Geometric clustering in the normed plane

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Geometric clustering

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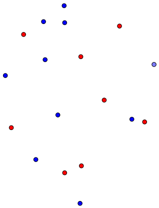

Geometric clustering

$k = 1$, minimizing the radius of a enclosing disc:

- Elzinga-Hearn and Shamos-Hoey (Euclidean plane).
- Alonso-Martini-Spirova and Jahn (general normed plane).

$k = 2$, minimizing the maximum Euclidean diameter of the clusters:

- Avis, $O(n^2 \log n)$.
- Asano-Bhattacharya-Keil-Yao, $O(n \log n)$.

$k = 2$, minimizing the sum of the two Euclidean diameters:

- Monma-Suri, $O(n^2)$.

$k = 2$, μ a measure, $\mu_1 > 0$ and $\mu_2 > 0$, splitting S into two clusters A and B such that $\mu(A) \leq \mu_1$ and $\mu(B) \leq \mu_2$:

- Hershberger and Suri,
  - $\mu =$Euclidean diameter, $O(n \log n)$.
  - $\mu =$area, perimeter, or diagonal of the smallest rectangle with sides parallel to the coordinates axes ($O(n \log n)$ time).
  - $\mu =$radius of the smallest enclosing sphere with the norms L_1 ($O(n \log n)$ time) or the Euclidean norm ($O(n^2 \log n)$ time)

Geometric clustering

$k = 2$, the 2-center problem: cover S by (the union of) two congruent closed disks whose radius is as small as possible.

- Eppstein and Sharir (1997), near linear time cost (Euclidean case).

$k = 3$, minimizing the maximum Euclidean diameter

- Hagauer-Rote, $O(n^2 \log^2 n)$

Any k, minimizing any monotone function $\mathcal{F}$ ($\mathcal{F} : \mathbb{R}^k \to \mathbb{R}$) of the Euclidean diameters or the Euclidean radii of the clusters.
Examples of $\mathcal{F}$:

- The sum of the diameters (or the radii)
- The maximum of the diameters (or the radii)
- The sum of the squares of the diameters (or the radii).

- Capoyleas-Rote-Woeginger, polynomial time.

Hagauer-Rote and Capoyleas-Rote-Woeginger obtain their results from this theorem

Theorem (Capoyleas-Rote-Woeginger)

Let A and B be two sets of points in the Euclidean plane. Then, there are two linearly separable sets A' and B' such that $\mathrm{diam}(A') \leq \mathrm{diam}(A)$, $\mathrm{diam}(B') \leq \mathrm{diam}(B)$, *and* $A' \cup B' = A \cup B$.

Figure: Non linearly separable (left) and linarly separable sets (right)

sets 2.pdf

Linear separation of clusters

This first statement is used in the proof of the Theorem:
In every triangle with an obtuse angle, the side lying opposite to the obtuse angle is the (Euclidean) longest side in the triangle.

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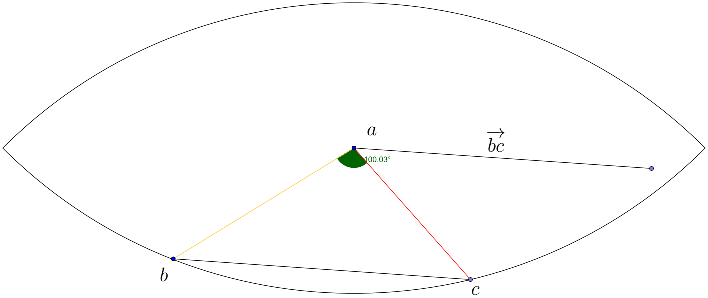


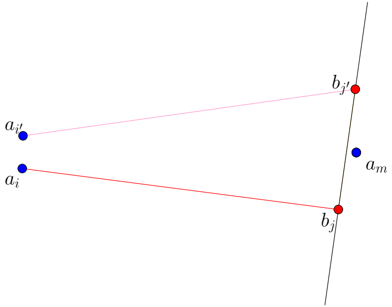
Figure: The side opposite to the obtuse angle is not the longest side in in the triangle $\triangle abc$.

Linear separation of clusters

This second statement is used in the proof of Theorem:

1.$\mathrm{diam}(A) \geq \mathrm{diam}(B)$
2.$\{a_i, a_i', a_m\} \subset A, \{b_j, b_j'\} \subset B$
Clockwise order: $a_{i'}, b_{j'}, a_m, b_j, a_i$
3. $< b_j, b_{j'} >$ separates $\{a_i, a_{i'}\}$ from a_m.

$\Longrightarrow_{(\mathbb{E}^2)}$ $\{\|a_i - b_j\|, \|a_{i'} - b_{j'}\|\} \leq \mathrm{diam}(A)$.



Linear separation of clusters

But this point configuration is possible in a general normed plane:

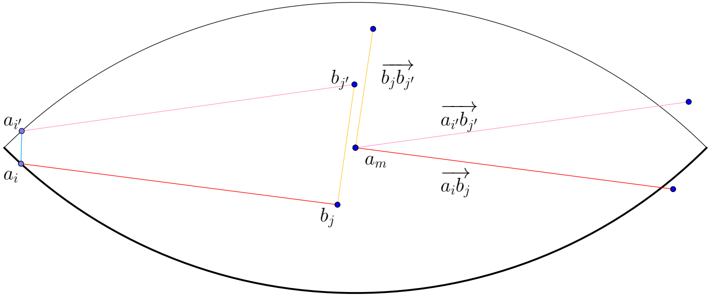
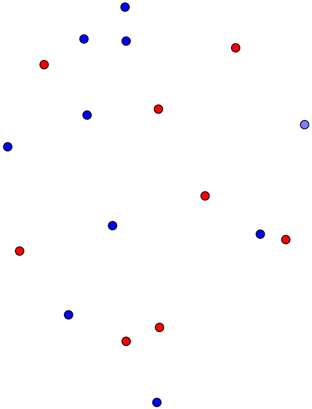


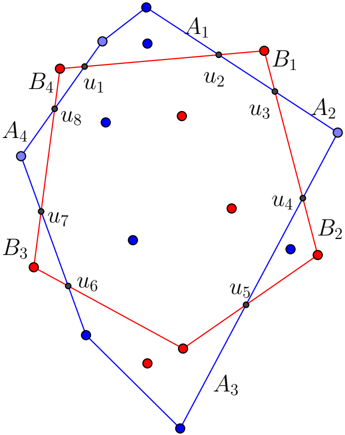
Figure: $\|a_i - b_j\|$ and $\|a_{i'} - b_{j'}\|$ are longer than the diameter of A.

Objective: to prove the Theorem for any normed plane.



Linear separation of clusters

Step 1: $\{u_1, u_2, \ldots, u_{2k}\} = \partial(\mathrm{conv}(A)) \cap \partial(\mathrm{conv}(B))$.



We can assume that $\mathrm{diam}(A) \geq \mathrm{diam}(B)$

We say that...

- (A_i, B_j) is a *bad pair* if $\mathrm{diam}(A_i \cup B_j) > \mathrm{diam}(A)$.

  Then, A_i and B_j are *bad partners*.

- $a_i \in A_i$ and $b_j \in B_j$ are *bad points* if $\|a_i - b_j\| > \mathrm{diam}(A)$.

  Then, a_i and b_j are *bad partners*,

  and the segment $\overline{a_i b_j}$ is a *bad segment*.

Lemma

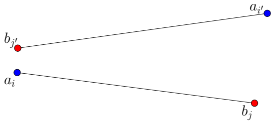
Let (A_i, B_j) and $(A_{i'}, B_{j'})$ two disjoint bad pairs. Let us choose $a_i \in A_i, b_j \in B_j, a_{i'} \in A_{i'}, b_{j'} \in B_{j'}$ such that $\overline{a_i b_j}$ and $\overline{a_{i'} b_{j'}}$ are bad segments. Then, either these bad segments intersect, or any point $a \in A_m$ belonging to the halfplane defined by $< b_j b_{j'} >$ where a_i and $a_{i'}$ are not contained, is not bad.

Lemma

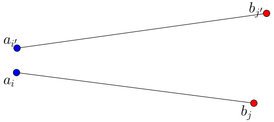
Let (A_i, B_j) and $(A_{i'}, B_{j'})$ two disjoint bad pairs. Let us choose $a_i \in A_i, b_j \in B_j, a_{i'} \in A_{i'}, b_{j'} \in B_{j'}$ such that $\overline{a_i b_j}$ and $\overline{a_{i'} b_{j'}}$ are bad segments. Then, either these bad segments intersect, or any point $a \in A_m$ belonging to the halfplane defined by $< b_j b_{j'} >$ where a_i and $a_{i'}$ are not contained, is not bad.

Skecth of the proof. Possible clockwise order (up to symmetries):

Case 1: $a_i, b_{j'}, a_{i'}, b_j$



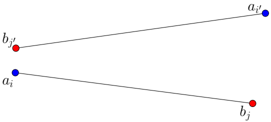
Case 2: $a_i, a_{i'}, b_{j'}, b_j$



Linear separation of clusters

Case 1: clockwise order

$a_i, b_{j'}, a_{i'}, b_j$



We get a contradiction:

$$\mathrm{diam}(A) + \mathrm{diam}(B) \geq \|a_i - a_{i'}\| + \|b_j - b_{j'}\| \geq \|a_i - b_j\| + \|a_{i'} - b_{j'}\| > 2\,\mathrm{diam}(A).$$

Linear separation of clusters

Case 2: clockwise order $a_i, a_{i'}, b_{j'}, b_j$:

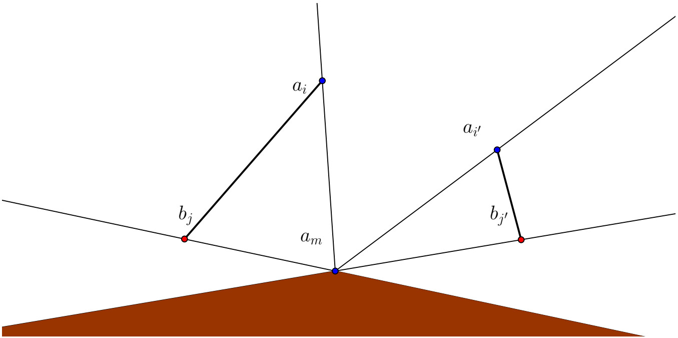


Figure: (a_i, b_j), $(a_{i'}, b_{j'})$ are bad partners $\Longrightarrow \nexists$ any bad partner for a_m

Linear separation of clusters

Case 2: clockwise order $a_i, a_{i'}, b_{j'}, b_j$:

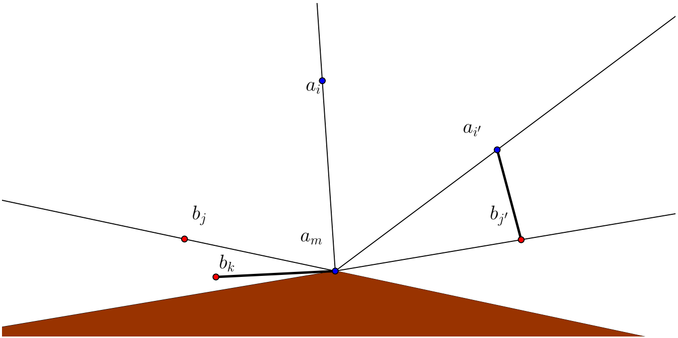
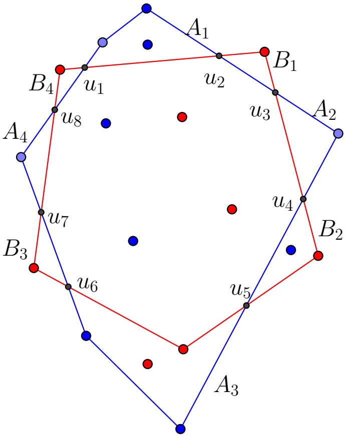


Figure: (a_i, b_j) and $(a_{i'}, b_{j'})$ bad partners $\Longrightarrow \nexists$ any bad partner for a_m

Step 2: Maximal cyclic subsequences of polygons.



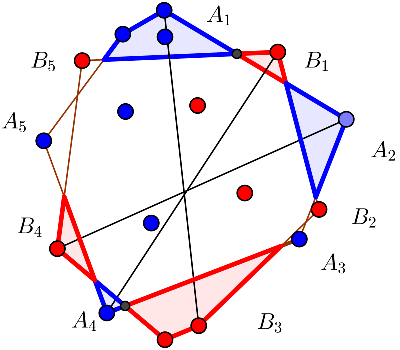
Linear separation of clusters

Step 2: Maximal cyclic subsequences of polygons.

- Consider maximal cyclic subsequences of adjacent bad polygons A_i.
  - No "good" polygon A_k belongs to one of this maximal cyclic subsequences of bad A_i-polygons.
  - Some intervening "good" polygon B_j can belong to this maximal cyclic subsequences of A_i-polygons.
- Similarly with adjacent bad polygons B_j.
- These maximal cyclic sequences are noted by $\bar{\mathbf{A}}_1, \bar{\mathbf{A}}_2, \ldots, \bar{\mathbf{A}}_p$ and $\bar{\mathbf{B}}_1, \bar{\mathbf{B}}_2, \ldots, \bar{\mathbf{B}}_q$.

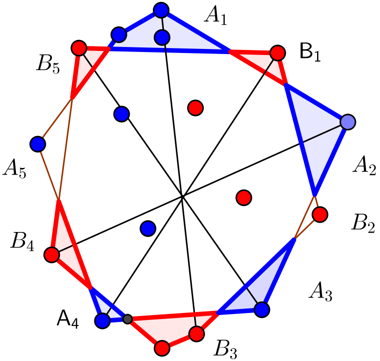
Example with 3 maximal cyclic subsequences of A_i-polygons and 3 maximal subsequences of B_j-polygons:

$\bar{\mathbf{A}}_1 = \{A_1\}$
$\bar{\mathbf{B}}_1 = \{B_1\}$
$\bar{\mathbf{A}}_2 = \{A_2\}$
$\bar{\mathbf{B}}_2 = \{B_3\}$
$\bar{\mathbf{A}}_3 = \{A_4\}$
$\bar{\mathbf{B}}_3 = \{B_5\}$



Example with 3 maximal cyclic subsequences of A_i-polygons, 3 maximal subsequences of B_j-polygons, and "good" intervening polygons:

$\bar{\mathbf{A}}_1 = \{A_1\}$
$\bar{\mathbf{B}}_1 = \{B_1\}$
$\bar{\mathbf{A}}_2 = \{A_2, B_2, A_3\}$
$\bar{\mathbf{B}}_2 = \{B_3\}$
$\bar{\mathbf{A}}_3 = \{A_4\}$
$\bar{\mathbf{B}}_3 = \{B_4, A_5, B_5\}$



Linear separation of clusters

Properties

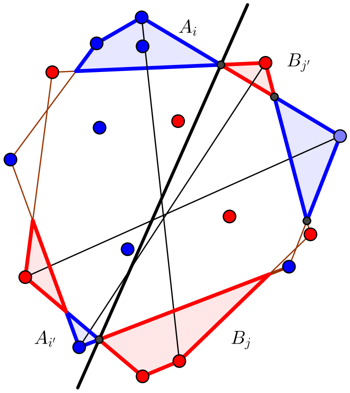
- Let (A_i, B_j) and $(A_{i'}, B_{j'})$ be two disjoint bad pairs. Then

$$A_i, A_{i'} \in \bar{\mathbf{A}}_k \Longrightarrow B_j, B_{j'} \in \bar{\mathbf{B}}_t$$

- The number of maximal cyclic sequences of adjacent bad A_i-polygons and B_j-polygons is the same.
- If $(\bar{\mathbf{A}}_i, \bar{\mathbf{B}}_j)$ and $(\bar{\mathbf{A}}_{i'}, \bar{\mathbf{B}}_{j'})$ are disjoint bad pairs of maximal subsequences, then there exist two (one from every pair) bad-crossing segments.
- There is an odd number of subsequences from each cluster, and they must be completely interlacing.

Linear separation of clusters

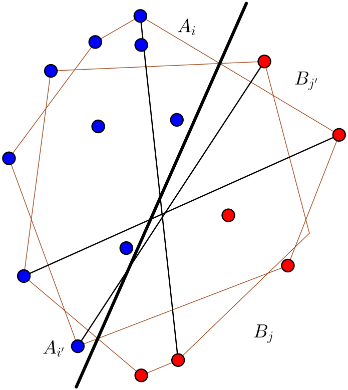
Step 3: Separate the sets.



Linear separation of clusters

- Let A_i be the last polygon of a maximal cyclic subsequence (in clockwise order)
- Let B_j be the last bad partner of A_i.
- Let $B_{j'}$ be the first bad polygon after A_i
- let $A_{i'}$ be the first bad partner of $B_{j'}$.
- Choose the line L going through the point just before B_j and the point just after $B_{j'}$.
- Define B' to be the points in $A \cup B$ lying on the same side of L as B_j and $B_{j'}$, and A' as the remaing points.

Linear separation of clusters



Linear separation of clusters

Proposition

$$\mathrm{diam}(A') \leq \mathrm{diam}(A), \qquad \mathrm{diam}(B') \leq \mathrm{diam}(B).$$

Theorem

Let A and B be two sets of points in a general normed plane. Then, there are two linearly separable sets A' and B' such that $\mathrm{diam}(A') \leq \mathrm{diam}(A)$, $\mathrm{diam}(B') \leq \mathrm{diam}(B)$, and $A' \cup B' = A \cup B$.

Corollary

In the construction in the Theorem,

$$\mathrm{perimeter}(A) + \mathrm{perimeter}(B) \geq \mathrm{perimeter}(A') + \mathrm{perimeter}(B')$$

holds. If $\mathrm{conv}(A) \cap \mathrm{conv}(B) \neq \emptyset$, then the inequality is strict.

Some consequences

The 2-clustering problem for diameter respect to the minimum: Dividing S in two sets minimizing the maximum diameter of the sets.

Theorem

Given a set S of n points in a normed plane, the 2-clustering problem for diameter respect to the minimum can be computed in $O(n^2 \log^2 n)$ time.

- Sort the distances d_i between the points of S into increasing order.
- By a binary search, locate the minimum d_i that admits a *stabbing line* for the set of segments meeting point of S at distance greater than d_i.

Some consequences

The k-clustering problem for diameter respect to a function $\mathcal{F}$ (for example, $\mathcal{F}$ can be the *maximum*, the *sum*, or the *sum of squares*):

Dividing S in k sets minimizing a function $\mathcal{F}$ of the diameters of the sets.

Theorem

Consider the optimal k-clustering problem for the diameter respect to a monotone increasing function $\mathcal{F}$ of such as diameters. For every set S of n points in a general normed plane,

- *There is an optimal k-clustering such that each pair of clusters is linearly separable.*
- *The problem is solvable by an algorithm in polynomial time.*

Thank you very much!