Differentiability of the convolution.

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Definition

Let $L_1[-1,1]$ be the set of the 2-periodic functions f so that $\int_{-1}^{1} |f(t)| \, \mathrm{dt} < \infty$.

Given $f, g \in L_1[-1,1]$, the **convolution** of f and g is the function

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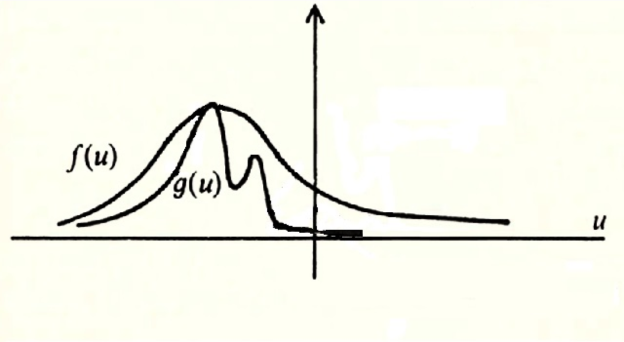

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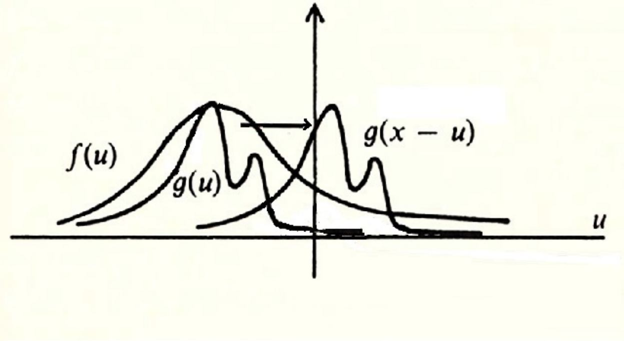

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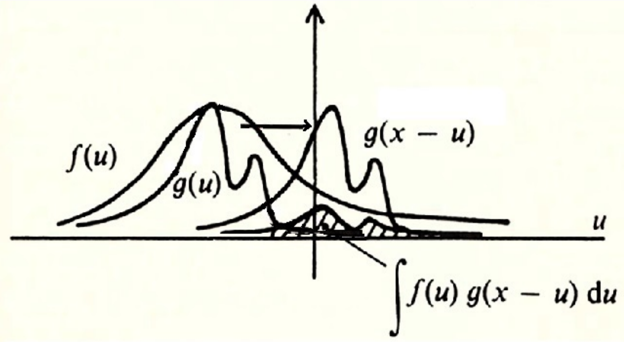

Lemma

Let $f, g \in L_1[-1,1]$.

- *If* f *is* k *times differentiable* ($k \geq 0$), *with* $\frac{d^k f}{dx^k}$ *continuous, then* $f * g$ *is* k *times differentiable, with* $\frac{d^k}{dx^k}(f * g)(x) = (\frac{d^k f}{dx^k} * g)(x)$.

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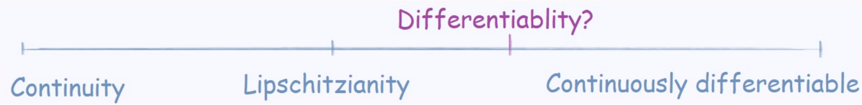

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What if we just ask **differentiability**?

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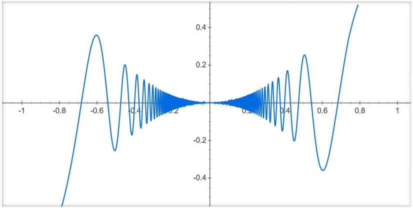

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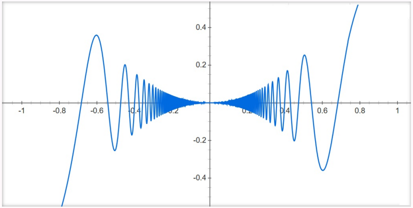

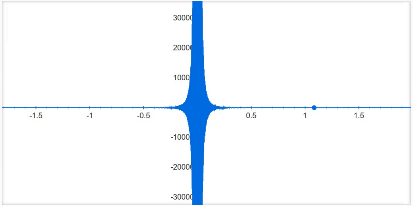

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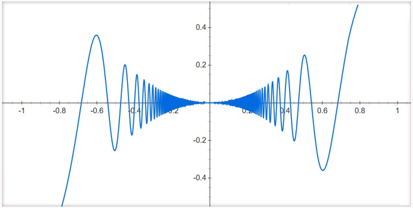

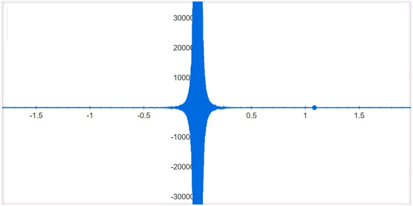

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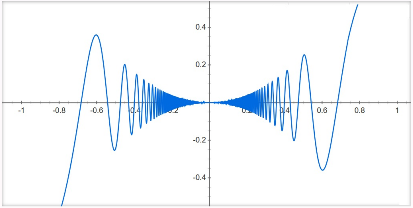

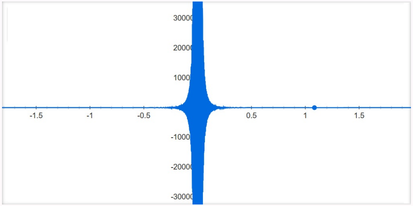

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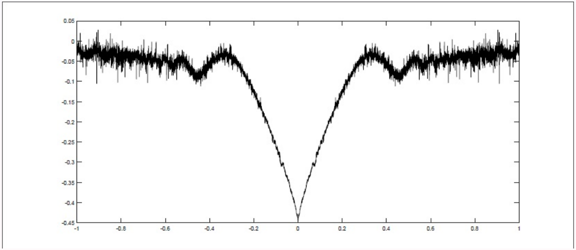

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$$f * g(0) = \int_{-1}^{1} t^2 \sin\left(\frac{1}{t^3}\right) \sin\left(\frac{1}{(-t)^3}\right) \mathrm{dt} = \int_{-1}^{1} t^2 \sin^2\left(\frac{1}{t^3}\right) \mathrm{dt}$$

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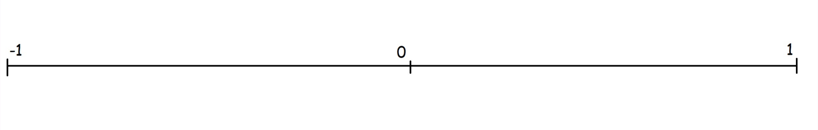

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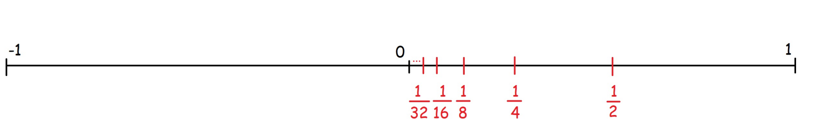

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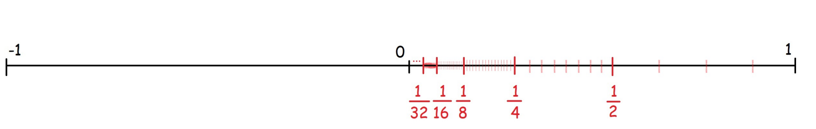

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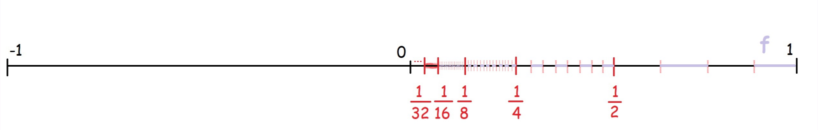

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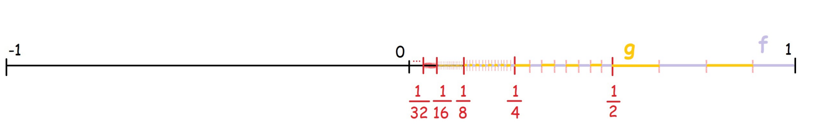

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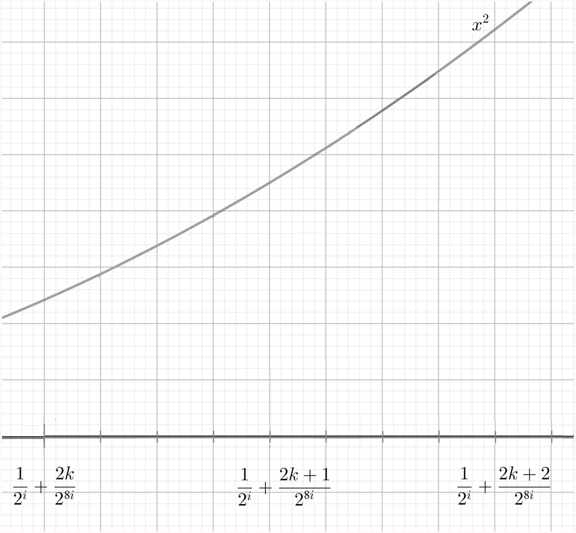

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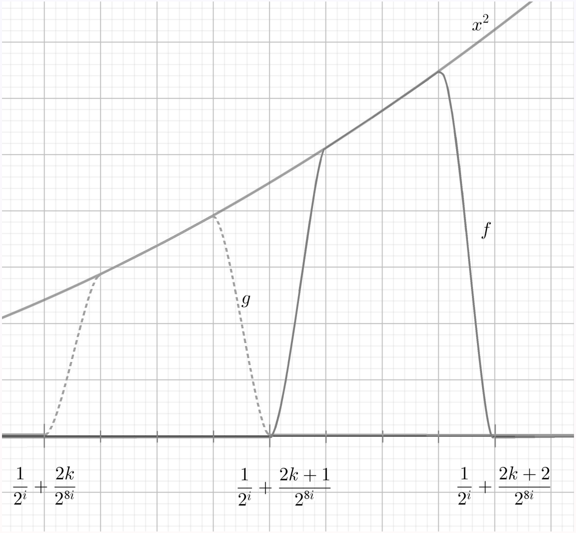

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$$\text{supp } \phi_{i,k} \subseteq \left(\frac{1}{2^i} + \frac{2k+1}{2^{8i}}, \frac{1}{2^i} + \frac{2k+2}{2^{8i}}\right),$$

$$\text{supp } \psi_{i,k} \subseteq \left(\frac{1}{2^i} + \frac{2k}{2^{8i}}, \frac{1}{2^i} + \frac{2k+1}{2^{8i}}\right),$$

$$\phi_{i,k}(x) = 1 \text{ for } \frac{1}{2^i} + \frac{8k+5}{2^{8i+2}} \leq x \leq \frac{1}{2^i} + \frac{8k+7}{2^{8i+2}} \text{ and}$$

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$$f(x) = x^2 \sum_{i=1}^{\infty} \sum_{k=0}^{2^{7i-1}-1} \phi_{i,k}(x),$$

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Theorem

The functions f and g defined above are differentiable functions for which $f * g$ is not differentiable at 0.

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In fact, for the differentiable functions f and g in the previous theorem, one can prove that $f' * g(x)$ is well-defined for every $-1 < x < 1$.

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Theorem 7.2. *Suppose that f is differentiable and the convolutions $f * g$ and $f' * g$ are well-defined. Then $f * g$ is differentiable and $(f * g)' = f' * g$. Likewise, if g is differentiable, then $(f * g)' = f * g'$.*

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In particular, there exists a differentiable function h so that $h * h$ is not differentiable at 0.

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What algebraic structure does this problem admit?

Theorem

There exist two algebras $\mathfrak{A}$ and $\mathfrak{B}$ generated by two respective non-numerable sets of differentiable functions so that, if $f \in \mathfrak{A} \setminus \{0\}$ and $g \in \mathfrak{B} \setminus \{0\}$, then $f * g$ is not differentiable at 0.

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There exist two closed cones $\mathfrak{V}$ and $\mathfrak{W}$ generated by two respective non-numerable sets of differentiable functions so that, if $f \in \mathfrak{V} \setminus \{0\}$ and $g \in \mathfrak{W} \setminus \{0\}$, then $f * g$ is not differentiable at 0.

For the algebras, consider

$$f_\lambda(x) = |x|^\lambda \sum_{i=1}^{\infty} \sum_{k=0}^{2^{i^2-i-1}-1} \phi_{i,k}^{\lambda}(|x|) \quad \text{and}$$

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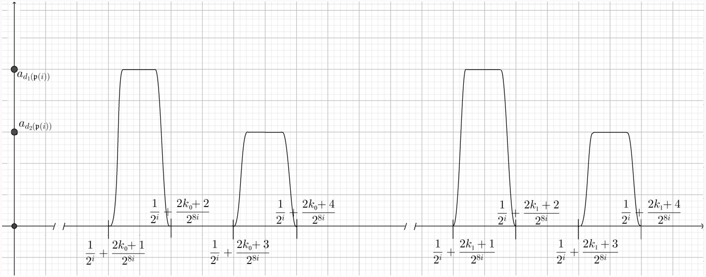

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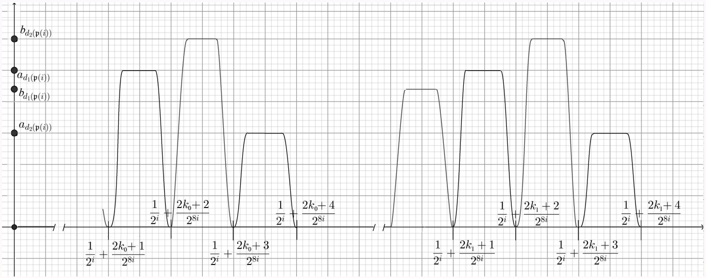

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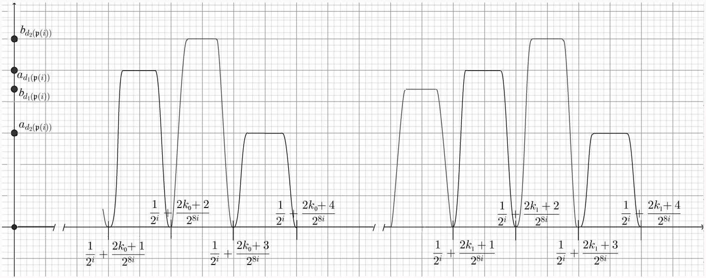


For the algebras, consider

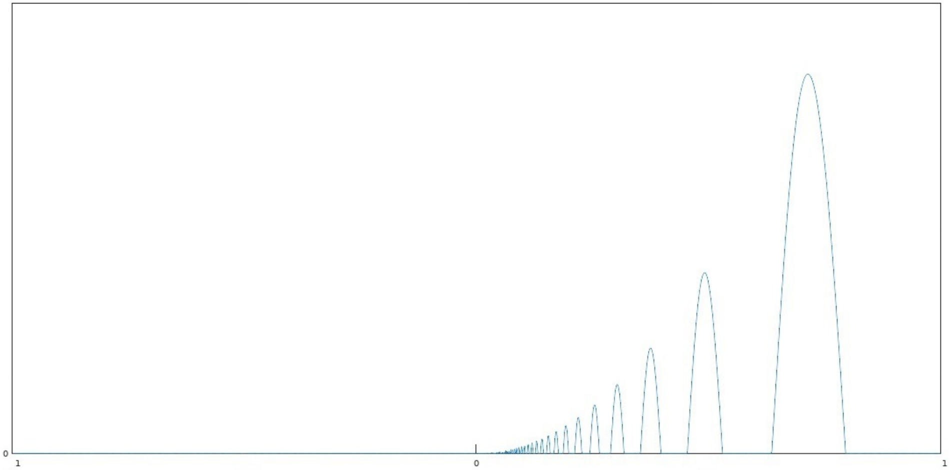
$$f_\lambda(x) = |x|^\lambda \sum_{i=1}^{\infty} \sum_{k=0}^{2^{i^2-i-1}-1} \phi_{i,k}^\lambda(|x|) \quad \text{and}$$

$$g_\lambda(x) = |x|^\lambda \sum_{i=1}^{\infty} \sum_{k=0}^{2^{i^2-i-1}-1} \psi_{i,k}^\lambda(|x|),$$

Theorem (V.I. Gurariy)

If $V \subseteq D[-1,1]$ is a closed vector space, then V is of finite dimension.

What about more points?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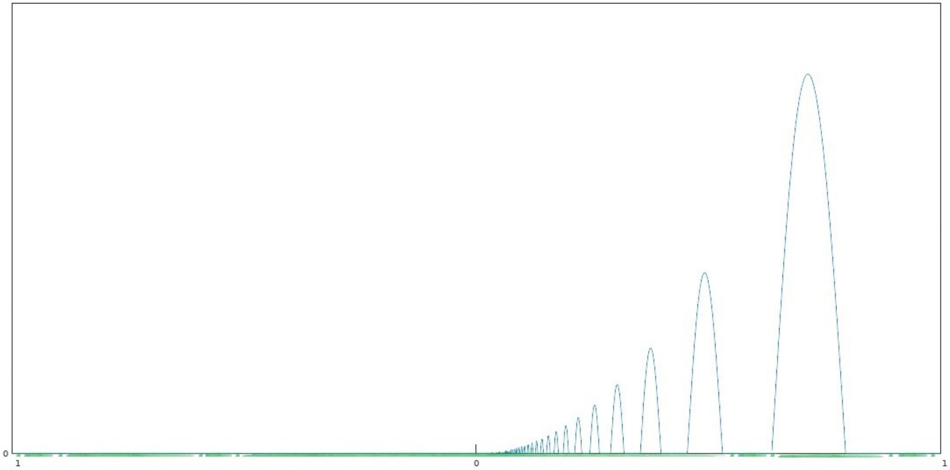
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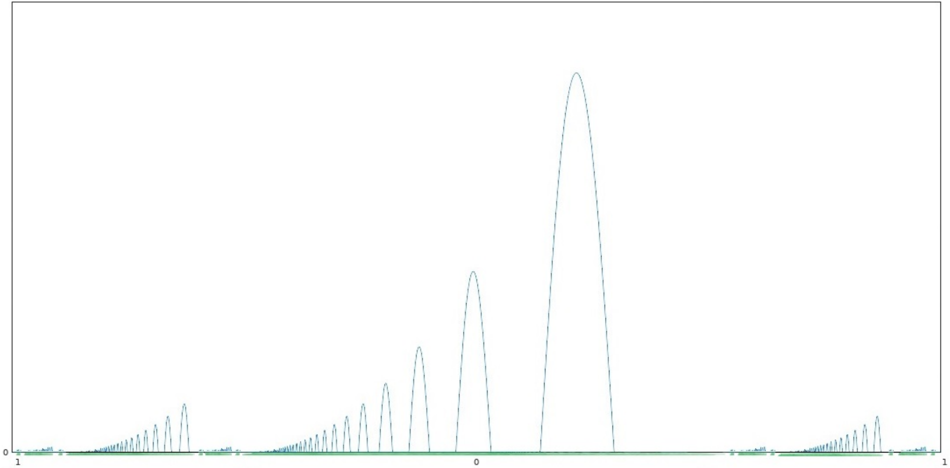
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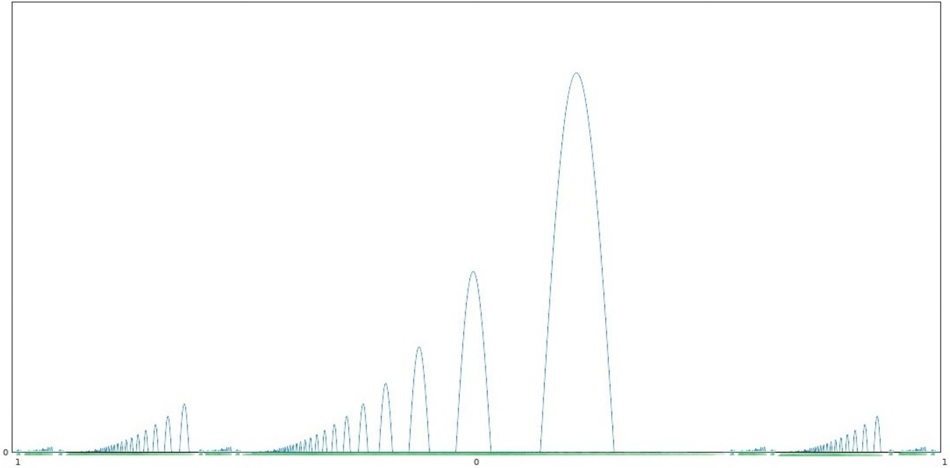
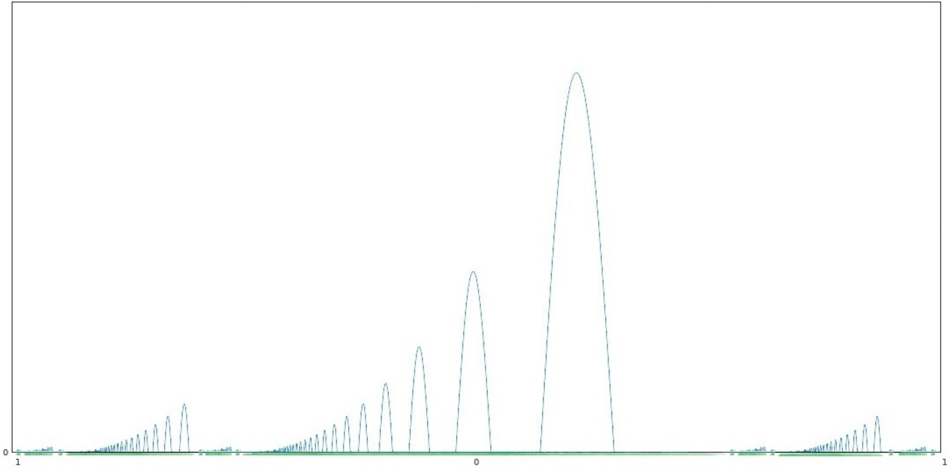


There exist differentiable functions f and g so that $f * g$ is not differentiable on a Perfect set



There exist differentiable functions f and g so that $f * g$ is not differentiable on a Perfect set (of zero measure)

Any positive result?

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Theorem

Let f be a differentiable function,
$\Delta_f = \{x \in [-1,1] \, : \, f' \text{ is locally unbounded}\}$.

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Any positive result?

Theorem

Let f be a differentiable function,
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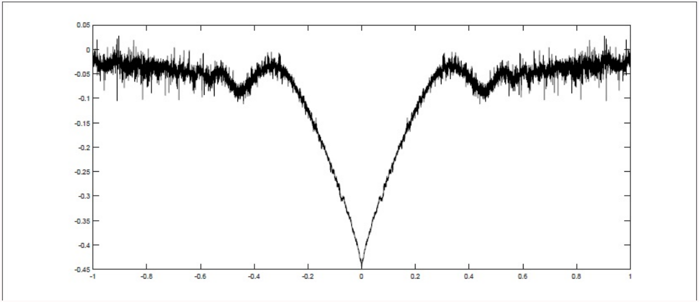

Any positive result?

Theorem

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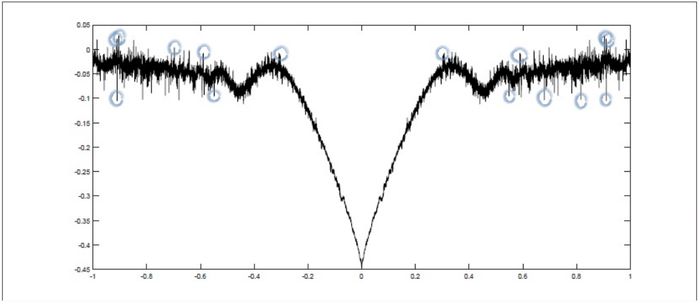

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Some open questions.

1. Let f and g be differentiable functions. How big can the set where $f * g$ is not differentiable be? A dense set? Of positive measure? The whole $[-1,1]$?
2. What are the weakest conditions over f' to ensure that $f * g$ is differentiable?
3. If f and g are such that $f * g$ is differentiable at x and $f' * g(x)$ is well-defined, is it $(f * g)'(x) = f' * g(x)$?

PUBLICATIONS

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.C. Ciesielski, P.J.R., G.A. Muñoz-Fernández and J.B. Seoane-Sepúlveda, *Nondifferentiability of the convolution of differentiable real functions.*, **preprint.**.

Thank you for your attention!!