Encompassing weakly compact sets of $C[0,1]$

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Joint work with J. López-Abad

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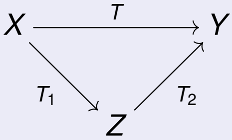
Motivation

Theorem (Davies-Figiel-Johnson-Pelczynski 1974)

Given Banach spaces X, Y and a weakly compact operator $T : X \to Y$, there is a reflexive Banach space Z and operators T_1, T_2 such that



Question: If X, Y are Banach lattices, can we make Z a (reflexive) Banach lattice?

Answers:

- Yes, under some conditions (Aliprantis-Burkinshaw 1984).
- In general, NO (Talagrand 1986).

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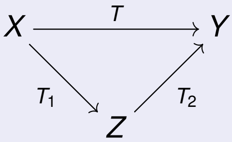

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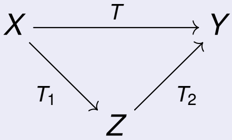

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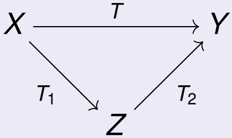

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Encompassable sets

Theorem (Davies-Figiel-Johnson-Pelczynski)

Let X be a Banach space, $K \subset X$ weakly compact. There is a reflexive Banach space Z and an operator $T : Z \to X$ such that $K \subseteq T(B_Z)$.

Definition

Let X be a Banach space. A weakly compact set $K \subset X$ is *encompassable by a reflexive Banach lattice* if there is a reflexive Banach lattice E and an operator $T : E \to X$ such that $K \subset T(B_E)$.

Theorem (Aliprantis-Burkinhaw)

Under any of the following assumptions

- *X is a space with an unconditional basis, or*
- *X is a Banach lattice which does not contain c_0,*

every weakly compact set $K \subseteq X$ is encompassable by a reflexive Banach lattice.

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Talagrand's question

Theorem (Talagrand)

There is a (countable) weakly compact set $K_{\mathcal{T}} \subseteq C[0,1]$ which is unencompassable by any reflexive Banach lattice.

$K_{\mathcal{T}}$ is homeomorphic to ω^{ω^2}.

Question: What is the smallest ordinal α such that there exists a weakly compact set $K \subseteq C[0,1]$ homeomorphic to α which is unencompassable by any reflexive Banach lattice?

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The lower bound

Theorem

Let $K \subseteq C[0,1]$ be a weakly compact set homeomorphic to $\alpha < \omega^\omega$. Then K is encompassable by a reflexive Banach lattice.

Sketch of proof:

1. Let $\phi : C[0,1]^* \to C(K)$ be given by $\phi(\mu)(k) = \int k d\mu$.
2. $C(K)$ is isomorphic to c_0.
3. There is a reflexive lattice E such that

$$\begin{array}{ccccc} C[0,1]^* & \xrightarrow{\phi} & C(K) & \xrightarrow{\simeq} & c_0 \\ & {\scriptstyle T}\searrow & & \nearrow{\scriptstyle S} & \\ & & E & & \end{array}$$

4. $T^*(\text{“}\delta_k\text{”}) = k$ for every $k \in K$.

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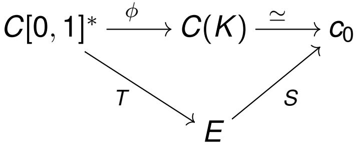

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The upper bound

Consider the Schreier family and its "square"

$$\mathcal{S} = \{s \subset \mathbb{N} : \sharp s \leq \min s\},$$

$$\mathcal{S}_2 = \mathcal{S} \otimes \mathcal{S} = \{\bigcup_{i=1}^{n} s_i : n \leq s_1 < \ldots < s_n,\ s_i \in \mathcal{S} \text{ for } 1 \leq i \leq n\}.$$

$S, S_2 \subset \mathcal{P}^{<\infty}(\mathbb{N})$ are compact and homeomorphic to ω^ω and ω^{ω^2} respectively.
Each element $s \in \mathcal{S}_2$ has a unique decomposition

$$s = s[0] \cup s[1] \cdots \cup s[n],$$

where $s[0] < s[1] < \cdots < s[n]$, $\{\min s[i]\}_{i \leq n} \in \mathcal{S}$, $s[n] \in \mathcal{S}$ and $\min s[i] = \sharp s[i]$ for $0 \leq i < n$.

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Let $\Theta : \mathcal{S} \to C(\mathcal{S}_2)$ be the mapping that for $s = \{m_0 < \cdots < m_k\} \in \mathcal{S}$ for every $t = t[0] \cup \cdots \cup t[l] \in \mathcal{S}_2$,

$$\Theta(s)(t) := \frac{1}{2}\Big((-1)^{\#(\{0 \leq i \leq \min\{k,l\} : m_i \in t[i]\})} + 1\Big).$$

$\Theta : \mathcal{S} \to C(\mathcal{S}_2)$ is well-defined and continuous.
Let $K_\omega := \Theta(\mathcal{S}) \subseteq C(\mathcal{S}_2)$ is weakly compact and homeomorphic to ω^ω (and extending its elements by zero we get $K_\omega \subset C[0,1]$).

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$K_\omega \subset C(\mathcal{S}_2)$ is unencompassable by any reflexive Banach lattice.

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$A \subset X$ is a Banach-Saks set if every $(x_n) \subset A$ has a Cesaro convergent subsequence.

Proposition (Flores-T. 2008)

Talagrand's weakly compact K_T is a Banach-Saks set.

Corollary

K_T is unencompassable by any Banach lattice with the Banach-Saks property.

It can be seen that the compact K_ω constructed before fails the Banach-Saks property.

Question: What is the smallest ordinal α such that there exists a Banach-Saks set $K \subseteq C[0,1]$ homeomorphic to α which is unencompassable by any Banach lattice with the Banach-Saks property?

Answer: ω [LópezAbad-Ruiz-T. 2014]

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