# On Hermite-Hadamard inequalities and some applications

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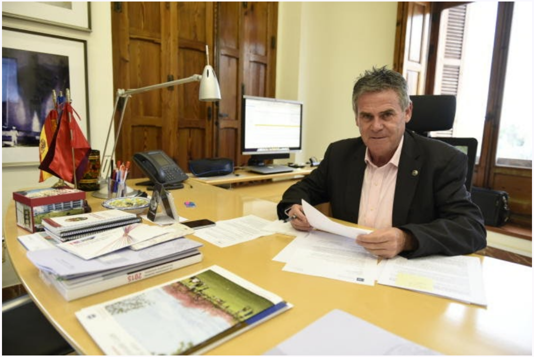

To the memory of Bernardo Cascales

# Hermite-Hadamard inequalities

C. Hermite

J. Hadamard

**Theorem 1 (Hermite 1881 & Hadamard 1893)**

Let $f : \mathbb{R} \to \mathbb{R}$ concave. Then

$$\frac{f(-a)}{2} + \frac{f(a)}{2} \leq \frac{1}{2a} \int_{-a}^{a} f(x)dx \leq f\left(\frac{-a}{2} + \frac{a}{2}\right) = f(0).$$

S.S. Dragomir, C.E.M. Pearce, Selected Topics on Hermite-Hadamard Inequality and Applications

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$$\frac{1}{|K|}\int_K f(x)dx = \int_K f(x)\frac{dx}{|K|} \le f\left(\int_K x\frac{dx}{|K|}\right) = f(c_K).$$

# Hermite-Hadamard for $f(x)^m$

**Theorem 2 (Milman & Pajor '00)**

Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be s.t. $\log f$ is concave and $\mu : \mathbb{R}^n \to \mathbb{R}_+$ a probability measure. Then

$$\int_{\mathbb{R}^n} f(x) d\mu(x) \leq f\left( \int_{\mathbb{R}^n} x \frac{f(x)}{\int f(z) d\mu(z)} d\mu(x) \right)$$

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**Corollary**

Let $K \in \mathcal{K}^n$, $f : K \to \mathbb{R}_+$ concave, and $m \in \mathbb{N}$. Then

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where $x_f = \int_K x \frac{f(x)^m}{\int f(z)^m dz} dx$.

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**Theorem 3 (G.M.+19, Dragomir '00)**

Let $f : B_2^n \to \mathbb{R}_+$ concave and $m \in \mathbb{N}$. Then

$$\frac{1}{|B_2^n|}\int_{B_2^n} f(x)^m dx \leq \frac{2^{m+n}}{(m+n)!}\Gamma\left(\frac{2m+n+1}{2}\right)\Gamma\left(\frac{n+2}{2}\right) f(0)^m.$$

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**Theorem 4 (G.M.+19)**

Let $K \in \mathcal{K}^n$ with $K = -K$, $f : K \to \mathbb{R}_+$ concave, and $m \in \mathbb{N}$. Then

$$\frac{1}{|K|}\int_K f(x)^m dx \leq \frac{2^m}{m+1} f(0)^m.$$

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Equality holds iff $f$ is affine and if moreover $m \geq 2$ then $K$ is a generalized cylinder s.t. $f \equiv 0$ in one of its basis.

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STEP 1: Replace $f$ by $g$ affine bounding from above.

# Proof of Theorem 4

$$\frac{1}{|K|}\int_K \frac{f(x)^m}{f(0)^m}dx \leq \frac{1}{|K|}\int_K \frac{g(x)^m}{g(0)^m}dx = \frac{1}{|S|}\int_S \frac{g(x)^m}{g(0)^m}dx$$

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# Reverse Hermite-Hadamard

**Theorem 5 (Alonso-Gutiérrez, Hernández Cifre, Roysdon, Yepes Nicolás, Zvavitch '18, G.M.+19)**

Let $0 \in K \in \mathcal{K}^n$, $f : K \to \mathbb{R}_+$ concave and $m \in \mathbb{N}$. Then

$$\binom{m+n}{n}^{-1} f(0)^m \leq \frac{1}{|K|} \int_K f(x)^m dx.$$

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$$\binom{m+n}{n}^{-1} f(0)^m \leq \frac{1}{|K|} \int_K f(x)^m dx.$$

Equality holds iff the graph of $f$ is a cone with basis $K \times \{0\}$ and apex $(0, f(0))$.

# Application

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**Theorem 6 (Brunn 1887 & Minkowski 1896)**

Let $K, C \in \mathcal{K}^n$. Then

$$|(1-\lambda)K + \lambda C|^{\frac{1}{n}} \geq (1-\lambda)|K|^{\frac{1}{n}} + \lambda |C|^{\frac{1}{n}}$$

for any $\lambda \in [0,1]$.

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for any $\lambda \in [0,1]$. Equality holds if $K = x + tC$, for $x \in \mathbb{R}^n$ and $t \geq 0$.

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**Theorem 7 (Rogers & Shephard '58)**

Let $K \in \mathcal{K}^n$ and $H \in \mathcal{L}_i^n$. Then

$$\binom{n}{i}^{-1} |P_H K| \cdot |K \cap H^{\perp}| \leq |K|.$$

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**Theorem 8 (Fubini's formula)**

Let $K \in \mathcal{K}^n$ and $H \in \mathcal{L}_i^n$. Then

$$|K| \leq |P_H K| \max_{x \in H} |K \cap (x + H^{\perp})|.$$

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**Corollary (Spingarn '93, Milman & Pajor '00)**

Let $K \in \mathcal{K}^n$ and $H \in \mathcal{L}_i^n$. Then

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**Corollary (Jensen 1906)**

Let $K \in \mathcal{K}^n$ and $H \in \mathcal{L}_{n-1}^n$. Then

$$|K| \leq |P_H K| \cdot |K \cap (x_{P_H K} + H^\perp)|.$$

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**Corollary (G.M.+19)**

Let $K \in \mathcal{K}^n$ and $H \in \mathcal{L}_i^n$ with $P_H K = B_2^i$. Then

$$|K| \leq \frac{2^n}{\pi^{\frac{1}{2}} n!} \Gamma\left(\frac{2n-i+1}{2}\right) \Gamma\left(\frac{i+2}{2}\right) |P_H K| \cdot |K \cap H^{\perp}|.$$

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**Corollary (G.M.+19)**

Let $K \in \mathcal{K}^n$ and $H \in \mathcal{L}_i^n$ with $P_H K = -P_H K$. Then

$$|K| \leq \frac{2^{n-i}}{n-i+1} |P_H K| \cdot |K \cap H^\perp|.$$

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*Proof of Corollary.* By Fubini's formula

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$$\frac{|K|}{|P_H K|} = \frac{1}{|P_H K|} \int_{P_H K} f(x)^{n-i} dx$$

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# Bibliography


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Thank you for your attention!!