About Mazur's rotations problem

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Terminology[1]

The results presented here are joint work with Christian Rosendal, from the University of Illinois at Chicago.

In this talk all spaces are complete, all Banach spaces are unless specified otherwise, separable, infinite dimensional, and, for expositional ease, assumed to be complex.

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Contents

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1. Mazur's rotations problem
2. Transitivity and maximality of norms in Banach spaces
3. Applications to the Hilbert space

Definition

- $\mathrm{Isom(X)}$ *is the group of linear surjective isometries on a Banach space* X.
- *The group* $\mathrm{Isom(X)}$ *acts* *transitively* *on the unit sphere* S_X *of* X *if for all* x, y *in* S_X, *there exists* T *in* $\mathrm{Isom}(X)$ *so that* $Tx = y$.

Introduction: Mazur's rotations problem

Definition

- $\mathrm{Isom(X)}$ *is the group of linear surjective isometries on a Banach space X.*
- *The group $\mathrm{Isom(X)}$ acts transitively on the unit sphere S_X of X if for all x, y in S_X, there exists T in $\mathrm{Isom}(X)$ so that $Tx = y$.*

Fact

The group $\mathrm{Isom(H)}$ acts transitively on any Hilbert space H.

Conversely if $\mathrm{Isom(X)}$ acts transitively on a Banach space X, must it be linearly isomorphic? isometric to a Hilbert space?

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Answers:

(a) if $\dim X < +\infty$: YES to both
(b) if $\dim X = +\infty$ is separable: ???
(c) if $\dim X = +\infty$ is not separable: NO to both

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Proof.

(a) $X = (\mathbb{C}^n, \|.\|)$. Choose an inner product $< ., . >$ such that $\|x_0\| = \sqrt{< x_0, x_0 >}$ for some x_0. Define

$$[x, y] = \int_{T \in \mathrm{Isom}(X, \|.\|)} < Tx, Ty > dT,$$

This a new inner product for which the T still are isometries, and $\|x\| = \sqrt{[x, x]}$, since holds for x_0 and by transitivity. □

Introduction: Mazur's rotations problem

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Proof.

(c) Prove that for $1 \leq p < +\infty$, the orbit of any norm 1 vector in $L_p([0,1])$ under the action of the isometry group is dense in the unit sphere.

Then note that any ultrapower of $L_p([0,1])$ is a non-hilbertian space on which the isometry group acts transitively. □

Introduction: Mazur's rotations problem

So we have the next unsolved problem which appears in Banach's book "Théorie des opérations linéaires", 1932.

Problem (Mazur's rotations problem, first part)

If $X, \|.\|$ is separable and transitive, must X be linearly isomorphic to the Hilbert space?

Problem (Mazur's rotations problem, second part)

Assume $X, \|.\|$ is linearly isomorphic to a Hilbert and transitive, must X be (isometric to) a Hilbert space?

Contents

1. Mazur's rotation problem
2. Transitivity and maximality of norms in Banach spaces
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Principles of renorming theory

Mazur's rotations problem is extremely difficult. Let us be more modest and look at the:

General objectives of renorming theory: replace the norm on a given Banach space X by a better one (i.e. an equivalent one with more properties).

Principles of renorming theory

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In general, one tends to look for an equivalent norm which make the unit ball of X

- smoother: e.g. $x \mapsto \|x\|$ must have differentiability properties,
- more symmetric: i.e. the norm induces more isometries.

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- smoother: e.g. $x \mapsto \|x\|$ must have differentiability properties,
- more symmetric: i.e. the norm induces more isometries.

Let us concentrate on the second aspect.

Introduction: transitive and maximal norms

In 1964, Pełczyński and Rolewicz looked at Mazur's rotations problem and defined properties of a given norm $\|.\|$.
In what follows $\mathcal{O}_{\|.\|}(x)$ represents the orbit of the point x of X, under the action of the group $\mathrm{Isom}(X, \|.\|)$, i.e.
$\mathcal{O}_{\|.\|}(x) = \{Tx, T \in \mathrm{Isom}(X, \|.\|)\}$.

Definition

Let X be a Banach space and $\|.\|$ an equivalent norm on X. Then $\|.\|$ is

(i) *transitive if $\forall x \in S_X$, $\mathcal{O}_{\|.\|}(x) = S_X$.*

(ii) *quasi transitive if $\forall x \in S_X$, $\mathcal{O}_{\|.\|}(x)$ is dense in S_X.*

(iii) *maximal if there exists no equivalent norm $|||.|||$ on X such that $\mathrm{Isom}(X, \|.\|) \subseteq \mathrm{Isom}(X, |||.|||)$ with proper inclusion.*

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Of course $(i) \Rightarrow (ii)$, and also $(ii) \Rightarrow (iii)$ (Rolewicz).

Transitive and maximal norms

Definition

Let X be a Banach space and $\|.\|$ an equivalent norm on X. Then $\|.\|$ is

(i) transitive *if* $\forall x \in S_X$, $\mathcal{O}_{\|.\|}(x) = S_X$.

(ii) quasi transitive *if* $\forall x \in S_X$, $\mathcal{O}_{\|.\|}(x)$ *is dense in* S_X.

(iii) maximal *if there exists no equivalent norm* $|||.|||$ *on* X *such that* $\mathrm{Isom}(X, \|.\|) \subsetneq \mathrm{Isom}(X, |||.|||)$ *with proper inclusion.*

Examples of (i): ℓ_2, of (ii): $L_p(0,1)$, of (iii): ℓ_p.

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Definition

A subgroup G of $GL(X)$ is bounded if $\sup_{g \in G} \|g\| < +\infty$.

Observation

(iii) $\Leftrightarrow \mathrm{Isom}(X, \|.\|)$ *is a maximal bounded subgroup of* $\mathrm{GL}(X)$.

(indeed if $\mathrm{Isom}(X, \|.\|) \subset G$ then any G-invariant norm $|||.|||$ satisfies $\mathrm{Isom}(X, \|.\|) \subseteq \mathrm{Isom}(X, |||.|||)$)

Transitive and maximal norms

Questions (Wood, 1982)

Does every Banach space admit an equivalent maximal norm? If yes, is every bounded group of isomorphisms on a Banach space contained in a maximal one?

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Transitive and maximal norms

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Question (Deville-Godefroy-Zizler, 1993)

Does every superreflexive Banach space admit an equivalent quasi-transitive norm?

Note that a positive answer to DGZ would imply that a space X with

i) a norm with modulus of convexity of "type" p and
ii) a norm whose dual norm has modulus of convexity of type q

would admit a norm with both properties i) and ii)

Recent solutions to Wood and DGZ problems

Theorem (F. - Rosendal, 2013)

There exists a separable superreflexive Banach space X without an equivalent maximal norm. Equivalently there is no maximal bounded subgroup of $GL(X)$.

Theorem (Dilworth - Randrianantoanina, 2014)

Let $1 < p < +\infty, p \neq 2$. Then

- *ℓ_p does not admit an equivalent quasi-transitive norm.*
- *there exists a bounded group of isomorphisms on ℓ_p which is not contained in any maximal one.*

Open questions

Question

Let $1 < p < +\infty, p \neq 2$. Show that $L_p([0,1])$ does not admit an equivalent transitive norm.

Question

Find a superreflexive space which admits i) a norm with modulus of convexity of power type p and ii) a norm whose dual norm has modulus of convexity of power type q, but does not admit a norm with both properties.

Transitivity and ultrahomogeneity

Definition

A norm $\|.\|$ on a Banach space X is (resp. approximately) ultrahomogeneous if for any isometry t between finite dim. subspaces F and G of X, there exists a surjective isometry T on X such that $T_{|F} = t$ (resp. such that $\|T_{|F} - t\| \leq \varepsilon$ given).

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Fact

1. *Any Hilbert norm is ultrahomogeneous*
2. *The usual norm on $L_p([0,1])$ is approximately ultrahomogeneous if (and only if) $p \neq 4, 6, 8, \ldots$*

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As far as I know the answers to the following are unknown:

Question

Show that $L_p([0,1]), p \neq 2$ does not admit an equivalent ultrahomogeneous norm. Show that every ultrahomogeneous norm on a separable space is a Hilbert norm.

Contents

Bounded groups of isomorphisms

Recall that a subgroup G of $GL(X)$ is bounded if $\sup_{g \in G} \|g\| < +\infty$. Note that this does not depend on the choice of an equivalent norm. Isometry groups are bounded, and conversely:

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When $X = H$ Hilbert, then this norm is not a priori a Hilbert norm, so we also consider, in the language of representations:

Definition

A bounded representation $\pi : \Gamma \to GL(H)$ is unitarizable if there is some equivalent Hilbert norm on H for which $\pi(\gamma)$ is an isometry (equivalently, a unitary) for all γ.

Transitive groups of isomorphisms

Definition

A bounded subgroup G of $GL(X)$ is transitive if there exists an equivalent norm $\|.\|$ on X such that

1. $\|.\|$ *is G-invariant.*
2. *for any $\|x\| = \|y\| = 1$, there exists $T \in G$ such that $Tx = y$.*

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If G is quasi-transitive on X, $\|.\|$ satisfies 1. 2. and $\|.\|'$ satisfies 1., then there exists $\lambda > 0$ s.t. $\|x\|' = \lambda\|x\|$ for all x.

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Proof.

Given $x_0 \neq 0$ let λ be s.t. $\|x_0\|' = \lambda\|x_0\|$. By 1., for any $T \in G$, $\|Tx_0\|' = \lambda\|Tx_0\|$. By 2. for $\|.\|$ this holds for all x. □

Transitive groups of isomorphisms

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Fact

If $\pi : \Gamma \to GL(H)$ is unitarizable then $\pi(\Gamma)$ extends to a transitive maximal bounded subgroup of $GL(H)$.

Proof.

If $\|.\|$ is a $\pi(\Gamma)$-invariant Hilbert norm, then $U(H, \|.\|)$ is transitive and maximal bounded. □

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Proof.

If $\|.\|$ is a $\pi(\Gamma)$-invariant Hilbert norm, then $U(H, \|.\|)$ is transitive and maximal bounded. □

To study Part 2 of Mazur's rotations problem, we should therefore look at a non-unitarizable representation π of a group Γ on H. if $\pi(\Gamma)$ is included in some maximal bounded group, then there exists a maximal non-Hilbert norm on ℓ_2. Then we should ask whether it can be quasi-transitive or transitive.

Theorem (Day-Dixmier, 1950)

Any bounded representation of an amenable group on the Hilbert space is unitarizable.

This does not extend to all (countable) groups:

Theorem (Ehrenpreis-Mautner, 1955)

The free group F_∞ admits a bounded non-unitarizable representation on H.

Question (Dixmier's unitarizability problem)

Suppose G is a countable group all of whose bounded representations on H are unitarizable. Is G amenable?

Digression: Day-Dixmier

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We shall work with a specific non-unitarizable representation of F_∞ based on its regular representation.

Example: twisting by unbounded linear operator

(see Ozawa, Pisier, ...) Let λ be the left regular unitary representation of F_∞ on $H = \ell_2(F_\infty)$, i.e.

$$\lambda(\gamma)(\sum a_s 1_s) = \sum a_s 1_{\gamma s}.$$

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Definition

Let $L\colon \ell_1(F_\infty) \longrightarrow \ell_1(F_\infty)$ be the "Left Shift", i.e. the bounded linear operator satisfying $L(1_e) = 0$ and $L(1_s) = 1_{\hat{s}}$ for $s \neq e$, where $\hat{s}$ is the predecessor of s.

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So L is a densely defined unbounded linear operator on $H = \ell_2(F_\infty)$.

Example: twisting by unbounded linear operator

Definition

Let

$$\lambda'(\gamma) := \begin{pmatrix} Id & -L \\ 0 & Id \end{pmatrix} \begin{pmatrix} \lambda(\gamma) & 0 \\ 0 & \lambda(\gamma) \end{pmatrix} \begin{pmatrix} Id & L \\ 0 & Id \end{pmatrix}$$

$$= \begin{pmatrix} \lambda(\gamma) & \lambda(\gamma)L - L\lambda(\gamma) \\ 0 & \lambda(\gamma) \end{pmatrix}$$

defined on $\ell_1(F_\infty) \oplus \ell_1(F_\infty)$. Check that this defines a bounded operator on $H = \ell_2(F_\infty) \oplus \ell_2(F_\infty)$.

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Note that if L were bounded, λ' would be unitarizable.

Proposition

λ' *is a bounded, non unitarizable representation of* F_∞ *on* $H \oplus H$.

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Proposition

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The theory of twisted sums suggests that non-linear bounded homogeneous maps could play a role here.

Twisting by uniformly continuous map

Proposition (F. - Rosendal 2015)

Suppose that $\lambda_d : \Gamma \longrightarrow \mathrm{GL}(H \oplus H)$ *is a representation of a group* Γ *on* $H \oplus H$ *leaving the first copy of* H *invariant, or equivalently*

$$\lambda_d(\gamma) = \begin{pmatrix} u(\gamma) & d(\gamma) \\ 0 & v(\gamma) \end{pmatrix}$$

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Then there exists $\psi : H \to H$ *homogeneous, uniformly continuous on bounded sets, such that* $d(\gamma) = u(\gamma)\psi - \psi v(\gamma)$ *for all* $\gamma \in \Gamma$*, or equivalently*

$$\lambda_d(\gamma) := \begin{pmatrix} Id & -\psi \\ 0 & Id \end{pmatrix} \begin{pmatrix} u(\gamma) & 0 \\ 0 & v(\gamma) \end{pmatrix} \begin{pmatrix} Id & \psi \\ 0 & Id \end{pmatrix}$$

This applies to the previous example.

Proof

1. The norm $|||x||| = \sup_{\gamma \in \Gamma} \|\lambda_d(\gamma)x\|_2$ is an equivalent $\lambda_d(\Gamma)$-invariant norm on $H \oplus H$ which is uniformly convex.
2. This implies that the nearest point map p in the first copy of H, $p : H \oplus H \to H$, is uniformly continuous on bounded sets.
3. From the isometry and translation invariance of this map, $p(x, y) = x + p(0, y)$ and $p(\lambda_d(x)) = \lambda_d(p(x))$ so
$$p(\lambda_d(0, y)) = p(d(\gamma)y, v(\gamma)y) = d(\gamma)y + p(0, v(g)y)$$
$$\text{and } p(\lambda_d(0, y)) = \lambda_d(p(0, y)) = u(p(0, y)).$$
4. Set
$$\psi(y) = p(0, y).$$

Twisting by Lipschitz map

Proposition (F. - Rosendal 2015)

Suppose that $\lambda_d : \Gamma \longrightarrow \mathrm{GL}(H \oplus H)$ is a representation of a group Γ on $H \oplus H$ leaving the first copy of H invariant, and assume $\lambda_d(\Gamma)$ extends to a quasi transitive group on $H \oplus H$. Then $\exists \psi : H \to H$ Lipschitz and homogeneous, s.t. for all $\gamma \in \Gamma$,

$$\lambda_d(\gamma) := \begin{pmatrix} Id & -\psi \\ 0 & Id \end{pmatrix} \begin{pmatrix} u(\gamma) & 0 \\ 0 & v(\gamma) \end{pmatrix} \begin{pmatrix} Id & \psi \\ 0 & Id \end{pmatrix}$$

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Proof.

1. The norms $\|x\|_0 = \sup_{g \in G} \|gx\|_2$ on $H \oplus H$ and $\|x^*\|_1 = \sup_{g \in G} \|g^* x^*\|_2$ on $(H \oplus H)^*$ have modulus of convexity of type p=2.
2. By almost transitivity the dual norm to $\|.\|_0$ is a multiple of $\|.\|_1$ and therefore has modulus of convexity of type $q = 2$.

Twistings by Lipschitz isomorphisms

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Proof.

3. So $\|.\|_0$ and $\|.\|_0^*$ have moduli of convexity of type $p = q = 2$.
4. By classical results the modulus of uniform continuity of the nearest $\|.\|_0$ point map has power type $p(1 - 1/q) = 1$, which means that the map is Lipschitz.

Conclusion

Corollary

Let $\lambda_d : F_\infty \longrightarrow \mathrm{GL}(H \oplus H)$ be the non-unitarizable representation of F_∞ on $H \oplus H$ defined earlier. Assume $\lambda_d(F_\infty)$ extends to a quasi transitive subgroup G of $\mathrm{GL}(H \oplus H)$. Then there exists $\psi : H \to H$ Lipschitz, non-linear, such that for all $\gamma \in F_\infty$,

$$\lambda_d(\gamma) := \begin{pmatrix} Id & \psi \\ 0 & Id \end{pmatrix} \begin{pmatrix} \lambda(\gamma) & 0 \\ 0 & \lambda(\gamma) \end{pmatrix} \begin{pmatrix} Id & -\psi \\ 0 & Id \end{pmatrix}$$

Question

- *Use linearization techniques of Lipschitz maps to show that such a ψ cannot exist.*
- *Or, show that $\lambda_d(F_\infty)$ extends to a quasi transitive group and identify ψ.*

Conclusion

The following questions remain open:

Question

Show that $L_p(0,1), 1 < p < +\infty, p \neq 2$ does not admit an equivalent transitive norm.

Question

Find a non-unitarizable, maximal bounded, subgroup of $\mathrm{GL}(H)$.

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