Banach-Stone theorem for free Banach lattices

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Based on joint work with T. Oikhberg, M. Taylor and V. G. Troitsky

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Banach lattice = Banach space + vector lattice + $\left[|x| \leq |y| \Rightarrow \|x\| \leq \|y\|\right]$

T lattice homomorphism: T linear + $|Tx| = T|x|$

Examples:

- $C(K)$
- $L_p(\mu)$ (and other function spaces such as Orlicz, Lorentz...)
- ℓ_p, c_0... (any space with unconditional basis)

Non-examples:

- James quasi-reflexive space.
- Bourgain-Delbaen spaces.
- Hereditarily indecomposable spaces.

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The free Banach lattice generated by a Banach space

Let E be a Banach space. $FBL[E]$ is a Banach lattice, with a linear isometric embedding $\delta : E \to FBL[E]$:

$\forall$ Banach lattice X and $T : E \to X$ $\exists$ unique lattice homomorphism $\hat{T} : FBL[E] \to X$ making the diagram commute and $\|\hat{T}\| = \|T\|$.

- $FBL[\ell_1(A)]$ for any set A. [de Pagter-Wickstead, 2015]
- $FBL[E]$ for every Banach space E. [Avilés-Rodríguez-T, 2018]
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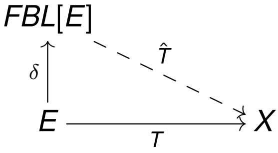

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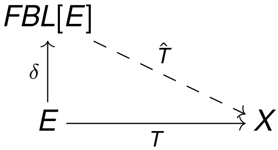

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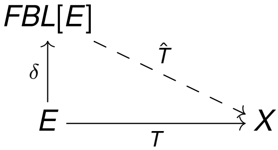

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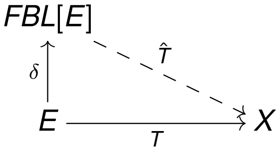

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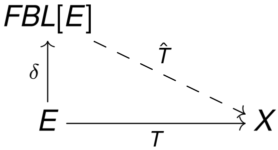

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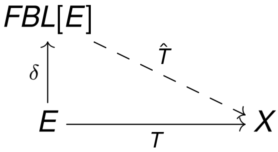

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Let $C_{ph}(B_{E^*})$ be the space of positively homogeneous w*-continuous functions on B_{E^*}.
For $f \in C_{ph}(B_{E^*})$, let

$$\|f\|_{FBL[E]} = \sup\Big\{ \sum_{i=1}^n |f(x_i^*)| : \sup_{x\in B_E} \sum_{i=1}^n |x_i^*(x)| \leq 1 \Big\}.$$

For $x \in E$, let $\delta_x \in C_{ph}(B_{E^*})$, given by $\delta_x(x^*) = x^*(x)$.

Theorem (Avilés-Rodríguez-T, 2018)

$FBL[E]$ is the $\|\cdot\|_{FBL[E]}$-closed sublattice generated by $(\delta_x)_{x\in E}$ in $C_{ph}(B_{E^*})$.

Notice:

- $\delta : E \to FBL[E]$ with $\delta(x) = \delta_x$ is a linear isometry.
- $\|f\|_\infty \leq \|f\|_{FBL[E]}$, so $FBL[E] \hookrightarrow C_{ph}(B_{E^*})$
- $FBL[E] = C_{ph}(B_{E^*})$ iff $dim(E) < \infty$.

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Every linear operator $T : E \to F$ between Banach spaces extends uniquely to a lattice homomorphism $\overline{T}$ as follows

$$\begin{array}{ccc} FBL[E] & \xrightarrow{\quad \overline{T} \quad} & FBL[F] \\ \delta_E \uparrow & & \uparrow \delta_F \\ E & \xrightarrow[T]{\qquad\qquad} & F \end{array}$$

Proposition

1. T is injective iff $\overline{T}$ is injective.
2. T is surjective iff $\overline{T}$ is surjective.

In particular, if E and F are linearly isomorphic (resp. isometric), then $FBL[E]$ and $FBL[F]$ are lattice isomorphic (resp. isometric).

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Suppose $T : FBL[E] \to FBL[F]$ is a lattice homomorphism.

Note $\varphi \in FBL[E]^*$ is a lattice homomorphism iff $\varphi = \widehat{x^*}$ for some $x^* \in E^*$ (i.e. $\varphi(f) = f(x^*)$)

Hence, for every $y^* \in F^*$ the composition $\widehat{y^*} \circ T$ corresponds to $\widehat{x^*}$ for some $x^* \in E^*$.

Thus, we can define $\Phi_T : F^* \to E^*$ by

$$\Phi_T y^*(x) := T\delta_x(y^*) \qquad y^* \in F^*, x \in E.$$

Lemma (Laustsen-T)

$\Phi_T : F^* \to E^*$ satisfies

1. *is positively homogeneous, $\Phi_T(\lambda y^*) = \lambda \Phi_T(y^*)$ for $\lambda \geq 0$;*
2. *is weak* to weak* continuous on bounded sets;*
3. *For $(y_i^*)_{i=1}^m \subset F^*$:*
$$\max_{\varepsilon_i \in \{-1,1\}} \left\| \sum_{i=1}^m \varepsilon_i \Phi_T(y_i^*) \right\| \leq \|T\| \max_{\varepsilon_i \in \{-1,1\}} \left\| \sum_{i=1}^m \varepsilon_i y_i^* \right\|$$
4. *$Tf = f \circ \Phi_T$ for $f \in FBL[E]$.*

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Theorem

Suppose E, F are Banach spaces so that E^*, F^* are smooth. $T : FBL[E] \to FBL[F]$ is a surjective lattice isometry iff $T = \overline{U}$, for some surjective isometry $U : E \to F$.

Since E^* is smooth, for every $x^* \in E^* \setminus \{0\}$ there is a unique $f_{x^*} \in E^{**}$ such that $\|f_{x^*}\| = \|x^*\| = \sqrt{f_{x^*}(x^*)}$.

Let us define the semi-inner product on E^*:

$$[y^*, x^*] = \begin{cases} f_{x^*}(y^*) & x^* \neq 0 \\ 0 & x^* = 0. \end{cases}$$

Theorem (Ilišević-Turnšek '20)

X, Y smooth spaces, $F : X \to Y$ surjective with $|[Fx, Fy]| = |[x, y]|$ for all $x, y \in X$. Then there is a phase $\sigma : X \to \{-1, 1\}$ and a linear surjective isometry $U : X \to Y$ with

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Theorem

Suppose E, F are Banach spaces so that E^*, F^* are smooth. $T : FBL[E] \to FBL[F]$ is a surjective lattice isometry iff $T = \overline{U}$, for some surjective isometry $U : E \to F$.

Since E^* is smooth, for every $x^* \in E^*\backslash\{0\}$ there is a unique $f_{x^*} \in E^{**}$ such that $\|f_{x^*}\| = \|x^*\| = \sqrt{f_{x^*}(x^*)}$.

Let us define the semi-inner product on E^*:

$$[y^*, x^*] = \begin{cases} f_{x^*}(y^*) & x^* \neq 0 \\ 0 & x^* = 0. \end{cases}$$

Theorem (Ilišević-Turnšek '20)

X, Y smooth spaces, $F : X \to Y$ surjective with $|[Fx, Fy]| = |[x, y]|$ for all $x, y \in X$. Then there is a phase $\sigma : X \to \{-1, 1\}$ and a linear surjective isometry $U : X \to Y$ with

$$F(x) = \sigma(x)Ux.$$

Lemma

Let $x^*, y^* \in E^*$ of norm 1.

$$\lim_{t\to 0} \frac{\max_{\pm} \|x^* \pm ty^*\| - \|x^*\| - |[y^*, x^*]t|}{t} = 0$$

Proposition

$$|[\Phi_T x^*, \Phi_T y^*]| = |[x^*, y^*]|$$

... $\Phi_T = \pm U^*$ for some linear isometry $U : E \to F$.

Lemma

Let $x^*, y^* \in E^*$ of norm 1.

$$\lim_{t\to 0} \frac{\max_{\pm} \|x^* \pm ty^*\| - \|x^*\| - |[y^*, x^*]t|}{t} = 0$$

Proposition

$$|[\Phi_T x^*, \Phi_T y^*]| = |[x^*, y^*]|$$

... $\Phi_T = \pm U^*$ for some linear isometry $U : E \to F$.

Lemma

Let $x^*, y^* \in E^*$ *of norm 1.*

$$\lim_{t \to 0} \frac{\max_{\pm} \|x^* \pm t y^*\| - \|x^*\| - |[y^*, x^*]t|}{t} = 0$$

Proposition

$$|[\Phi_T x^*, \Phi_T y^*]| = |[x^*, y^*]|$$

... $\Phi_T = \pm U^*$ for some linear isometry $U : E \to F$.

Thank you for your attention!

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