterizes the RNP in dual Banach spaces, suggest us to look for another sort of property of Banach spaces with no copies of c_0.

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Moreover, we can provide a "basis free" approach to several result involving c_0, as for instance, James' nondistortability of c_0.

References

- A. Pérez, M. Raja, A Bourgain-like property of Banach spaces with no copies of c_0, *RACSAM*, to appear 2016.
- M. Raja, On asymptotically uniformly smooth Banach spaces, *J. Funct. Anal.* 264 (2013), no. 2, 479–492.

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