Maximal averaging operators: from geometry to boundedness through duality

Arraz Almirall, Alexis (Universitat de Barcelona)
Cobollo Gómez, Christian (Universitat de València)
Jardón Sánchez, Héctor (Universidá d'Uviéu)
Martín Murillo, Cristina (Universidad de Extremadura)
Quilis Sandemetrio, Andrés (Universitat Politècnica de València)
Ribera Baraut, Pol (Universitat de Barcelona)

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Introduction

Motivation Question

Lebesgue Differentiation Theorem

$$\frac{1}{|Q(x,r)|}\int_{Q(x,r)} f \xrightarrow[r\to 0]{} f(x) \quad \text{a.e } x\in\mathbb{R}^n, \quad \text{for all } f\in L^p(\mathbb{R}^n)$$

$Q(x,r)$ is a cube centered at x and sidelength r.

Some Notation

Definition

$\mathcal{B} :=$ {Collection of bounded open sets on $\mathbb{R}^n$}

$\mathcal{R}_n :=$ {Open rectangles on $\mathbb{R}^n$}

$\mathcal{Q}_n :=$ {Open cubes on $\mathbb{R}^n$}

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They are all homothecy invariant.

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Maximal operator associated to $\mathcal{B}$:

$$M_{\mathcal{B}}f(x) := \sup_{\substack{B\in\mathcal{B} \\ x\in B}} \frac{1}{|B|}\int_B |f|, \quad f \in L^p(\mathbb{R}^n), \quad x \in \mathbb{R}^n.$$

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- $L^p(\mathbb{R}^n) := \left\{ f \text{ measurable} : \left(\int_{\mathbb{R}^n} |f|^p \right)^{\frac{1}{p}} =: ||f||_{L^p(\mathbb{R}^n)} < \infty \right\}$

For all $f \in L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ we have

$$|\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| \leq \frac{C_0^p}{\lambda^p} \qquad (*)$$

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- $||f||_{L^{p,\infty}} := \{\text{The smallest } C_0 \text{ that verifies (*)}\}$
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$$|\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| \leq \frac{C_0^p}{\lambda^p} \qquad (*)$$

Definition

We say that T is strong-type (p, p) if $T : L^p \longrightarrow L^p$ is bounded.
We say that T is weak-type (p, p) if $T : L^p \longrightarrow L^{p,\infty}$ is bounded,

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \leq \frac{C^p}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p dx, \quad \lambda > 0,$$

for some $C > 0$.

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Lemma

T is of weak-type (p,p) $(1 < p < \infty)$ if and only if for every E with $0 < |E| < \infty$

$$\left| \int_E Tf \right| \leq C_p \|f\|_{L^p(\mathbb{R}^n)} |E|^{\frac{1}{p'}}.$$

Strong Maximal Theorem

Strong Maximal Theorem (Jessen, Marcinkiewicz, Zygmund 1935)

The estimate

$$|\{x \in \mathbb{R}^n : M_{\mathcal{R}_n} f(x) > \lambda\}| \leq C_n \int_{\mathbb{R}^n} \frac{|f|}{\lambda} \left(1 + \log^+ \frac{|f|}{\lambda}\right)^{n-1}$$

holds for every $\lambda > 0$, with $\log^+ t = \max\{0, \log t\}$. It follows that $\mathcal{R}_n$ differentiates functions f for which

$$\int_K |f|(1 + \log^+ |f|)^{n-1} < \infty,$$

for every compact set $K \subset \mathbb{R}^n$.

Duality link between analysis and geometry

Duality approach

Duality approach for general $\mathcal{B}$ (A. Córdoba - R. Fefferman).

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Definition (Covering property V_q)

Let $1 \leq q \leq \infty$. We say that $\mathcal{B}$ has the *covering property* V_q if there exist $c_1, c_2 > 0$ such that for every finite collection $\{B_j\}_{j=1}^N \subset \mathcal{B}$ there exists a finite subcollection $\{\tilde{B}_k\}_{k=1}^M$ such that

(i) $\left| \bigcup_{j=1}^N B_j \right| \leq c_1 \left| \bigcup_{k=1}^M \tilde{B}_k \right|$ (We don't lose much of the measure)

(ii) $\left\| \sum_{k=1}^M \mathbf{1}_{\tilde{B}_k} \right\|_{L^q(\mathbb{R}^n)} \leq c_2 \left| \bigcup_{j=1}^N B_j \right|^{\frac{1}{q}}$ (Control of the overlap)

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Duality approach for general $\mathcal{B}$ (A. Córdoba - R. Fefferman).

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Proposition

Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. The maximal operator $M_{\mathcal{B}}$ is of weak-type (p, p) if and only if $\mathcal{B}$ has the covering property $V_{p'}$.

Proof: $V_{p'} \Rightarrow$ weak-type (p, p)

Consider

$$E_\lambda = \{x \in \mathbb{R}^n : M_{\mathcal{B}} f(x) > \lambda\}$$

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such that

$$\frac{1}{|B_x|} \int_{B_x} |f(y)| dy > \lambda, \quad x \in B_x.$$

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By the property (i) of $V_{p'}$

$$|K| \leq \left| \bigcup_{j=1}^{N} B_j \right| \leq c_1 \left| \bigcup_{k=1}^{M} \tilde{B}_k \right|.$$

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Consider

$$K \subset \bigcup_{j=1}^{N} B_j$$

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By Hölder's inequality and (ii) of $V_{p'}$:

$$|K| \leq \left| \bigcup_{j=1}^{N} B_j \right| \leq c_1 \left| \bigcup_{k=1}^{M} \tilde{B}_k \right| \leq \frac{c_1}{\lambda} \int_{\mathbb{R}^n} \sum_{k=1}^{M} \mathbf{1}_{\tilde{B}_k}(y) |f(y)| dy$$

$$\overset{\text{Hölder}}{\leq} \frac{c_1}{\lambda} \left\| \sum_{k=1}^{M} \mathbf{1}_{\tilde{B}_k} \right\|_{L^{p'}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} \overset{(ii)}{\leq} \frac{c_1 c_2}{\lambda} \left| \bigcup_{j=1}^{N} B_j \right|^{\frac{1}{p'}} \|f\|_{L^p(\mathbb{R}^n)}.$$

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Consider

$$K \subset \bigcup_{j=1}^{N} B_j$$

such that

$$\frac{1}{|B_j|} \int_{B_j} |f(y)| dy > \lambda$$

With a few more simple computations:

$$|K| \leq \frac{c_1^p c_2^p}{\lambda^p} \|f\|_p^p$$

and let $K \nearrow E_\lambda$.

Proof: weak-type $(p, p) \Rightarrow V_{p'}$

Let $\{B_j\}_{j=1}^N \subset \mathcal{B}$, and suppose without loss of generality that

$$|B_1| \geq \cdots \geq |B_N|.$$

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We extract a subcollection as follows:

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- $\tilde{B}_1 = B_1$
- Selected $\{\tilde{B}_k\}_{k=1}^m$, $m < N$, choose the first $B \in \{B_j\}_{j=1}^N \setminus \{\tilde{B}_k\}_{k=1}^m$ verifying:

$$\left| B \cap \bigcup_{k \leq m} \tilde{B}_k \right| \leq \frac{1}{2}|B|$$

The overlapping is less than 50% in measure.

Proof: weak-type $(p, p) \Rightarrow V_{p'}$

We have selected $\{\tilde{B}_k\}_{k=1}^M$ such that, defining

$$\tilde{E}_k = \tilde{B}_k \setminus \bigcup_{j<k} \tilde{B}_j,$$

it holds

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To prove (i):

$$\bigcup_{\substack{B \text{ not} \\ \text{selected}}} B \subset \bigcup \left\{ B : \frac{|B \cap \bigcup \tilde{B}_k|}{|B|} > \frac{1}{2} \right\} \subset \left\{ M_{\mathcal{B}}(\mathbf{1}_{\bigcup \tilde{B}_k}) > \frac{1}{2} \right\}.$$

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Proof: weak-type $(p, p) \Rightarrow V_{p'}$

To prove (ii) we define the linear and weak-type (p, p) operator

$$Tf(x) := \sum_{k=1}^{M} \left(\frac{1}{|\tilde{B}_k|} \int_{\tilde{B}_k} f(y)dy \right) \mathbf{1}_{\tilde{E}_k}(x) \leq M_{\mathcal{B}} f(x).$$

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To prove (ii) we define the linear and weak-type (p,p) operator

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Computing the adjoint of T and evaluating at $\mathbf{1}_{\bigcup_k \tilde{B}_k}$ we get

$$T^*(\mathbf{1}_{\bigcup_k \tilde{B}_k}) = \sum_{k=1}^{M} \frac{|\tilde{E}_k|}{|\tilde{B}_k|} \mathbf{1}_{\tilde{B}_k} \geq \frac{1}{2} \sum_{k=1}^{M} \mathbf{1}_{\tilde{B}_k}.$$

Proof: weak-type $(p, p) \Rightarrow V_{p'}$

Therefore

$$\left| \int_{\mathbb{R}^n} \sum_{k=1}^{M} \mathbf{1}_{\tilde{B}_k} f \right| \leq 2 \left| \int_{\mathbb{R}^n} T^*(\mathbf{1}_{\bigcup_k \tilde{B}_k}) f \right| = 2 \left| \int_{\bigcup_k \tilde{B}_k} Tf \right| \leq 2 C_p \|f\|_{L^p(\mathbb{R}^n)} \left| \bigcup_{k=1}^{M} \tilde{B}_k \right|^{\frac{1}{p'}}.$$

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Therefore

$$\left|\int_{\mathbb{R}^n}\sum_{k=1}^{M}\mathbf{1}_{\tilde{B}_k}f\right| \le 2\left|\int_{\mathbb{R}^n}T^*(\mathbf{1}_{\bigcup_k \tilde{B}_k})f\right| = 2\left|\int_{\bigcup_k \tilde{B}_k}Tf\right| \le 2C_p\|f\|_{L^p(\mathbb{R}^n)}\left|\bigcup_{k=1}^{M}\tilde{B}_k\right|^{\frac{1}{p'}}.$$

Taking the supremum over all $f \in L^p(\mathbb{R}^n)$ with $\|f\|_{L^p(\mathbb{R}^n)} \le 1$ we obtain

$$\left\|\sum_{k=1}^{M}\mathbf{1}_{\tilde{B}_k}\right\|_{L^{p'}(\mathbb{R}^n)} \le 2C_p\left|\bigcup_{k=1}^{M}\tilde{B}_k\right|^{\frac{1}{p'}}.$$

□

Some remarks

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The covering property V_∞ implies weak-type $(1,1)$.

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Remark 2

The differentiation basis given by all cubes $\mathcal{Q}_n$ verifies the property V_∞ (Vitali's covering lemma). We conclude that $\mathcal{Q}_n$ differentiates $L^1(\mathbb{R}^n)$.

Strong Maximal Theorem

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- Setting: rectangles in $\mathbb{R}^n$. For simplicity, we will restrict to $\mathbb{R}^2$. Let $\mathcal{R}_2$ be the set of all rectangles in $\mathbb{R}^2$ with sides parallel to the axes.

Strong Maximal Theorem

- Setting: rectangles in $\mathbb{R}^n$. For simplicity, we will restrict to $\mathbb{R}^2$. Let $\mathcal{R}_2$ be the set of all rectangles in $\mathbb{R}^2$ with sides parallel to the axes.
- $M_{\mathcal{R}_2}$ is **not** weak-type $(1,1)$. The following does **not** hold for any c

$$|\{x \in \mathbb{R}^2 : M_{\mathcal{R}_2} f(x) > \lambda\}| \leq \frac{c}{\lambda} \int_{\mathbb{R}^2} |f(y)| dy.$$

Strong Maximal Theorem

New covering property $V_{\exp}$: we say that $\mathcal{B}$ has the *covering property* $V_{\exp}$ if there exist $c_1, c_2 > 0$ such that for every finite collection $\{R_j\}_{j=1}^N \subset \mathcal{B}$ there exists a finite subcollection $\{\tilde{R}_k\}_{k=1}^M$ such that

i) $$\left|\bigcup_{j=1}^N R_j\right| \leq c_1 \left|\bigcup_{k=1}^M \tilde{R}_k\right|,$$

ii) $$\int_{\mathbb{R}^2} \left[\exp\left(\theta \sum_{k=1}^M \mathbf{1}_{\tilde{R}_k}\right) - 1\right] \leq \theta c_2 \left|\bigcup_{k=1}^M \tilde{R}_k\right| \quad \text{for } \theta \text{ small.}$$

Strong Maximal Theorem: $V_{\exp} \Rightarrow$ *boundedness*

$V_{\exp}$ implies the strong maximal theorem in $\mathbb{R}^2$.

Theorem

If $\mathcal{B}$ has the covering property $V_{\exp}$, then

$$\left|\{x \in \mathbb{R}^2 : M_{\mathcal{B}} f(x) > \lambda\}\right| \leq C \int_{\mathbb{R}^2} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \frac{|f(x)|}{\lambda}\right) dx,$$

where $\log^+ t := \max\{0, \log t\}$.

Strong Maximal Theorem: $V_{\exp} \Rightarrow$ *boundedness*

Sketch of proof:

- Instead of Hölder's, use Young's inequality (Hölder's is a particular case):

$$st \leq c_\theta s\left(1 + \log^+ s\right) + \exp\left(\theta t\right) - 1,$$

where $s, t \geq 0$ and $\theta > 0$.

$$|K| \leq c_1 \left|\bigcup_{k=1}^{M} \tilde{R}_k\right| \leq c_1 \int_{\mathbb{R}^2} \left(\sum_{k=1}^{M} \mathbf{1}_{\tilde{R}_k}(y)\right) \frac{|f(y)|}{\lambda} dy$$

$$\overset{Young}{\leq} c_1 \int_{\mathbb{R}^2} \left[c_\theta \frac{|f(y)|}{\lambda}\left(1 + \log^+ \frac{|f(y)|}{\lambda}\right) + \exp\left(\theta \sum_{k=1}^{M} \mathbf{1}_{\tilde{R}_k}(y)\right) - 1\right] dy$$

$$\overset{(ii)}{\leq} \quad \dots \quad + c_1 c_2 \theta \left|\bigcup_{k=1}^{M} \tilde{R}_k\right|.$$

Strong Maximal Theorem: $\mathcal{R}_2$ *has* $V_{\exp}$

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The differentiation basis of rectangles $\mathcal{R}_2$ *satisfies the* $V_{\exp}$ *covering property.*

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Sketch of proof (inductive selection scheme): Let $\Pi_1(R), \Pi_2(R)$ denote the projections of $R \in \mathcal{R}_2$ on the x and y axis (resp.).

- Let $\{R_j\}_{j=1}^N \subset \mathcal{R}_2$ ordered by $|\Pi_2(R_1)| \geq \cdots \geq |\Pi_2(R_N)|$.

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- Fix $\tilde{R}_1 := R_1$.

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- Let $\{R_j\}_{j=1}^N \subset \mathcal{R}_2$ ordered by $|\Pi_2(R_1)| \geq \cdots \geq |\Pi_2(R_N)|$.
- Fix $\tilde{R}_1 := R_1$.
- Having selected $\{\tilde{R}_k\}_{k=1}^m$, $m < N$, choose the first $R \in \{R_j\}_{j=1}^N \setminus \{\tilde{R}_k\}_{k=1}^m$ verifying either

$$\left| R \cap \left(\cup_{j<k} \tilde{R}_k^* \right) \right| \leq \frac{1}{2} |R|,$$

or $R \cap \left(\cup_{j<k} \tilde{R}_k \right) = \emptyset$.

Strong Maximal Theorem: $\mathcal{R}_2$ *has* V_{exp}

Sketch of proof: claim $\{\tilde{R}_k\}_{k=1}^m$ satisfies the V_{exp} estimates.

Strong Maximal Theorem: $\mathcal{R}_2$ *has* $V_{\exp}$

Sketch of proof: claim $\{\tilde{R}_k\}_{k=1}^m$ satisfies the $V_{\exp}$ estimates.

- The selection scheme allows us to show that the 1D slices of the $\{\tilde{R}_k\}_{k=1}^m$ satisfy similar sparseness properties.

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- The selection scheme allows us to show that the 1D slices of the $\{\tilde{R}_k\}_{k=1}^m$ satisfy similar sparseness properties.
- Fix $y \in \Pi_2(R)$, R not selected. Then,

$$\text{(2D)} \ \left| R \cap \left(\cup_{j<k} \tilde{R}_k^* \right) \right| > \frac{1}{2}|R| \Rightarrow$$

$$\Rightarrow \text{(1D)} \ \left| \Pi_1^y(R) \cap \left(\cup_{j<k} \Pi_1^y(\tilde{R}_k) \right) \right| > \frac{1}{2}|\Pi_1^y(R)|,$$

and these are 1D averages.

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and these are 1D averages.

$$\bigcup_{\substack{R \text{ not} \\ \text{selected}}} \Pi_1^y(R) \subset \left\{ x \in \mathbb{R} : M_{\mathcal{R}_1} \mathbf{1}_{\bigcup_{j<k} \Pi_1^y(\tilde{R}_k)}(x) > \frac{1}{2} \right\}$$

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Sketch of proof: i) $\left|\bigcup_{j=1}^{N} R_j\right| \leq C_1 \left|\bigcup_{k=1}^{M} \tilde{R}_k\right|.$

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Sketch of proof: i) $\left|\bigcup_{j=1}^{N} R_j\right| \leq C_1 \left|\bigcup_{k=1}^{M} \tilde{R}_k\right|.$

- Maximal operator $M_{\mathcal{R}_1}$ is Hardy-Littlewood maximal operator M_1.
- weak-type properties of M_1 prove the estimates

$$\left|\bigcup_{\substack{R \text{ not} \\ \text{selected}}} \Pi_1^y(R)\right| \leq C_1 \left|\bigcup_{k=1}^{M} \Pi_1^y(\tilde{R}_k)\right|.$$

Strong Maximal Theorem: $\mathcal{R}_2$ *has* $V_{\exp}$

Sketch of proof: i) $\left|\bigcup_{j=1}^{N} R_j\right| \leq C_1 \left|\bigcup_{k=1}^{M} \tilde{R}_k\right|.$

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- Integrate in y.

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Sketch of proof: ii) $$\int_{\mathbb{R}^2} \left[\exp\left(\theta \sum_{k=1}^{M} \mathbf{1}_{\tilde{R}_k} \right) - 1 \right] \leq \theta C_2 \left| \bigcup_{k=1}^{M} \tilde{R}_k \right|.$$

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- Take the exponential series expansion

$$\int_{\mathbb{R}^2}\sum_{p=1}^{\infty}\frac{\theta^p}{p!}\left(\sum_{k=1}^{M}\mathbf{1}_{\tilde{R}_k}\right)^p=\sum_{p=1}^{\infty}\frac{\theta^p}{p!}\left\|\sum_{k=1}^{M}\mathbf{1}_{\tilde{R}_k}\right\|_{L^p(\mathbb{R}^2)}^p.$$

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- Estimate the $L^p(\mathbb{R}^2)$ norms of the overlaps from the slices

$$\left\| \sum_{k=1}^{M} \mathbf{1}_{\Pi_1^y(\tilde{R}_k)} \right\|_{L^p(\mathbb{R}^2)}^p \leq c^p \left| \bigcup_{k=1}^{M} \Pi_1^y(\tilde{R}_k) \right| .$$

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- Boundedness of maximal operator equivalent to covering properties of the differentiation basis. Equivalence is provided by duality.

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- Rectangles $\neq$ cubes in covering terms, but boundedness properties still hold: strong maximal theorem.

Thanks for your attention!