A CHARACTERIZATION OF THE DENSITY PROPERTY FOR TRANSLATION INVARIANT BASES

XV Encuentro de la Red de Análisis Funcional y Aplicaciones Bilbao, 7 - 8 de marzo de 2019

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(joint work with Paul A. Hagelstein)

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Friday 8th March, 2019

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DENSITY PROPERTY FOR TI BASES

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DIFFERENTIATION BASES AND THE BUSEMANN-FELLER THEOREM.

- 2 The density property for translation invariant bases
- 3 A COUNTEREXAMPLE TO A CENTERED DE GUZMÁN CONJECTURE
- A few words about the proof

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A basis B is a collection of open bounded sets in Rⁿ. Some important examples of bases are:
We say that B differentiates a class of functions X if

$\int_{B} f := \frac{1}{|B|} \int_{B} f(y) dy \xrightarrow[\text{Berg Berg}]{} f(x), \quad \text{for non } x \in \mathbb{R}^{n}, \\ \underset{(\text{diam}(B) \to 0)}{\text{diam}(b) \to 0}$

- We say that \mathfrak{B} is a *density basis* if \mathfrak{B} differentiates the class $X^{\text{dens}} := \{1_E : E \subseteq \mathbb{R}^n, 0 < |E| < \infty\}.$
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 The differentiation properties of B are controlled by the mapping properties of the corresponding maximal operator:

• The typical estimate we are looking for is of the type $M_2 : D \to D^{p,m}$ for some $1 \le p < \infty$

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• The differentiation properties of \mathfrak{B} are controlled by the mapping properties of the corresponding maximal operator:

The non-centered maximal function of f with respect to $\mathfrak B$

$$M_{\mathfrak{B}}f(x) \coloneqq \sup_{\substack{B \in \mathfrak{B} \\ B \ni x}} \int_{B} |f(y)| dy, \quad x \in \bigcup_{B \in \mathfrak{B}} B, \quad f \in L^{1}_{\mathrm{loc}}(\mathbb{R}^{n}).$$

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When does B differentiate L^{oo}?

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There exists differentiation bases
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⁶ which differentiate L[∞] (so they are density bases) but do not differentiate L^p for any p < ∞ (Hayes '50s and also subsequent slides).

We call $C_{2}(\alpha)$ the halo function of \mathfrak{B} .

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The BUSEMANN-FELLER THEOREM (1939) If 3 is a homotheory invariant basis then 3 is a density basis if and only if for all $a \in (0,1)$ $C_{0}(a) = \sum_{i=1}^{n} [a \in \mathbb{R}^{n} \circ d(a \lfloor a(r) \to a)] < +\infty.$

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We call C_B(α) the halo function of B. Note that C(α) < ∞ is automatically satisfied for all α if M_B is weak-type (p, p) for some p < ∞.

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- Let us now consider a basis B which is translation invariant but not necessarily scale invariant.

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- The Busemann-Feller theorem gives a quite satisfactory characterization of density bases in the homothecy invariant case.
- Let us now consider a basis \mathfrak{B} which is *translation invariant* but not necessarily scale invariant. Immediately one discovers many pathological examples simply by introducing bad sets in large scales. Clearly these sets do not affect the differentiation properties of \mathfrak{B} but descree the boundedness properties of the

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 $M_{\mathfrak{B},r,f}(x) \coloneqq \sup_{\substack{B \in \mathfrak{B}(x) \\ \operatorname{diam}(B) < r}} \int_{B} |f(y)| dy, \qquad \mathfrak{B} \coloneqq \bigcup_{x \in \mathbb{R}^{n}} \mathfrak{B}(x).$

Here $\mathfrak{B}(x)$ is a given collection of sets containing $x \in \mathbb{R}^n$. For example $\mathfrak{B}(x)$ is the collection of all balls centered at x.

 A counterexample to De Guzmán's conjecture in the centered case can be constructed as follows. For k ∈ N let

 $\mathfrak{B}_{k}(0) := \left\{ \left(-\frac{\delta}{2}, \frac{\delta}{2}\right) \cup (\epsilon, s+2\gamma^{k}\delta) : 0 < \delta < 2\gamma^{k}, \dots, s \in (2\gamma^{k}, 2\gamma^{k}+2\gamma^{k}) \right\}.$ (Then external DV translation invariance $\mathfrak{B} := [1 - \gamma s, 1 + 2\gamma^{k})$

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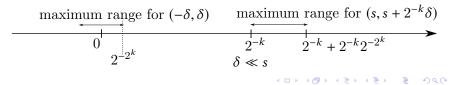
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$$\frac{|E \cap (I_1 \cup I_2)|}{|I_1 \cup I_2|} > \alpha \implies \frac{|I_1 \cap E|}{|I_1|} > \alpha/2.$$

- By the Hardy-Littlewood maximal theorem we then get that $C_{\mathfrak{B}_k}(\alpha) \leq 1/\alpha$ if $2^{-k} \leq \alpha$. Thus for every $\alpha \in (0, 1)$ there exists $r = r(\alpha)$ (!) such that $C_{\mathfrak{B},r}(\alpha) < \infty$.
- We show however that it is impossible to choose a *uniform* r for every $\alpha \in (0, 1)$ so that $C_{\mathfrak{B}, r}(\alpha) < \infty$.
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- DIFFERENTIATION BASES AND THE BUSEMANN-FELLER THEOREM.
- 2 The density property for translation invariant bases
- 3 A COUNTEREXAMPLE TO A CENTERED DE GUZMÁN CONJECTURE
- A FEW WORDS ABOUT THE PROOF

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• Assuming our conclusion fails (for contradiction) there exists $\alpha \in (0, 1)$ and sets $S_{k,\ell} \subset \mathbb{R}^n$ of finite and positive measure with

 $|\{x\in \mathbb{R}^n:\, M_{2^{-k}}\mathbf{1}_{S_{k,\ell}}(x)>\alpha\}|\geq 2^\ell|S_{k,\ell}|,\qquad \forall (k,\ell)\in \mathbb{N}^2.$

• By pigeonholing and using that $M_{2^{-k}}$ is a local operator we can then find a unit cube Q and for each $\ell \in \mathbb{N}$ a set $E_{\ell} := S_{\ell,\ell} \cap 3Q$ of positive and finite measure such that

 $|F_{\ell} := \{ x \in Q : M_{2^{-\ell}} \mathbf{1}_{E_{\ell}}(x) > \alpha \} | \gtrsim_n 2^{\ell} |E_{\ell}|, \qquad |E_{\ell}| \lesssim_n 2^{-\ell} |F_{\ell}|.$

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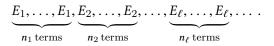
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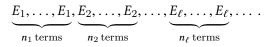
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With these definitions we can conclude (remember $n_{\ell}|F_{\ell}| = 1$) $\sum_{m} |\overline{E}_{m}| = \sum_{\ell} n_{\ell}|E_{\ell}| \leq \sum_{\ell} n_{\ell} 2^{-\ell}|F_{\ell}| \leq 1, \quad \sum_{m} |\overline{E}_{m}| = +\infty$

- By a Borel-Cantelli type of argument we can now find translations $\{\tau_m\}_m$ such that almost every point of \mathbb{R}^n is contained in infinitely many of the sets $\{\tau_m \widetilde{F}_m\}_m$.
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• With these definitions we can conclude (remember $n_{\ell}|F_{\ell}| \approx 1$)

$$\sum_{m} |\widetilde{E}_{m}| = \sum_{\ell} n_{\ell} |E_{\ell}| \lesssim \sum_{\ell} n_{\ell} 2^{-\ell} |F_{\ell}| \lesssim 1, \quad \sum_{m} |\widetilde{F}_{m}| = +\infty$$

- By a Borel-Cantelli type of argument we can now find translations $\{\tau_m\}_m$ such that almost every point of \mathbb{R}^n is contained in infinitely many of the sets $\{\tau_m \widetilde{F}_m\}_m$.
- By this construction and translation invariance we can now find a strictly increasing sequence $\{m_j\}_j$ such that almost every $x \in \mathbb{R}^n$ satisfies

$$\begin{aligned} x \in \bigcap_{j=1}^{\infty} \left\{ Q : M_{2^{-m_{j}}} \mathbb{1}_{\tau_{m_{j}}E_{m_{j}}} > \alpha \right\} &\subseteq \bigcap_{j=1}^{\infty} \left\{ Q : M_{2^{-j}} \mathbb{1}_{\tau_{m_{j}}E} > \alpha \right\}, \\ E &\coloneqq \bigcup_{m} \tau_{m} \widetilde{E}_{m}, \qquad |E| < +\infty. \end{aligned}$$

• We have proved that if the necessity statement of the theorem is negated then for almost every $x \in \mathbb{R}^n$ we have

$$x \in \bigcap_{j=1}^{\infty} \{Q : M_{2^{-j}} \mathbb{1}_E > \alpha\}, \qquad |E| < \infty.$$

• We conclude that there exists $\alpha > 0$ such that, for almost every $x \in \mathbb{R}^n$ there exists a sequence of sets $\{R_{x,j}\}_j \subset \mathfrak{B}$ with $\operatorname{diam}(R_{x,j}) \leq 2^{-j}$, and such that

$$\frac{|R_{x,j} \cap E|}{|R_{x,j}|} > \alpha, \qquad \forall j \ge 1.$$

• But \mathfrak{B} is a density basis so for almost all $x \in E^{\mathsf{c}}$ we must have

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