Generalized Black–Scholes PDEs by bisectorial operators

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The Black–Scholes equation is a degenerate parabolic PDE, given by

$$u_t = x^2 u_{xx} + x u_x = Bu, \qquad t, x > 0, \tag{BS}$$

where $(Bf)(x) = x^2 f''(x) + x f'(x)$. (BS) is used in mathematical finance for the modelling of the price of European-style options.

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C_0-semigroups on a complex Banach space I

Definition (C_0-semigroup of operators)

A family $T = (T(t))_{t\in\mathbb{R}^+}$ in $\mathcal{B}(X)$ is called a semigroup if the following properties are satisfied:

1. $T(0) = 1_X$, the identity operator on X.
2. $T(t+s) = T(t)T(s)$, for every $t, s \in \mathbb{R}^+$.

We will say that T is a C_0-semigroup (or strongly continuous) if in addition, it holds that

3. $\lim_{t\to 0} \|T(t)x - x\| = 0$, for all $x \in X$.

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Definition (Infinitesimal generator)

Let $(T(t))_{t\in\mathbb{R}^+}$ be a semigroup of operators. Then, we set the infinitesimal generator $\Lambda : \mathcal{D}(\Lambda) \to X$,

$$\Lambda x := \lim_{\varepsilon \to 0} \frac{T(\varepsilon)x - x}{\varepsilon}.$$

where $\mathcal{D}(\Lambda) := \{x \in X \mid \text{limit above exists}\}$.

C_0-semigroups on a complex Banach space II

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where $\mathcal{D}(\Lambda) := \{x \in X \,|\, \text{limit above exists}\}$.

One has that

$$T(t)x \to x \text{ as } t \to 0 \iff x \in \overline{\mathcal{D}(\Lambda)}.$$

So T is strongly continuous if and only if $\overline{\mathcal{D}(\Lambda)} = X$.

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Abstract Cauchy problem

Let B be a closed operator on a Banach space X, and consider the following abstract Cauchy problem:

$$\begin{cases} u \in C^1((0,\infty);X), \quad u(t) \in \mathcal{D}(B), \quad t>0, \\ u'(t) = Bu(t), \qquad t>0, \\ \lim_{t\downarrow 0} u(t) = f \in X, \end{cases} \tag{ACP_0}$$

We say that the Cauchy problem (ACP_0) is well-posed if for for any $f \in X$, there exists a unique solution u.

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Theorem

The abstract Cauchy problem ACP_0 is well-posed if and only if B is the infinitesimal generator of a C_0-semigroup T. In that case, the solution is given by $u(t) = T(t)f$.

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Black Scholes equation

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In [ADP02], (BS) is studied in order continuous norm ($L^1 - L^\infty$)-interpolation spaces on $(0, \infty)$ by means of the relation

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Proposition

If A generates a C_0-group, then A^2 generates a C_0-semigroup.

For $\alpha > 0$, one can define the generalized Cesàro operator $\mathcal{C}_\alpha, \mathcal{C}_\alpha^*$,

$$(\mathcal{C}_\alpha f)(x) := \frac{\alpha}{x^\alpha} \int_0^x (x-y)^{\alpha-1} f(y)\, dy, \quad x > 0,$$

$$(\mathcal{C}_\alpha^* f)(x) := \alpha \int_x^\infty \frac{(x-y)^{\alpha-1}}{y^\alpha} f(y)\, dy, \quad x > 0,$$

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If $\alpha = 1$, one obtains the classical Cesàro operator $\mathcal{C}_1$ and its adjoint $\mathcal{C}_1^*$

$$(\mathcal{C}_1 f)(x) = \frac{1}{x} \int_0^x f(y)\, dy, \quad (\mathcal{C}_1^* f)(x) = \int_x^\infty \frac{f(y)}{y}\, dy, \quad x > 0.$$

Generalized Black–Scholes equations

The Black–Scholes operators satisfies that

$$B = (1 - \mathcal{C}_1^{-1})^2 = (\mathcal{C}_1^*)^{-2} = (\mathcal{C}_1^*)^{-1}(1 - \mathcal{C}_1^{-1}).$$

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With an arbitrary $\alpha > 0$, one obtains the following PDEs

$$u_t = \frac{1}{\Gamma(\alpha+1)^2} D^\alpha(x^\alpha D^\alpha(x^\alpha u)) - \frac{2}{\Gamma(\alpha+1)} D^\alpha(x^\alpha u) + u,$$
$$u_t = \frac{1}{\Gamma(\alpha+1)^2} x^\alpha W^\alpha(x^\alpha W^\alpha u),$$
$$u_t = -\frac{1}{\Gamma(\alpha+1)^2} D^\alpha(x^{2\alpha} W^\alpha u) + \frac{1}{\Gamma(\alpha+1)} x^\alpha W^\alpha u,$$

where D^α and W^α stand for the Riemann-Liouville and Weyl fractional derivatives of order α, respectively.

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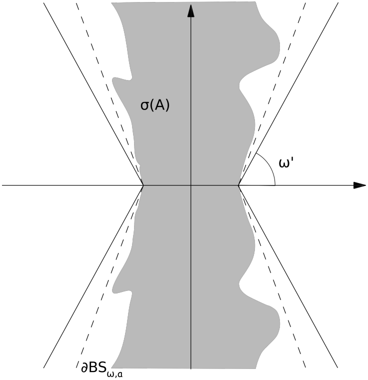
Bisectorial operators

Next, for any $\omega \in (0, \pi/2]$ and $a \geq 0$ a real number, we set

$$BS_{\omega,a} := \begin{cases} (-a + S_{\pi-\omega}) \cap (a - S_{\pi-\omega}) & \text{if } \omega < \pi/2 \text{ or } a > 0, \\ i\mathbb{R} & \text{if } \omega = \pi/2 \text{ and } a = 0. \end{cases}$$



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Bisectorial operators

Definition

Let $\omega \in (o, \pi/2]$, $a \geq 0$, $A \in \mathcal{C}(X)$. A is a **bisectorial operator** $\in \mathrm{BSect}(\omega, a)$ if:

- $\sigma(A) \subset \overline{BS_{\omega,a}}$.
- Fix any $\omega' \in (0, \omega)$. Then

$$\|R(\lambda, A)\|_{\mathcal{B}(X)} \leq \frac{K_{\omega'}}{\min\{|\lambda - a|, |\lambda + a|\}} \quad \lambda \notin \overline{BS_{\omega',a}},$$

for some $K_{\omega'} > 0$.

Natural functional calculus of bisectorial operators I

Idea: For $A \in \mathrm{BSect}(\omega, a)$ set

$$\mathcal{E}(A) := \left\{ f \in H^{\infty}(BS_{\varphi,a}) \ : \ \int_{\Gamma} \left| \frac{f(z)}{\min\{|\lambda - a|, |\lambda + a|\}} \right| |dz| < \infty, \varphi < \omega \right.$$
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and set

$$f(A) := \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A)\, dz \in \mathcal{B}(X), \qquad f \in \mathcal{E}(A).$$

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Natural functional calculus of bisectorial operators II

Definition

Let $A \in \mathrm{BSect}(\omega, a)$ and $\varphi < \omega$. Let $f \in \mathcal{M}(BS_{\varphi,a})$ be such that there exists $e \in \mathcal{E}(A)$ for which

1. $ef \in \mathcal{E}(A)$.
2. $e(A)$ is injective.

Then we say that f is regularizable, $f \in \mathcal{M}(A)$ and set

$$f(A) := e(A)^{-1}(ef)(A).$$

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Generalized Black–Scholes through functional calculus

The generalized Black–Scholes equations

$$u_t = \frac{1}{\Gamma(\alpha+1)^2} D^\alpha(x^\alpha D^\alpha(x^\alpha u)) - \frac{2}{\Gamma(\alpha+1)} D^\alpha(x^\alpha u) + u,$$

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can be respectively written as

$$u_t = (1 - (\alpha\mathbb{B}(I-A,\alpha))^{-1})^2 u := g_1^\alpha(A)u,$$
$$u_t = (\alpha\mathbb{B}(A,\alpha))^{-2} u := g_2^\alpha(A)u,$$
$$u_t = (\alpha\mathbb{B}(A,\alpha))^{-1}(1 - (\alpha\mathbb{B}(I-A,\alpha))^{-1})u := g_3^\alpha(A)u,$$

where $\mathbb{B}$ denotes the Beta-Euler function.

Holomorphic semigroups

For $\varphi \in (0, \pi)$, set the sector $S_\varphi := \left\{ z \in \mathbb{C} : |\arg(z)| < \varphi \right\}$, and $S_0 := (0, \infty)$.

Definition

Let $\delta \in (0, \pi/2]$. A mapping $T : S_\delta \to \mathcal{B}(X)$ is called an **exponentially bounded holomorphic semigroup** (of angle δ):

1. $T(z)T(z') = T(z + z')$ for all $z, z' \in S_\delta$.
2. $T : S_\delta \to \mathcal{B}(X)$ is holomorphic.
3. For each $\delta' \in (0, \delta)$,

$$\|T(w)\|_{\mathcal{B}(X)} \leq M_{\delta'} e^{\rho_{\delta'} \Re w}, \quad w \in S_{\delta'}.$$

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If $\rho_{\delta'} = 0$, then T is a **bounded holomorphic semigroup**.

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Sectorial operators

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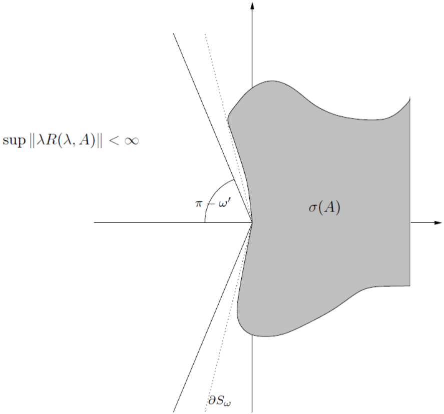
with domain $\mathcal{D}(B)$ is said to be a
sectorial operator (of angle φ) if



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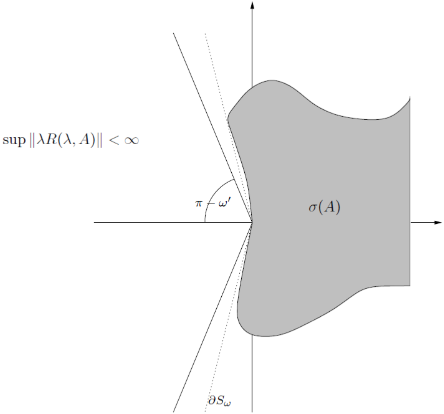

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2. Fix $\varphi' \in (\varphi, \pi)$. Then

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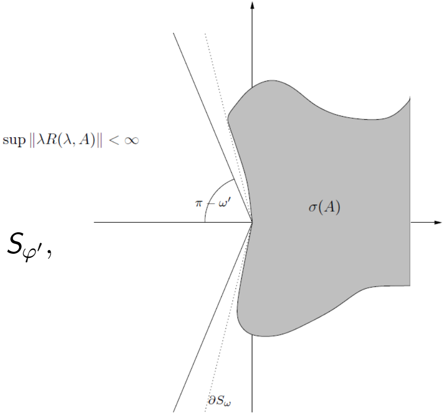

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Holomorphic semigroups and sectorial operators

Theorem

A closed operator A generates a bounded holomorphic semigroup of angle $\delta \in (0, \pi/2]$ if and only if $-A$ is a sectorial operator of angle $\frac{\pi}{2} - \delta$.

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2. (Scaling property) If A is a **sectorial** operator of angle β, then A^α is a sectorial operator of angle $\alpha\beta$ for $\alpha \in \left(0, \frac{\pi}{\beta}\right)$.

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The functions above satisfy that

$$g_i^\alpha(z) \sim \begin{cases} (-z)^{2\alpha} & \text{if } i = 1, \\ z^{2\alpha} & \text{if } i = 2, \\ z^\alpha(-z)^\alpha & \text{if } i = 3, \end{cases} \qquad \text{as } z \to \infty.$$

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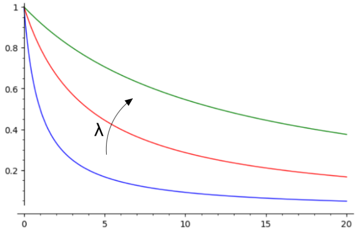
Idea of the method

Spectral properties are good enough, so the generalized Black–Scholes operators may be sectorial. We need to uniformly bound their resolvent.

Sketch of the problem: $\int_0^\infty \frac{1}{1+s}\frac{\lambda}{\lambda+s^\alpha}\, ds$ is not uniformly bounded as $\lambda \to \infty$.



Idea: $\left\| \frac{\lambda'}{\lambda'+A} \right\| = \|\lambda' R(\lambda', -A)\|$ is uniformly bounded since A is bisectorial. Then, try, to uniformly bound

$$\|\lambda' R(\lambda', -A)\| + \int_0^\infty \frac{1}{1+s}\left(\frac{\lambda}{\lambda+s^\alpha} - \frac{\lambda'}{\lambda'+s}\right) ds.$$

Key tool

We say that $(f^\lambda)_{\lambda\notin\overline{S_\beta}} \subset \mathcal{M}[BS_{\omega,a}]_A$ make $\left(\frac{\lambda}{\lambda-g(z)}\right)_{\lambda\notin\overline{S_\beta}}$ $\varepsilon-$uniformly bounded at $d\in M_A=\{-a,a,\infty\}\cap\widetilde{\sigma}(A)$ if, for any $\varepsilon\in(0,\pi-\beta)$,

1. $|f^\lambda(z)|$ is uniformly bounded for all $z\in\mathcal{D}(f^\lambda)$ and $\lambda\notin\overline{S_{\beta+\varepsilon}}$.
2. $\|f^\lambda(A)\|_{\mathcal{B}(X)}$ is uniformly bounded for all $\lambda\notin\overline{S_{\beta+\varepsilon}}$.
3. There exists a neighbourhood Ω_d containing d for which

$$\sup_{\lambda\notin\overline{S_{\beta+\varepsilon}}}\int_{\Gamma\cap\Omega_d}|h_g^\lambda(z)-f^\lambda(z)|\|R(z,A)\|_{\mathcal{B}(X)}\,|dz|<\infty.$$

4. There exists a neighborhood $\Omega_{d'}$ containing each $d'\in M_A\backslash\{d\}$ for which

$$\sup_{\lambda\notin\overline{S_{\beta+\varepsilon}}}\int_{\Gamma\cap\Omega_{d'}}|f^\lambda(z)|\|R(z,A)\|_{\mathcal{B}(X)}\,|dz|<\infty \quad \text{for each } d'\in M\backslash\{d\}.$$

Exactly polynomially behaviour

Let $A \in \mathrm{BSect}(\omega, a)$, $f \in \mathcal{M}(A)$. We say that f has exactly polynomial limits at $d \in \overline{\mathcal{D}(f)}$ if

$$|f(z)| \sim |z-d|^{\alpha}, \qquad \text{as } z \to d,\ d \neq \infty,$$
$$|f(z)| \sim |z|^{\alpha}, \qquad \text{as } z \to \infty,\ d = \infty$$

for some $\alpha \in \mathbb{R}$.

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Main result

Theorem (O.-M., M. Warma)

Let $\omega \in (0, \pi/2]$, $a \geq 0$ *and* $\beta \in [0, \pi)$. *Let* $A \in \mathrm{BSect}(\omega, a) g \in \mathcal{M}(A)$. *If*

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Corollary (O.-M., M. Warma)

Let $a \geq 0$, $A \in \mathrm{BSect}(\pi/2, a)$, *and let* $\alpha \in \mathbb{R}^+ \setminus \{1, 3, 5, ...\}$. *Then* $(-1)^n(A + aI)^\alpha$ *is quasi-sectorial operator of angle* $\pi\left|\frac{\alpha}{2} - n\right|$ *for* $\alpha \in (2n-1, 2n+1)$.

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Domain properties

Proposition (O.-M., M. Warma)

Let A, g *be as above. If* $g^{-1}(\infty) \cap M_A = \emptyset$, *then* $\overline{\mathcal{D}(g(A))} = X$. *Otherwise,*

$$\overline{\mathcal{D}(g(A))} = \bigcap_{d \in g^{-1}(\infty) \cap M_A} \overline{\mathcal{R}(dI - A)},$$

where $\mathcal{R}(\infty - A) := \mathcal{D}(A)$.

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where $\mathcal{R}(\infty - A) := \mathcal{D}(A)$.

Theorem (O.-M., M. Warma)

Let β, A, g *be as above with* $\beta \in \left(0, \frac{\pi}{2}\right)$, *so* $-g(A)$ *generates a holomorphic semigroup* T_g *of angle* $\frac{\pi}{2} - \beta$. *Then, for any* $w \in S_{\pi/2-\beta}$, *we have that* $\exp_{-w} \circ g \in \mathcal{M}_A$ *and*

$$T_g(w) = (\exp_{-w} \circ g)(A).$$

Let E be a $(L^1 - L^\infty)$-interpolation space. E is said to have an **order continuous norm** if $\|f_n\|_E \to 0$ for every sequence of functions $E \supset |f_n| \downarrow 0$ a.e.

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$$\underline{\alpha}_E := -\lim_{t\to\infty} \frac{\log \| T_E(-t)\|}{t}, \quad \overline{\alpha}_E := \lim_{t\to\infty} \frac{\log \| T_E(t)\|}{t}.$$

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Then ([ADP02])

1. $\sigma(A_E) = \{\lambda \,|\, \underline{\alpha}_E \le \Re\lambda \le \overline{\alpha}_E\}$.
2. T_E is strongly continuous if and only if E has order continuous norm.

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Generalized Black–Scholes equation on order continuous norm interpolation spaces

Theorem (O.-M., M. Warma)

Let E be a $(L^1 - L^\infty)$-interpolation space with order continuous norm and let $n \in \mathbb{N}$ and $\alpha > 0$. Then the following assertions hold.

1. *If $\overline{\alpha}_E < 1$ and $\alpha \in \left(n - \frac{1}{2}, n + \frac{1}{2}\right)$, then (ACP_0) is well-posed with $B_E = (-1)^{n+1} g_1^\alpha(A_E)$.*
2. *If $\underline{\alpha}_E > 0$ and $\alpha \in \left(n - \frac{1}{2}, n + \frac{1}{2}\right)$, then (ACP_0) is well-posed with $B_E = (-1)^{n+1} g_2^\alpha(A_E)$.*
3. *If $\overline{\alpha}_E < 1$ and $\underline{\alpha}_E > 0$, then (ACP_0) is well-posed with $B_E = g_3^\alpha(A_E)$.*

In any case, identifying $u(t,x) = u(t)(x)$, we obtain that $u \in C^\infty((0,\infty) \times (0,\infty))$.

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The End

$$\left(T_{(-1)^{n+1}g_1^{\alpha}(A_E)}(w)f\right)(x) = \frac{1}{2\pi}\int_0^{\infty}\frac{f(s)}{s}\int_{-\infty}^{\infty}\left(\frac{s}{x}\right)^{iu} \exp\left((-1)^{n+1}w\left(1-\frac{1}{\alpha\mathbb{B}(1-iu,\alpha)}\right)^2\right)\,duds,$$

$$\left(T_{(-1)^{n+1}g_2^{\alpha}(A_E)}(w)f\right)(x) = \frac{1}{2\pi}\int_0^{\infty}\frac{f(s)}{s}\int_{-\infty}^{\infty}\left(\frac{s}{x}\right)^{iu+\delta} \exp\left((-1)^{n+1}w\left(\alpha\mathbb{B}(iu+\delta,\alpha)\right)^{-2}\right)\,duds,$$

$$\left(T_{g_3^{\alpha}(A_E)}(w)f\right)(x) = \frac{1}{2\pi}\int_0^{\infty}\frac{f(s)}{s}\int_{-\infty}^{\infty}\left(\frac{s}{x}\right)^{iu+\delta} \exp\left(\frac{w}{\alpha\mathbb{B}(\delta+iu,\alpha)}\left(1-\frac{1}{\alpha\mathbb{B}(1-\delta-iu,\alpha)}\right)\right)\,du$$

Classical Black–Scholes

$$
\begin{aligned}
\left(T_{B_E}(w)f\right)(x) &= \frac{1}{2\pi}\int_0^\infty \frac{f(s)}{s}\int_{-\infty}^{\infty}\left(\frac{s}{x}\right)^{iu}\exp\left(-wu^2\right)\,duds \\
&= \frac{1}{\sqrt{4\pi w}}\int_0^\infty \exp\left(-\frac{(\log x-\log s)^2}{4w}\right)\frac{f(s)}{s}\,ds, \qquad x>0,
\end{aligned}
$$

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Regularity conditions and spectral inclusion

1. We say that a function f is regular at $d \in \{-a, a\}$ if $\lim_{z \to d} f(z) =: c_d \in \mathbb{C}$ exists and, for some $\varepsilon > 0$, $\varphi < \omega$,

$$\int_{\partial(BS_{\omega',a} \cap \{|z-d|<\varepsilon\}} \left| \frac{f(z) - c_d}{z - d} \right| |dz| < \infty, \quad \text{for all } \omega' \in \left(\varphi, \frac{\pi}{2}\right).$$

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2. We say that f is regular at ∞ if $\lim_{z\to\infty} f(z) =: c_\infty \in \mathbb{C}$ exists and, for some $R > 0$, $\varphi < \omega$,
$$\int_{\partial BS_{\omega',a},|z|>R} \left|\frac{f(z)-c}{z}\right| |dz| < \infty, \quad \text{for all } \omega' \in \left(\varphi, \frac{\pi}{2}\right).$$

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3. We say that f is quasi-regular at $d \in M_A$ if f or $1/f$ is regular at d.

Proposition

Let $A \in \mathrm{BSect}(\omega, a)$, and take $f \in \mathcal{M}_A$ to be quasi-regular at M_A. Then

$$\widetilde{\sigma}(f(A)) \subset f(\widetilde{\sigma}(A)).$$

Köthe dual

Theorem above does not hold for a $(L^1 - L^\infty)$-interpolation space E which has no order continuous norm. Consider the Köthe dual $E^\star$ of E, given by

$$E^\star := \Big\{ g : (0,\infty) \to \mathbb{C} \text{ measurable and } \int_0^\infty |f(x)g(x)|\, dx < \infty \quad \text{for all } f \in E \Big\}.$$

Every $g \in E^\star$ defines a bounded (order continuous) linear functional φ_g on E, given by

$$\langle f, \varphi_g \rangle_{E,E^\star} := \int_0^\infty f(x)g(x)\, dx \ \text{ for all } \ f \in E.$$

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Another abstract Cauchy problem

Next, consider the following abstract Cauchy problem:

$$\begin{cases} u \in C^1((0,\infty); E), \quad u(t) \in \mathcal{D}(B_E), \quad t > 0, \\ u'(t) = B_E u(t), \qquad t > 0, \\ \lim_{t \downarrow 0} \langle u(t), \varphi \rangle_{E,E^\star} = \langle f, \varphi \rangle_{E,E^\star}, \quad f \in E \text{ and for all } \varphi \in E^\star. \end{cases} \tag{ACP_1}$$

Again, we say that (ACP_1) is well-posed if, for any $f \in E$, there exists a unique u which is solution of (ACP_1).

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Generalized Black–Scholes equation on arbitrary interpolation spaces

Theorem (O.-M., M. Warma)

Let E be a $(L^1 - L^\infty)$-interpolation space and let $n \in \mathbb{N}$ and $\alpha > 0$. Then the following assertions hold.

1. *If $\overline{\alpha}_E < 1$ and if $\alpha \in \left(n - \frac{1}{2}, n + \frac{1}{2}\right)$, then ($ACP_1$) is well-posed with $B_E = (-1)^{n+1} g_1^\alpha(A_E)$.*
2. *If $\underline{\alpha}_E > 0$ and if $\alpha \in \left(n - \frac{1}{2}, n + \frac{1}{2}\right)$, then ($ACP_1$) is well-posed with $B_E = (-1)^{n+1} g_2^\alpha(A_E)$.*
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In any case, identifying $u(t,x) = u(t)(x)$, we obtain that $u \in C^\infty((0,\infty) \times (0,\infty))$.

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