# A characterization of the density property for translation invariant bases

XV Encuentro de la Red de Análisis Funcional y Aplicaciones
Bilbao, 7 - 8 de marzo de 2019

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Friday $8^{\text{th}}$ March, 2019

1. Differentiation bases and the Busemann-Feller theorem.
2. The density property for translation invariant bases
3. A counterexample to a centered de Guzmán conjecture
4. A few words about the proof

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# Contents

# Differentiation bases

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  - The collection $\mathfrak{Q}$ consisting of all *cubes* in $\mathbb{R}^n$ with sides // axes.
  - The collection $\mathfrak{b}$ consisting of all *Euclidean balls* in $\mathbb{R}^n$.
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$$\fint_B f := \frac{1}{|B|}\int_B f(y)dy \xrightarrow[\substack{B \ni x,\ B \in \mathfrak{B} \\ \mathrm{diam}(B) \to 0}]{} f(x), \quad \text{for a.e. } x \in \mathbb{R}^n.$$

- We say that $\mathfrak{B}$ is a *density basis* if $\mathfrak{B}$ differentiates the class $X^{\mathrm{dens}} := \{1_E : E \subseteq \mathbb{R}^n, 0 < |E| < \infty\}$.
- If we are working with an abstract basis $\mathfrak{B}$ we many times require additional structure. The basis $\mathfrak{B}$ is homothecy invariant:

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# The Maximal Operator

- The differentiation properties of $\mathfrak{B}$ are controlled by the mapping properties of the corresponding maximal operator:

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**The non-centered maximal function of $f$ with respect to $\mathfrak{B}$**

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# The Busemann-Feller theorem

- There exists differentiation bases $\mathfrak{B}$ which differentiate $L^\infty$ (so they are density bases) but do not differentiate $L^p$ for any $p < \infty$ (Hayes '50s and also subsequent slides).

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**The Busemann-Feller theorem (1939)**

If $\mathfrak{B}$ is a homothecy invariant basis then $\mathfrak{B}$ is a density basis if and only if for all $\alpha \in (0,1)$

$$C_{\mathfrak{B}}(\alpha) = \sup_{E:\, 0<|E|<\infty} \frac{1}{|E|} |\{x \in \mathbb{R}^n : M_{\mathfrak{B}}\chi_E(x) > \alpha\}| < +\infty.$$

- We call $C_{\mathfrak{B}}(\alpha)$ the *halo function of* $\mathfrak{B}$. Note that $C(\alpha) < \infty$ is automatically satisfied for all $\alpha$ if $M_{\mathfrak{B}}$ is weak-type $(p,p)$ for some $p < \infty$.

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# Translation invariant bases

- The Busemann-Feller theorem gives a quite satisfactory characterization of density bases in the homothecy invariant case.
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**Question: (M. de Guzman, 1975)**

If $\mathfrak{B}$ is just translation invariant is it true that $\mathfrak{B}$ is a density basis if and only if there exists (small) $r > 0$ such that for all $\alpha \in (0,1)$ we have

$$C_{\mathfrak{B}}(\alpha) = \sup_{0<|E|<\infty} \frac{1}{|E|} |\{x \in \mathbb{R}^n : M_{\mathfrak{B},r}\chi_E(x) > \alpha\}| < \infty ?$$

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- Let us now consider a basis $\mathfrak{B}$ which is *translation invariant* but *not* necessarily *scale invariant*. Immediately one discovers many pathological examples simply by introducing bad sets in large scales. Clearly these sets do not affect the differentiation properties of $\mathfrak{B}$ but destroy the boundedness properties of $M_{\mathfrak{B}}$.

**Question: (M. de Guzman, 1975)**

If $\mathfrak{B}$ is just translation invariant is it true that $\mathfrak{B}$ is a density basis if and only if there exists (small) $r > 0$ such that for all $\alpha \in (0,1)$ we have

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$$M_{\mathfrak{B},r}f(x) := \sup_{\substack{B \in \mathfrak{B}(x) \\ \operatorname{diam}(B) < r}} \fint_B |f(y)|dy, \qquad \mathfrak{B} := \bigcup_{x \in \mathbb{R}^n} \mathfrak{B}(x).$$

  Here $\mathfrak{B}(x)$ is a given collection of sets containing $x \in \mathbb{R}^n$. For example $\mathfrak{B}(x)$ is the collection of all balls centered at $x$.
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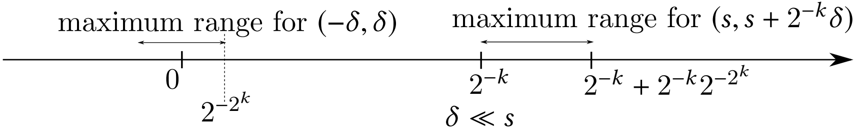

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- Let $\alpha \in (0,1)$, $k \in \mathbb{N}$ such that $2^{-k} < \alpha/2$, and $x \in \mathbb{R}$ such that $M_{\mathfrak{B}} 1_E(x) > \alpha$ for some $E$ with $0 < |E| < \infty$.
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- However, the positive direction of the theorem shows that $\mathfrak{B}$ is in fact a density basis.

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# A counterexample for centered bases, II

- We showed $M_{\mathfrak{B}_k} 1_E > 1/(1+2^{-k})$ on an interval of length $2^{-2^k}$, and $|E| \approx \delta \ll 2^{-2^k}$.
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# The necessity part of the theorem, I

- Assuming our conclusion fails (for contradiction) there exists $\alpha \in (0,1)$ and sets $S_{k,\ell} \subset \mathbb{R}^n$ of finite and positive measure with

$$|\{x \in \mathbb{R}^n : M_{2^{-k}} 1_{S_{k,\ell}}(x) > \alpha\}| \geq 2^{\ell} |S_{k,\ell}|, \qquad \forall (k,\ell) \in \mathbb{N}^2.$$

- By pigeonholing and using that $M_{2^{-k}}$ is a local operator we can then find a unit cube $Q$ and for each $\ell \in \mathbb{N}$ a set $E_\ell \coloneqq S_{\ell,\ell} \cap 3Q$ of positive and finite measure such that

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- With these definitions we can conclude (remember $n_\ell |F_\ell| \approx 1$)

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- By a Borel-Cantelli type of argument we can now find translations $\{\tau_m\}_m$ such that almost every point of $\mathbb{R}^n$ is contained in infinitely many of the sets $\{\tau_m \widetilde{F}_m\}_m$.
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$$\frac{|R_{x,j} \cap E|}{|R_{x,j}|} > \alpha, \qquad \forall j \geq 1.$$

- But $\mathfrak{B}$ is a density basis so for almost all $x \in E^{c}$ we must have

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