# Averaging operators, fixed points and partial differential equations

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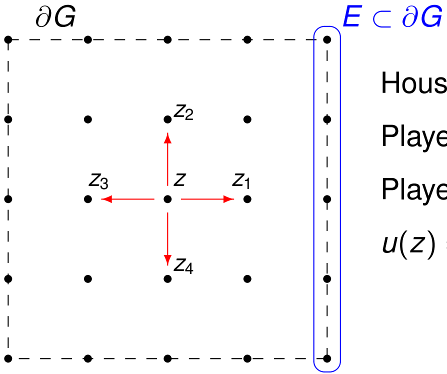
# Discrete games




House moves randomly the token

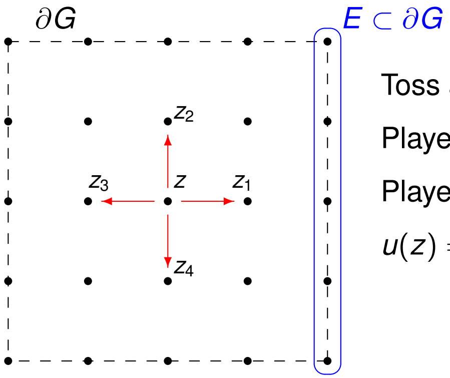
Player I wins when reaching $E$

Player II wins when reaching $\partial G \setminus E$

$u(z) = \mathbb{P}_z(\text{ Player 1 wins }) = \mathbb{P}_z(E)$

$$u(z) = \frac{1}{4}\sum_{i=1}^{4} u(z_i) \qquad \text{(Usual MVP)}$$

# Discrete games




Toss a fair coin to decide who moves

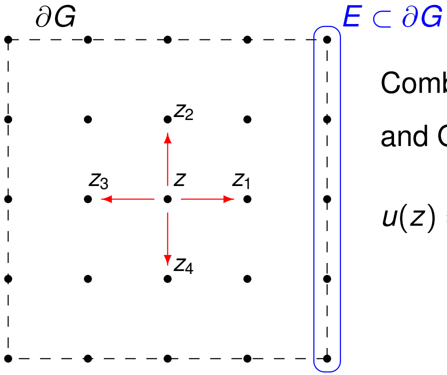
Player I wins when reaching $E$

Player II wins when reaching $\partial G$

$u(z) = \mathbb{P}_z(\text{ Player 1 wins }) = \mathbb{P}_z(E)$

$$u(z) = \frac{1}{2}\left(\max_i u(z_i) + \min_i u(z_i)\right)$$

# Discrete games




Combine Game 2 with probability $\alpha$

and Game 1 with probability $1 - \alpha$

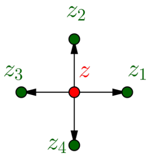
$$u(z) = \mathbb{P}_z(\text{ Player 1 wins }) = \mathbb{P}_z(E)$$

$$u(z) = \frac{\alpha}{2}\Big(\max_i u(z_i) + \min_i u(z_i)\Big) + \frac{1-\alpha}{4}\sum_{i=1}^{4} u(z_i)$$

# PDE's and MVP's on graphs

$u : G \to \mathbb{R}$ is **harmonic** iff

$$u(z) = \frac{1}{4}\sum_{i=1}^{4} u(z_i)\,, \quad (z \in G)$$



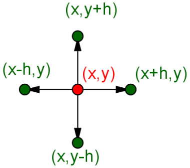

- Birkhoff (graph laplacian).
- Electrical networks.
- Discretization of PDE's.
- Image processing, image interpolation.

# Harmonic functions

Rewrite the **discrete MVP**

in terms of the horizontal and

vertical second differences:




$$u(x+h,y)+u(x-h,y)-2u(x,y)+$$
$$u(x,y+h)+u(x,y-h)-2u(x,y)=0$$

# Harmonic functions

$$u(x+h,y)+u(x-h,y)-2u(x,y)+$$
$$u(x,y+h)+u(x,y-h)-2u(x,y)=0$$

Now, remind that if $f:\mathbb{R}\to\mathbb{R}$, then

$$f''(a)=\lim_{h\to 0}\frac{f(a+h)+f(a-h)-2f(a)}{h^2}$$

# Harmonic functions

$$u(x+h,y)+u(x-h,y)-2u(x,y)+$$
$$u(x,y+h)+u(x,y-h)-2u(x,y)=0$$

Dividing by $h^2$ and taking limits as $h \to 0$ we formally obtain the Laplace equation in two variables

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

# Harmonic functions and the direct MVP


## The direct Mean Value Property (Gauss, 1840)

Let $u$ be harmonic in a domain $\Omega \subset \mathbb{R}^n$. Then for any ball $\overline{B}(x,r) \subset \Omega$, we have

- $u(x) = \fint_{\partial B(x,r)} u$ (**Spherical MVP**)
- $u(x) = \fint_{B(x,r)} u$ (**Volume MVP**)

# The converse MVP

## The basic converse MVP question

Given $u : \Omega :\to \mathbb{R}$, what sort of MVP does imply that $u$ is harmonic in $\Omega$?

## Different directions

- Requirements on $u$ and $\Omega$.
- How many radia?
- Asymptotic version of the MVP.
- Contributions due to **Cauchy, Darboux, Volterra, Vitali, Fréchet, Koebe, Sierpinski, Littlewood, Tonelli, Privalov, Banach...**

# Converse MVP results

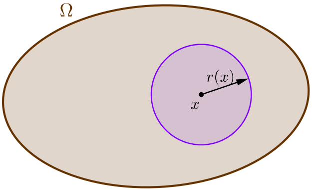
**Theorem (Koebe, 1906).**

Let $u \in \mathcal{C}(\Omega)$. If for each $x \in \Omega$ there is a sequence $r_n(x) \to 0$ such that $u$ satisfies the MVP at $x$ (volume or spherical) with radius $r_n(x)$ for all $n$, then $u$ is harmonic in $\Omega$.

# Converse MVP results

## One radius theorem (Volterra 1909, Kellogg 1928).

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $u \in \mathcal{C}(\overline{\Omega})$. Suppose that for each $x \in \Omega$ there is a single radius $r(x)$, with $0 < r(x) \leq dist(x, \partial\Omega)$ such that $u$ satisfies the MVP (either volume or spherical) at $x$ with radius $r(x)$. Then $u$ is harmonic in $\Omega$.



# Averaging operators

## Averaging Operators

Let $\Omega \subset \mathbb{R}^n$ be bounded and $r : \Omega \to \mathbb{R}_+$ such that $0 < r(x) \leq dist(x, \partial\Omega)$ for each $x \in \Omega$. Define the averaging operator $T_0$ associated to $r$ as follows

$$T_0 u(x) = \fint_{B(x,r(x))} u$$

## Remarks

- If $r$ is continuous then $T_0 : \mathcal{C}(\overline{\Omega}) \to \mathcal{C}(\overline{\Omega})$.
- Volterra-Kellogg Theorem is equivalent to saying that if $u \in \mathcal{C}(\overline{\Omega})$ is a **fixed point** of $T_0$ then $u$ is **harmonic** in $\Omega$.

# Lebesgue's approach to the Dirichlet Problem

Let $\Omega \subset \mathbb{R}^n$, bounded and regular. Choose $r(x) = dist(x, \partial\Omega)$ for $x \in \Omega$ and let $T_0$ be the corresponding averaging operator.

**Theorem (Lebesgue, 1912)**

Let $f \in \mathcal{C}(\partial\Omega)$ and $u_0 \in \mathcal{C}(\overline{\Omega})$ such that $u_0|_{\partial\Omega} = f$. Then

$$\{T_0^k u_0\} \to \widetilde{u} \text{ uniformly in } \overline{\Omega} \text{ as } k \to \infty$$

where $\widetilde{u}$ is the solution to the Dirichlet Problem

$$\begin{cases} \triangle u & = 0 \quad \text{in } \Omega \\ u & = f \quad \text{on } \partial\Omega \end{cases}$$

# Lebesgue's approach to the Dirichlet Problem

## Remarks

- $T_0 : \mathcal{C}(\overline{\Omega}) :\to \mathcal{C}(\overline{\Omega})$ is **linear** and **non-expanding**:

$$||T_0 u - T_0 v||_\infty \leq ||u - v||_\infty \;, \quad (u, v \in \mathcal{C}(\overline{\Omega}))$$

- If $\{T_0^k u_0\} \to \widetilde{u}$ uniformly in $\overline{\Omega}$ then $\widetilde{u}$ is a **fixed point** of $T_0$, therefore **harmonic** by Volterra-Kellogg.
- For $f \in \mathcal{C}(\partial\Omega)$, put

$$\mathcal{K}_f = \{u \in \mathcal{C}(\overline{\Omega}) : \; u|_{\partial\Omega} = f\}$$

Then $\mathcal{K}_f$ is closed in $\mathcal{C}(\overline{\Omega})$ and $T_0(\mathcal{K}_f) \subset \mathcal{K}_f$. To solve **Dirichlet Problem** with boundary data $f$ is equivalent to seek a **fixed point** of $T_0$ in $\mathcal{K}_f$.

# A nonlinear scenario

**The *p*-laplace operator**

For $1 < p < \infty$, define

$$\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$$

# A nonlinear scenario

### The $p$-laplace operator

For $1 < p < \infty$, define

$$\Delta_p u = div(|\nabla u|^{p-2} \nabla u)$$

- Euler-Lagrange equation associated to $p$-energy.
- (Weak) solutions are called $p$-harmonic functions.
- $p$-harmonic functions solve the Dirichlet Problem with continuous boundary data (in regular domains).
- $p$-harmonic functions are $\mathcal{C}^{1,\alpha}$ for some $0 < \alpha < 1$, not $\mathcal{C}^2$ in general.

# A nonlinear scenario

## Two relevant questions

1. Is there a "natural" **stochastic process** associated to the $p$-laplacian?
2. Is there a "natural" **MVP** related to the $p$-laplacian?

# A nonlinear scenario

## Two relevant questions

1. Is there a “natural” **stochastic process** associated to the $p$-laplacian?
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## Question 2: two main keys

- A representation of $\Delta_p$ in terms of $\Delta$ and $\Delta_\infty$.
- Averaging Taylor formula.

# Representing $\Delta_p$ in terms of $\Delta$ and $\Delta_\infty$

Assume $u \in \mathcal{C}^2(\Omega)$. Then, away from the critical set:

$$\Delta_p u = |\nabla u|^{p-2}\Big(\Delta u + (p-2)\frac{\Delta_\infty u}{|\nabla u|^2}\Big)$$

where

$$\Delta_\infty u = \sum_{i,j} u_{x_i} u_{x_j} u_{x_i,x_j} =< (Hu)\nabla u, \nabla u >$$

is the so called **infinity laplacian.**

# Averaging Taylor

Let $\Omega \subset \mathbb{R}^n$, $u \in \mathcal{C}^2(\Omega)$, $x \in \Omega$, $h \in B(0, r)$. Denote by $Hu(x)$ the **hessian** matrix of $u$ at $x$. By Taylor:

$$u(x+h) = u(x) + < \nabla u(x), h > + \frac{1}{2} < Hu(x)h, h > + o(r^2)$$

# Averaging Taylor

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$$u(x+h) = u(x) + < \nabla u(x), h > + \frac{1}{2} < Hu(x)h, h > + o(r^2)$$

Average over the ball $B(0,r)$ in two different ways:

- $\fint_{B(x,r)} u$ (usual average )
- $\frac{1}{2}\left( \sup_{B(x,r)} u + \inf_{B(x,r)} u \right)$ (mid-range average)

# Averaging taylor

## Usual averages

By elementary computation

$$\fint_{B(x,r)} u = u(x) + \frac{\Delta u(x)}{2(n+2)} r^2 + o(r^2)$$

Therefore

$$\boxed{\fint_{B(x,r)} u = u(x) + \frac{\Delta u(x)}{2(n+2)} r^2 + o(r^2)}$$

# Averaging taylor

## Mid-range averages

From Taylor:

$$u\left(x + r\frac{\nabla u(x)}{|\nabla u(x)|}\right) = \quad u(x) + r|\nabla u(x)| + \frac{\Delta_\infty u(x)}{2|\nabla u(x)|^2}\, r^2 + o(r^2)$$
$$u\left(x - r\frac{\nabla u(x)}{|\nabla u(x)|}\right) = \quad u(x) - r|\nabla u(x)| + \frac{\Delta_\infty u(x)}{2|\nabla u(x)|^2}\, r^2 + o(r^2)$$

where

$$\Delta_\infty u =< (Hu)\nabla u, \nabla u >= \sum_{i,j} u_{x_i} u_{x_j} u_{x_i,x_j}$$

# Averaging taylor

## Mid-range averages

Suppose

$$\sup_{B(x,r)} u \approx u\left(x + r\frac{\nabla u(x)}{|\nabla u(x)|}\right), \qquad \inf_{B(x,r)} u \approx u\left(x - r\frac{\nabla u(x)}{|\nabla u(x)|}\right)$$

Then

$$\frac{1}{2}\left(\sup_{B(x,r)} u + \inf_{B(x,r)} u\right) = u(x) + \frac{\Delta_\infty u(x)}{2|\nabla u(x)|^2} r^2 + o(r^2)$$

# Averaging taylor

### Conclusion

If $u \in \mathcal{C}^2(\Omega)$, $x \in \Omega$ and $\nabla u(x) \neq 0$ then

$$\begin{aligned} \fint_{B(x,r)} u &= u(x) + \frac{\Delta u(x)}{2(n+2)} r^2 + o(r^2) \\ \frac{1}{2}\left( \sup_{B(x,r)} u + \inf_{B(x,r)} u \right) &= u(x) + \frac{\Delta_\infty u(x)}{2|\nabla u(x)|^2} r^2 + o(r^2) \end{aligned}$$

# Averaging taylor

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\end{aligned}
$$

and, having in mind the representation

$$
\Delta_p u = |\nabla u|^{p-2} \Big( \Delta u + (p-2) \frac{\Delta_\infty u}{|\nabla u|^2} \Big)
$$

# Averaging taylor

## Conclusion

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we finally obtain that $\Delta_p u(x) = 0$ iff

$$u(x) = \frac{\alpha}{2}\Big( \sup_{B(x,r)} u + \inf_{B(x,r)} u \Big) + (1-\alpha) \fint_{B(x,r)} u + o(r^2)$$

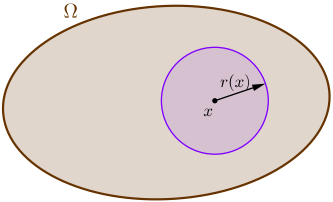
where $\alpha = \dfrac{p-2}{p+n}$.

# Recent approaches

- Optimal lipschitz extensions and $\Delta_\infty$ (Aronsson, 60's, Archer-LeGruyer...)
- Image processing (Caselles- Morel-Sbert 86...)
- Stochastic games (Peres-Shramm-Sheffield-Wilson 2009, Peres-Sheffield 2008, Manfredi-Parviainen-Rossi 2010-13)
- Asymptotic MVP (Manfredi-Parviainen-Rossi 2010, Lindqvist-Manfredi 2018, Arroyo-Llorente 2018).
- One radius MVP (Arroyo-Llorente 2015, 2018).

# A nonlinear one-radius MVP

- $\Omega \subset \mathbb{R}^n$ bounded.
- $r : \Omega \to \mathbb{R}_+$ continuous such that $0 < r(x) \leq dist(x, \partial\Omega)$ for each $x \in \Omega$.
- $\alpha \in [0, 1]$.



# A nonlinear one-radius MVP

## A nonlinear averaging operator

Define $T_\alpha : \mathcal{C}(\overline{\Omega}) \to \mathcal{C}(\overline{\Omega})$ by

$$T_\alpha u(x) = \frac{\alpha}{2}\left( \sup_{B(x,r(x))} u + \inf_{B(x,r(x))} u \right) + (1-\alpha) \fint_{B(x,r(x))} u$$

# A nonlinear one-radius MVP

$$T_\alpha u(x) = \frac{\alpha}{2}\left( \sup_{B(x,r(x))} u + \inf_{B(x,r(x))} u \right) + (1-\alpha) \fint_{B(x,r(x))} u$$

## Remarks

- No Volterra-Kellogg unless $\alpha = 0$. Fixed points of $T_\alpha$ and $p$-harmonic functions are different classes.
- $T_\alpha$ is not linear, unless $\alpha = 0$.
- $T_\alpha$ is non-expanding: $||T_\alpha u - T_\alpha v||_\infty \leq ||u - v||_\infty$.
- If $f \in \mathcal{C}(\partial\Omega)$ and $\mathcal{K}_f = \{u \in \mathcal{C}(\overline{\Omega}) : u|_{\partial\Omega} = f\}$ then $T_\alpha(\mathcal{K}_f) \subset \mathcal{K}_f$.

# The Dirichlet Problem for $T_\alpha$

## Problem

Let $\Omega \subset \mathbb{R}^n$ bounded, $r : \Omega \to \mathbb{R}_+$ continuous, with $0 < r(x) \leq dist(x, \partial\Omega)$ for $x \in \Omega$ and $\alpha \in [0, 1]$.

Which assumptions on $\Omega$, $r$ and $\alpha$ imply that the **Dirichlet Problem**

$$\begin{cases} T_\alpha u &= u \quad \text{in } \Omega \\ u &= f \quad \text{on } \partial\Omega \end{cases}$$

has a unique solution in $\mathcal{C}(\overline{\Omega})$?

# The Dirichlet Problem for $T_\alpha$

## Previous results

- $\alpha = 0$ (classical case).
- $\alpha = 1$. Archer-LeGruyer (1998).
- $0 \leq \alpha < 1$, $r(x) \equiv \epsilon$ (Manfredi-Parviainen-Rossi, Luiro-Saksman).

# The Dirichlet Problem for $T_\alpha$

**Theorem (Arroyo-Llorente, 2015, 2018)**

Let $\Omega \subset \mathbb{R}^n$ be strictly convex, and $0 \leq \alpha < 1$. Suppose that $r : \Omega \to \mathbb{R}_+$ is continuous and

$$\delta_1 \, dist(x, \partial\Omega) \leq r(x) \leq \delta_2 \, dist(x, \partial\Omega)$$

where $0 < \delta_1 \leq \delta_2 < 1 - \alpha$. Then the Dirichlet Problem

$$\begin{cases} T_\alpha u &= u \quad \text{in } \Omega \\ u &= f \quad \text{on } \partial\Omega \end{cases}$$

has a unique solution in $\mathcal{C}(\overline{\Omega})$.

# Existence: key steps

- Given $f \in \mathcal{C}(\partial\Omega)$, choose $u_0 \in \mathcal{C}(\overline{\Omega})$ such that $u_0|_{\partial\Omega} = f$.

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- Local equicontinuity. Archer-LeGruyer estimates for $T_1$ + local estimates for $T_0$.

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- Local equicontinuity. Archer-LeGruyer estimates for $T_1$ + local estimates for $T_0$.
- Boundary equicontinuity. Invariant convexity trick: to show that $co(G_{T_\alpha^k u}) \subset co(G_u)$, where $G_u$ stands for the graph of $u$.

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- Local equicontinuity. Archer-LeGruyer estimates for $T_1$ + local estimates for $T_0$.
- Boundary equicontinuity. Invariant convexity trick: to show that $co(G_{T_\alpha^k u}) \subset co(G_u)$, where $G_u$ stands for the graph of $u$.
- Final argument showing that non-expansiveness of $T_\alpha$ actually implies that $\{T_\alpha^k\}$ converges to a fixed point.

# Further comments and directions

- Relax the strict convexity assumption.
- Seek a more direct argument towards uniform convergence of the iterates $\{T_\alpha^k u_0\}$.
- Existence in metric measure spaces.
- If $r_\varepsilon(x) = \varepsilon\, r(x)$, suppose that $T_\alpha u_\varepsilon = u_\varepsilon$, where $u_\varepsilon \in \mathcal{C}(\overline{\Omega})$ and $u_\varepsilon|_{\partial\Omega} = f \in \mathcal{C}(\partial\Omega)$. Is it true that $u_\varepsilon \to u$ uniformly in $\overline{\Omega}$ as $\epsilon \to 0$, where $\Delta_p u = 0$ in $\Omega$?
- Asymptotic MVP for the $p$-laplacian in higher dimensions.