Heisenberg Uniqueness Pairs and Unique Continuation for the Helmholtz equation

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Joint work with Ph. Jaming (Bordeaux) and K. Gröchenig (Vienna)

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A problem coming from geometry

Problem

Given a distribution of points in $\mathbb{R}^2$, A, when is A uniquely determined from its information on certain lines?

Theorem (Cramér-Wold, 1936)

If, A and B are finite sets, and we define $\delta_A = \sum_{a\in A}\delta_a$.

$$\hat{\delta}_A = \hat{\delta}_B \text{ on } \mathbb{R}\theta,\ \forall\theta\in\mathbb{S}^1 \Longrightarrow A = B.$$

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Heisenberg Uniqueness Pairs

Question: Is there any way to reduce the number of lines if the sets of points are supported in a manifold?

Definition (Hedenmaln,Montes-Rodríguez, 2011)

Let $\mathcal{M} \subset \mathbb{R}^d$ *manifold*, $\Sigma \subset \mathbb{R}^d$. $(\mathcal{M}, \Sigma)$ is a Heisenberg Uniqueness Pairs (HUP) if the only finite measure μ supported on $\mathcal{M}$ such that $\hat{\mu} = 0$ in Σ is $\mu = 0$.

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Example:

$\mathcal{M} = \mathbb{T}$ unit circle in $\mathbb{R}^2$. $\Sigma = L_1 \cup L_2$, $L_i = \mathbb{R}(\cos\theta_i, \sin\theta_i)$
We consider $\alpha = \frac{1}{\pi} \times$(angle between L_1 and L_2).

Lev, Sjölin 2011

$$(\mathcal{M}, \Sigma) \text{ is a HUP} \iff \alpha \notin \mathbb{Q}$$

Jaming and Kellay generalize this result to different manifolds (hyperbola, polygon, ellipse...) and the same set Σ.

Jaming, Kellay 2013

There exists a set E of positive measure such that
$$(\theta_1, \theta_2) \in E \implies (\mathcal{M}, \Sigma) \text{ is HUP}$$

Gröchenig-Jaming 2016: Extension to $d > 2$ replacing lines by hyperplanes.

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Goal of this talk

We will look at this example $\mathcal{M} = \mathbb{S}^{d-1}$ from a PDE point of view, since μ supported on $\mathbb{S}^{d-1}$ implies that $u = \hat{\mu}$ solves $\Delta u + u = 0$.

Theorem (Cheng 1976, Lev,Sjölin 2011, Gröchenig-Jaming 2016)

$\theta_1, \theta_2 \in \mathbb{S}^{d-1}$ such that $\frac{1}{\pi}\arccos(\theta_1, \theta_2) \notin \mathbb{Q}$. Let u be solution of $\Delta u + k^2 u = 0$ on $\mathbb{R}^d$. Assume that there exists μ s.t.

$$\begin{cases} u = \hat{\mu}, \\ u = 0, \quad x \in \theta_1^{\perp} \cup \theta_2^{\perp}. \end{cases}$$

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We will consider the following problems:

- **P1:** Can we remove the condition $u = \hat{\mu}$?
- **P2:** Can we consider solutions on domains Ω (bounded, connected, $0 \in \Omega$)?
- **P3:** Can we replace the Dirichlet conditions by Neumann or Robin conditions?
- **P4:** Can we replace hyperplanes by other types of submanifolds?
- **P5:** Can we replace hyperplanes by lower dimensional submanifolds?

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Outline

First method: Schwarz reflection principle

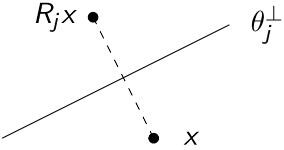
Theorem

*$d \geq 2$, Ω domain in $\mathbb{R}^d$, $\theta_1, \theta_2 \in \mathbb{S}^{d-1}$ s.t. $\frac{1}{\pi}\arccos(\theta_1, \theta_2) \notin \mathbb{Q}$.
Let u be solution of $\Delta u + k^2 u = 0$ on Ω s.t.*

$$\text{either} \begin{cases} u = 0 & \text{in } \theta_1^{\perp}, \\ u = 0 & \text{in } \theta_2^{\perp}, \end{cases} \quad \text{or} \begin{cases} u = 0 & \text{in } \theta_1^{\perp}, \\ \partial_n u = 0 & \text{in } \theta_2^{\perp}. \end{cases}$$

Then $u \equiv 0$.

Schwarz reflection principle:



$$u = 0 \text{ on } \theta_j^\perp \Rightarrow u(R_j x) = -u(x),$$
$$\partial_n u = 0 \text{ on } \theta_j^\perp \Rightarrow u(R_j x) = u(x).$$

Proof: On $\theta_1^\perp \cap B(0,r)$,
$u = 0 \Rightarrow u(R_1 R_2 x) = 0 \Rightarrow u((R_1 R_2)^n x) = 0.$
$R_1 R_2$ is a rotation of angle $2\arccos(\theta_1, \theta_2) \notin \pi\mathbb{Q}$, so the orbit of $\theta_1^\perp \cap B(0,r)$ is dense $\Rightarrow u = 0$ on $B(0,r) \Rightarrow u = 0$ on Ω.

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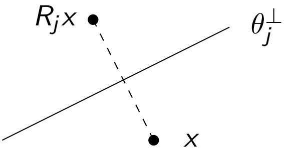

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Second method: Series expansions in $\mathbb{R}^2$

Theorem

*Ω domain in $\mathbb{R}^2$, $L_i = \mathbb{R}(\cos\theta_i, \sin\theta_i)$ with $\theta_1 - \theta_2 \notin \pi\mathbb{Q}$.
Let u be solution of $\Delta u + k^2 u = 0$ on Ω s.t. $u = 0$ on $L_1 \cup L_2$.
Then $u \equiv 0$.*

Remarks:

- Only Dirichlet conditions (not really a problem as we will see)
- We can replace lines by analytic curves in $\mathbb{R}^2$.

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- We can replace lines by analytic curves in $\mathbb{R}^2$.

Proof ($k = 0$): In a neighborhood of 0 the solution can be written, in polar coordinates, as

$$u(r, \theta) = \sum_{m \in \mathbb{Z}} c_m r^{|m|} e^{im\theta}.$$

We may assume wlog, $\theta_1 = 0$, $\theta_2 = \eta \notin \pi\mathbb{Q}$.
$c_0 = u(0, 0) = 0$. Assume $c_m = 0$ for $m = 0, \pm 1, \dots, \pm(n-1)$.
Then

$$\frac{u(r, \theta)}{r^n} = c_n e^{in\theta} + c_{-n} e^{-in\theta} + o(1) \Rightarrow \begin{cases} c_n + c_{-n} = 0, \\ c_n e^{in\eta} + c_{-n} e^{-in\eta} = 0. \end{cases}$$

This system has determinant $-2i \sin(n\pi\eta) \neq 0 \Rightarrow c_{\pm n} = 0$.

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Outline

A solution with two different Robin conditions

Theorem

Let θ_1, θ_2 such that $\theta_1 - \theta_2 \notin \pi\mathbb{Q}$. Let u be solution of

$$\begin{cases} \Delta u + k^2 u = 0, & (x,y) \in \Omega, \\ \alpha_1 u + \beta_1 \partial_n u = 0, & (x,y) \in L_1, \\ \alpha_2 u + \beta_2 \partial_n u = 0, & (x,y) \in L_2, \\ u(0,0) = 0. & \end{cases}$$

Then $u \equiv 0$.

Proof (k=0): On $L_j, \partial_n u(r,\theta_j) = \frac{1}{r}\partial_\theta u(r,\theta_j)$.Notice that if we use the same expansion as before, $u(0,0) = c_0 = 0$. Moreover,

$$u(r,\theta) = (c_1 e^{i\theta} + c_{-1}e^{-i\theta})r + \sum_{|m|\geq 2} c_m r^{|m|} e^{im\theta},$$

$$\frac{1}{r}\partial_\theta u(r,\theta) = i(c_1 e^{i\theta} - c_{-1}e^{-i\theta}) + i\sum_{|m|\geq 2} m c_m r^{|m|-1} e^{im\theta}.$$

First case, $\beta_i \neq 0$: Robin conditions imply $c_1 e^{i\theta_i} - c_{-1}e^{-i\theta_i} = 0$ for $i = 1,2 \Rightarrow c_{\pm 1} = 0$.
Second case, $\beta_1 = 0, \beta_2 \neq 0$:

$$\begin{cases} c_1 e^{i\theta_1} + c_{-1}e^{-i\theta_1} = 0 \\ c_1 e^{i\theta_2} - c_{-1}e^{-i\theta_2} = 0, \end{cases} \Rightarrow c_{\pm 1} = 0.$$

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What about the condition $u(0,0) = 0$?

If we do not assume a priori $u(0,0) = 0$ and $\beta_i \neq 0$ we cannot conclude $c_{\pm 1} = 0$,

$$\begin{cases} \alpha_1 c_0 + i\beta_1(c_1 e^{i\theta_1} - c_{-1} e^{-i\theta_1}) = 0, \\ \alpha_2 c_0 + i\beta_2(c_1 e^{i\theta_2} - c_{-1} e^{-i\theta_2}) = 0. \end{cases}$$

However $c_m = c_m(\theta_1, \theta_2, c_0)$. Hence, the space of solutions satisfying Robin conditions has dimension at most 1.

It might happen that the series expansion for the solution diverges. This seems to depend on how bad is $\frac{1}{\pi}(\theta_1 - \theta_2)$ approximated by rationals.

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A case where the condition is required

Definition

A *badly approximable* number is an x for which there is $c > 0$ such that

$$|mx - l| \geq \frac{c}{m}, \ \forall m, l \in \mathbb{Z}.$$

Theorem

If $\frac{1}{\pi}(\theta_1 - \theta_2)$ is badly approximable then the space of solutions to Helmholtz $(k > 0)$ equation satisfying Robin conditions on L_i has dimension exactly 1.

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¡Gracias!

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