La conjetura de Rudin

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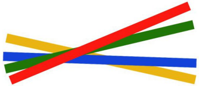
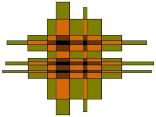

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RUDIN'S CONJECTURE

$$f \sim \sum a_n e^{2\pi i n^2 x} \in L^1(T) \Rightarrow f \in L^p(T), \quad \forall p < 4$$

$$\boxed{\text{More generally: } f \sim \sum a_n e^{2\pi i n^k x} \in L^1(T) \Rightarrow f \in L^p(T), \ \forall p < 2k}$$

An equivalent formulation arises substituting $L^1(T)$ by $L^2(T)$, because $\|f\|_2 \leq \|f\|_1^t \, \|f\|_p^{1-t}$ where $\frac{1}{2} = \frac{t}{1} + \frac{1-t}{2}$, $p > 2$.

In terms of the multiplier T_k: $T_k\left(\sum a_n e^{2\pi i n}\right) =: \sum a_{n^k} e^{2\pi i n^k x}$ we have:

$$\boxed{\text{CONJECTURE: } T_k \text{ maps } L^2(T) \text{ unto } L^p(T), \ \forall p < 2k.}$$

MOTIVATION (Lacunary Fourier Series).

$$f \sim \sum a_k e^{2\pi i n_k \cdot x}, \quad n_k / n_{k+1} \geq \rho > 1.$$

$\boxed{\text{Theorem: } f \in L^1(T) \implies f \in B.M.O.}$

B.M.O. $$\sup_{I \subset [0,1]} \frac{1}{\mu(I)} \int_I |f - f_I| = \|f\|_* < \infty, \quad f_I = \frac{1}{\mu(I)} \int_I f \, d\mu.$$

Theorem (John-Nirenberg).

$$f \in B.M.O. \implies \mu\{x \in I : |f(x) - f_I| \geq \lambda > 0\} \leq \mu(I) \, e^{-\frac{c\lambda}{\|f\|_*}}$$

where $c > 0$ is a universal constant.

Corollary: $f \in B.M.O. \implies f \in L^p, \ \forall p < \infty$

($\log|x| \in BMO$ but $\log|x| \notin L^\infty$).

Case $n=2$

Gaussian's sums: $G_N(x) = \sum_{0}^{N-1} e^{2\pi i n^2 x}$

Quadratic reciprocity.
Circle method (Hardy, Littlewood, Ramanujan).
Carleson's theorem.

$$G_N^*(x) = \sup_{k \le N} \left| \sum_{n=1}^{k} e^{2\pi i n^2 x} \right| \lesssim \frac{N}{\sqrt{q}} + \sqrt{q}, \quad \left|x - \frac{p}{q}\right| \le \frac{1}{q^2}.$$

$$\mu\{x : G_N^*(x) \ge \alpha > 0\} \lessapprox \frac{N^2}{\alpha^4}$$

$\boxed{\begin{array}{l}\textbf{Theorem.- } \text{Rudin's conjecture is true under the monotonicity} \\ \text{assumption about the coefficients: } a_0 \ge a_1 \ge \cdots \ge a_n \ge a_{n+1} \ge \cdots: \\ \quad f \sim \sum a_n e^{2\pi i n^2 x} \in L^1, \ \{a_n\} \text{ monotonic} \Rightarrow f \in L^p, \ \forall p < 4.\end{array}}$

SQUARES IN ARITHMETIC PROGRESSIONS

$$\{a_0 + nr\}_{n=0,1,2,\dots} \qquad a_0 \in \mathbb{Z}, \quad r \in \mathbb{Z}^+$$

$$\sigma(N; a_0, r) = \#\{m^2 = a_0 + nr \mid 0 \le r \le N-1\}$$

$$\sigma(N) = \sup_{a_0, r} \sigma(N; a_0, r)$$

$$\boxed{\text{ERDOS CONJECTURE: } \sigma(N) = O(N^{\frac{1}{2}+\varepsilon}), \quad \forall \varepsilon > 0}$$

Best known result: $\sigma(N) = O(N^{\frac{3}{5}+\varepsilon})$

(Bombieri, Zanner (2002)).

RUDIN'S CONJECTURE $\Longrightarrow$ ERDOS CONJECTURE

$$T_2 : L^2(\mathbb{T}) \longrightarrow L^p(\mathbb{T}), \text{ bounded } \forall p < 4$$

$\Longleftrightarrow$ (duality)

$$T_2 : L^q(\mathbb{T}) \longrightarrow L^2(\mathbb{T}), \text{ bounded } \forall q > \tfrac{4}{3}.$$

Let us consider $f(x) = \sum_0^{N-1} e^{2\pi i (a_0 + nr)x}$

Then $\|T_2 f\|_2^2 = \sigma(N; a_0, r)$

and $\int_0^1 |f(x)|^q \, dx = \int_0^1 \left| \sum_0^{N-1} e^{2\pi i n r x} \right|^q dx =$

$$= \frac{1}{r} \int_0^r \left| \sum_0^{N-1} e^{2\pi i n x} \right|^q dx = \int_0^1 |D_N(x)|^q dx \simeq \begin{cases} \log N, & q = 1 \\ N^{q-1}, & q > 1, \end{cases}$$

Therefore: $\sigma(N; a_0, r) = \|T_2 f\|_2^2 \lesssim N^{\frac{2}{q}(q-1)} = N^{2-\frac{2}{q}} = O(N^{\frac{1}{2}+\varepsilon}).$

Donde hay vino beben vino,
y donde no hay vino agua fresca.
Antonio Machado

Riemann's Fourier Series $(\delta > 1)$

$$\sum \frac{1}{n^\delta} e^{2\pi i n^k x} = F_{k,\delta}(x) + i\, G_{k,\delta}(x)$$

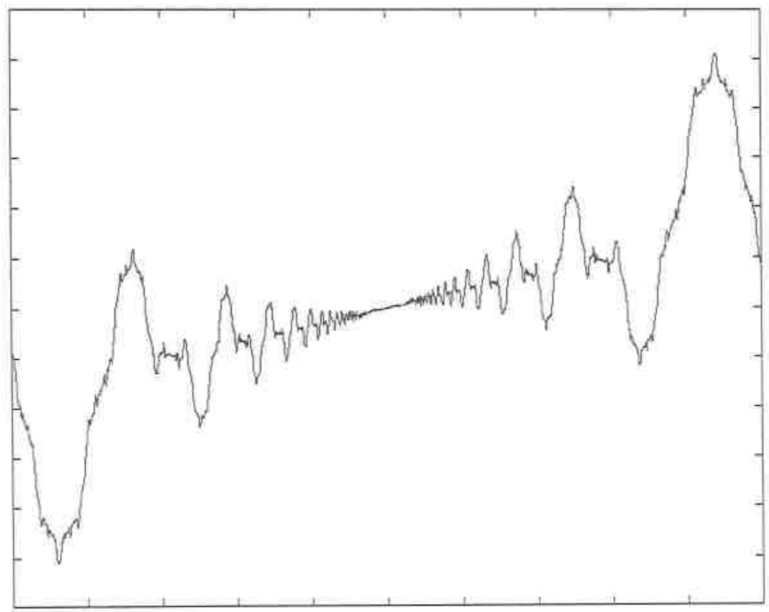
Theorem (F. Chamizo, A.C.).- The graphs of $F_{k,\delta}$, $G_{k,\delta}$ are fractals whose Minkowski dimension $d_{k,\delta}$ is:

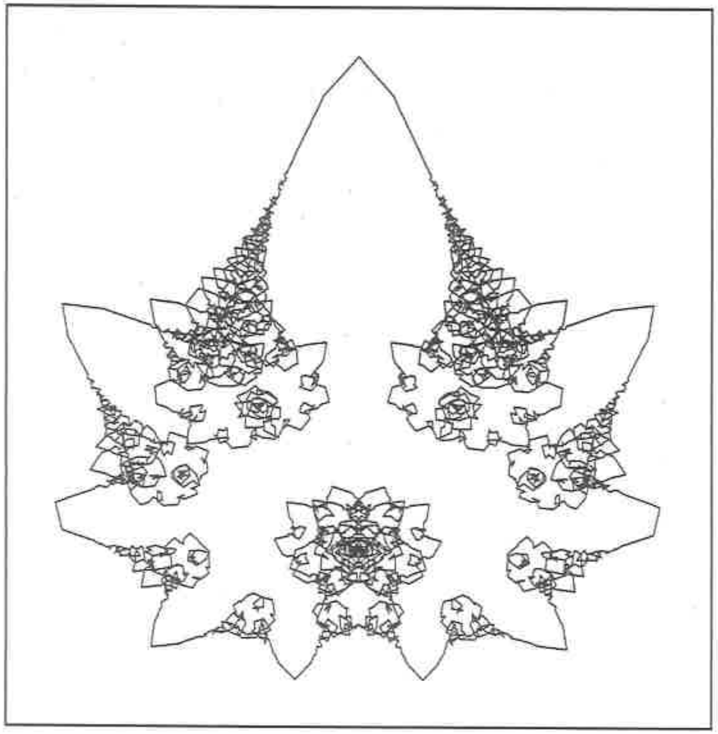
i) $\frac{7}{4} - \delta$ when $k = 2$.

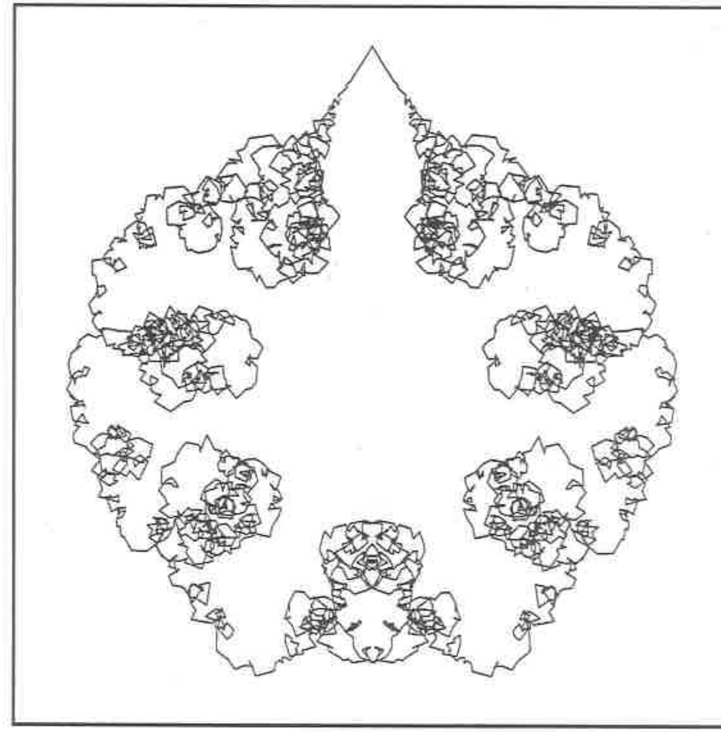
ii) $\max\left(1, 2 + \frac{1-2\delta}{2k}\right)$ for $k \geq 3$, $\delta \geq \frac{k+2}{4}$.

Under certain arithmetical conjectures (sharp Hua's inequality) ii) also holds for $\delta > 1$.

f







$$F(x) = \sum \frac{1}{n} e^{2\pi i n^2 x}$$

We know that:

(*) $F \in B.M.O.$

(**) Carleson's theorem $\Rightarrow$ series converges almost everywhere.

Gaussian's Sums:

$$G_{p/q} = \frac{1}{\sqrt{q}} \sum_{n=0}^{q-1} e^{2\pi i \frac{p}{q} n^2} \; ; \; |G_{p/q}| = \begin{cases} 2, & 4/q \\ 1, & q \text{ odd} \\ 0, & \text{otherwise.} \end{cases}$$

Then: The absolute value of the partial sums of $F(x)$ tends to infinity at p/q, (irreducible and $4 \nmid q-2$). In particular the series diverges on a dense subset of $[0,1]$ and nevertheless belong to BMO!

Theorem (F. Chamizo, A.C., Adrián Ubis) Math. Annalen 2021.
Let $x \in [0,1] \setminus \mathbb{Q}$ and let $\{p_j/q_j\}$ be the convergents of its continuous fraction expansion. Then the series F converges at x if and only if also does the series $\sum G_{p_j/q_j} \cdot \frac{1}{\sqrt{q_j}} \log \frac{q_{j+1}}{q_j}$.