Motivic Aspects of Degenerations

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June 2023

These are notes for the 8th seminar talk of the Seminar "Complex Geometry and Hodge Theory" organised by Prof. Dr. Daniel Huybrechts on the summer semester 2023 in the University of Bonn. The main references are [PS08, Chapter 11] and [Pet10, Lecture 7].

I would like to thank Dr. Evgeny Shinder for his generous help and assistance with the preparation of the talk, not only for the actual talk but also when I was studying the prerequisites on my own. Also Alejandro García Rivas for the pictures.

1 Introduction

Basic Idea: study what happens to the Hodge structure (MHS) of a projective manifold as it degenerates to a singular variety.

Let X be a complex manifold, $\Delta \subset \mathbb{C}$ a disk around 0 and $f : X \to \Delta$ be a proper holomorphic map that is smooth over the puncutured disk Δ^*.

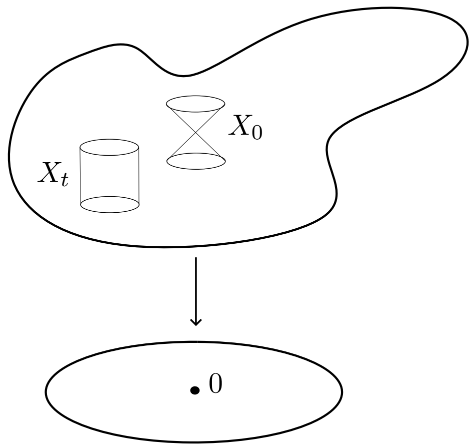


Figure 1: 1-parameter degeneration.

We know the cohomology of a generic fibre $X_t := f^{-1}(t)$ carries a pure Hodge structure with $H^n(X_t)$ having weight n. In 1970 Griffiths conjectured that the monodromy around 0 could be used to define a weight filtration W on $H_{\mathbb{Q}} := H^n(X_t, \mathbb{Q})$ with t close to 0 and a Hodge filtration F on $H_{\mathbb{Q}}$ so that the triple $(H_{\mathbb{Q}}, W, F)$ defines a mixed Hodge structure. This amounts to saying that F defines a pure Hodge structure of weight $k+n$ on the k-th graded piece of $Gr_k^W H_{\mathbb{Q}}$.

Most of this talk will be devoted to define this mixed Hodge structure, with some examples illustrating how it works and some simple applications, the next talks will expand upon this.

We have the following goals for now:

- Define a mixed Hodge structure on $H_{\mathbb{Q}}$,
- study the properties of this mixed Hodge structure to draw some connection between cohomology of X_0 and X_t.
- Enhance the construction to get a motivic construction.

2 Milnor fibres and the nearby cycle complex

We will now refine a bit our setup. As before let X be complex manifold and $f : X \to \Delta$ a proper surjective morphism with connected fibres such that over Δ^* the punctured disk the function is smooth.

Furthermore, using Mumford's semistable reduction theorem [KKMSD06, Ch. 2], after a series of blow-ups and base changes, normalising if necessary and further shrinking the base we may assume that the special fibre has reduced structure and satisfies

$$X_0 = \cup_{i\in I} E_i \text{ is a divisor with simple normal crossings (snc) on } X.$$

Moreover, by Ehresmann's theorem [PS08, Theorem C.10] we know that over the punctured disk f defines a locally trivial differential fibration. Finally, we will later assume that the fibres are projective, assuming Kähler fibres would actually be sufficient.

Let $x \in X_0$ we can define the **Milnor fibre** of f at x, $\mathrm{Mil}_{f,x}$, as a representative of

$$X_t \cap B(x, r)$$

for t very close to 0, $\eta := d(t, 0)$, $0 < \eta \ll r \ll 1$. Milnor showed that the diffeomorphism type of this manifold does not depend on r, η (cf. [Mil68]) Moreover, from his work we also get the following:

Proposition 1. *[PS08, Proposition C.11] Let X be a manifold, $f : X \longrightarrow \Delta$ be a proper map which is smooth over Δ^*. There is a fibrewise retraction $r : X \to X_0$. In particular the homotopy type of X is that of the central fibre.*

Let $i_t : X_t \hookrightarrow X$ be the inclusion. If x is an isolated singularity we may choose a retraction s.t. the inclusion $(r \circ i_t)^{-1}x \hookrightarrow \mathrm{Mil}_{f,x}$ is a homotopy equivalence.

This motivates the following construction: We define the **(topological) complex of sheaves of nearby cycles** as the complex of sheaves on X_0 defined as

$$(Rr_t)_* \underline{\mathbb{Q}}_{X_t}$$

here $(Rr_t)_*$ here denotes the right derived pushforward and $\underline{\mathbb{Q}}_{X_t}$ the constant sheaf on X_t. Now we can compute the hypercohomology of this complex as

$$\mathbb{H}^k(X_0, (Rr_t)_* \underline{\mathbb{Q}}_{X_t}) = \mathbb{H}^k(X_t, \underline{\mathbb{Q}}_{X_t}) = H^k(X_t; \mathbb{Q}). \tag{2.1}$$

This seems like a good step for our plan of relating the cohomology of the smooth fibre to the one of the singular one. By the previous proposition, we can relate the cohomology of the Milnor fibre and X_t. But first we make this construction more canonical, for this consider the specialization diagram:

$$\begin{array}{ccccc} X_\infty & \xrightarrow{k} & X & \xleftarrow{i} & X_0 \\ {\scriptstyle f_\infty}\big\downarrow & & {\scriptstyle f}\big\downarrow & & \big\downarrow \\ \mathfrak{h} & \xrightarrow[exp]{} & \Delta & \longleftarrow & \{0\} \end{array}$$

Here $\mathfrak{h}$ denotes the complex upper half plane and $X_\infty := X \times_{\Delta^*} \mathfrak{h}$ and exp is the map $z \mapsto e^{2\pi i z}$. Recall that $\mathfrak{h} \xrightarrow{exp} \Delta^*$ is a universal covering. It turns out that the manifold X_∞ actually retracts onto any of the smooth fibres, as f_∞ is differentiably a product and so can be seen as an object in the homotopy category which is canonically associated to the smooth part of the family.

As X_0 is snc for any point $x \in X_0$ we can choose a system of local coordinates on $U(x) \subset X$ centered at x such that $f(z_0, ..., x_n) = z_0 \cdots z_k$ and define

$$V_{r,\eta} = \{z \in U : \ ||z|| < r \text{ and } |f(z)| < \eta\}.$$

These form a fundamental system of neighbourhoods of x in X. The Milnor fibre embeds in $k^{-1}V_{r,\eta}$ via the map $z \mapsto \left(z, \frac{\log t}{2\pi i}\right)$ this can be seen to be a homotopy equivalence thus using what we've seen so far we get

$$(\mathbb{H}^q(X_0, (Rr_t)_* \underline{\mathbb{Q}}_{X_t}))_x = H^q(\mathrm{Mil}_{f,x}; \mathbb{Q}) \simeq \lim_{r,\eta} H^q(k^{-1}(V_{r,\eta})) = \Big((R^q k)_* \underline{\mathbb{Q}}_{X_\infty}\Big)_x .$$

So we can define a canonical object, the (analytic) **nearby cycle complex** on X_0:

$$\psi_f \underline{\mathbb{Q}}_{X_0} := i^* Rk_*(k^* \underline{\mathbb{Q}}_X)[1]. \tag{2.2}$$

Notice that $\mathbb{H}^q(i^* Rk_*(k^*\underline{\mathbb{Q}}_X))_x = ((R^q k)_* \underline{\mathbb{Q}}_{X_\infty})_x$ and as a consequence we get a cumbersome way to compute the cohomology of the smooth fibre:

$$\mathbb{H}^q(X_0, \psi_f \underline{\mathbb{Q}}_{X_0}) = H^k(X_t; \mathbb{Q}).$$

This same construction can be extended giving a functor $\psi_f : D^+(X) \to D^+(X_0)$, defined by $\mathcal{F}^\bullet \mapsto i^* Rk_*(k^* \mathcal{F}^\bullet)$. In particular we can define: $\psi_f \underline{\mathbb{C}}_{X_0}$.

We will now want to give a mixed Hodge structure[2] to this complex which will be called the **Hodge-theoretic nearby cycle complex**:

$$\psi_f^{Hdg} = ((\psi_f \underline{\mathbb{Q}}_{X_0}, W), (\psi_f \underline{\mathbb{C}}_{X_0}, W, F), \alpha).$$

We call the filtrations F and W the limit Hodge filtrations. We write $H^k(X_\infty)$ for the mixed Hodge structure which this complex puts on $\mathbb{H}^k(X_0, \psi_f \underline{\mathbb{Q}}_X)$, we will also call this the **limit mixed Hodge structure**.

[1] The notation might be a bit confusing but we choose it this way to highlight that it is a complex of sheaves on X_0.

[2] These were defined on an earlier talk see [PS08] for a definition.

3 Construction of the Hodge-theoretic nearby cycle complex.

To construct a MHS we are going to define a much more intuitive complex of sheaves quasi-isomorphic to the nearby cycle complex (cf. [PS08, Chapter 11] and [Ste76]). First we recall that that 2 talks ago for a divisor with simple normal crossings $Y \subset X$ we constructed the following:

Definition 2. *The* ***holomorphic de Rham complex with logarithmic poles along*** Y*, denoted by* $\Omega^\bullet_X(\log Y)$*, is given on* $U \subset X$ *by sections which are holomorphic forms on* $U \backslash Y$ *and meromorphic with at most simple poles on* Y*. If locally around* $P \in Y$ *the coordinate description is given by*

$$Y \cap U = \left\{ z \in U : \prod_{i=1}^{\nu} z_i = 0 \right\}$$

then at the stalk $(\Omega^p_X(\log Y))_P$ *is the module freely generated by* $\frac{dz_1}{z_1}, ..., \frac{dz_\tau}{z_\tau}, dz_{\nu+1}, ..., dz_n$ *over* $\mathcal{O}_{X,P}$*.*

On an earlier talk we also defined a weight filtration for this complex

$$W_k\Omega^p_X(\log Y) := \Omega^k_X(\log Y) \wedge \Omega^{p-k}_X.$$

This we may think of as the meromorphic forms on Y with exactly k terms of the form dz_i/z_i. Moreover we define as ascending filtration simply the trivial filtration

$$F^k\Omega^\bullet_X(\log Y) = \Omega^{\bullet \geq k}(\log Y). \tag{3.1}$$

This is a good approximation but it is still not enough as we would like to also encapsule the information coming from the fibration for this. So define the **relative de Rham complex on** X **with logarithmic poles along** X_0

$$\Omega^\bullet_{X/\Delta}(\log X_0) := \Omega^\bullet_X(\log X_0)/(f^*\Omega^1_\Delta(\log 0) \wedge \Omega^{\bullet-1}_X(\log X_0)).$$

It can be seen that $\psi_f\underline{\mathbb{C}}_X$ is actually quasi-isomorphic to $\Omega^\bullet_{X/\Delta}(\log X_0) \otimes O_{X_0}$. The proof is out of the scope of this talk for time reasons, but it is a very nice proof so we encourage listeners to take a look at it (cf. [PS08, Theorem 11.16])

Locally near x there is a system of coordinates such that $f(z_0, ..., z_n) = z_0 \cdots z_\nu$ then $\Omega^1_{X/\Delta}(\log X_0)$ we obtain as in the non-relative case the module freely generated by $\frac{dz_1}{z_1}, ..., \frac{dz_\tau}{z_\tau}, dz_{\nu+1}, ..., dz_n$ but subject to the relations $\sum \frac{dz_i}{z_i} = 0$ (cf. [Ste76, 1.4]). This gives us a much more geometric picture of the nearby cycle complex.

One could naively assume that the filtrations defined earlier would give us a MHS for $\Omega^\bullet_{X/\Delta}(\log X_0) \otimes O_{X_0}$, unfortunately this turns out to be impossible. Fortunately just taking it a step further is enough. Consider the bi-filtered double complex of sheaves on X_0

$$A^{p,q} := \Omega^{p+q+1}_X(\log X_0)/W_q\Omega^{p+q+1}_X(\log X_0), \quad p, q \geq 0$$

here the degree $(1, 0)$ differential is given by the usual de Rham differential and the other by $\omega \mapsto (dt/t) \wedge \omega$, here $t \in \Delta$ but it can be seen that this construction does not depend on t.

Let $A^\bullet = Tot(A^{\bullet,\bullet})$, the total complex. Then for the increasing filtration choose

$$F^rA^\bullet = \bigoplus_p \bigoplus_{q \geq r} A^{p,q}$$

pick the trivial or stupid filtration (as in 3.1) and for the weight filtration choose

$$W_k A^r = \bigoplus_{p+q=r} W_{2q+k+1} A^{p,q} = \bigoplus_{p+q=r} W_{2q+k+1}\Omega_X^{p+q+1}(\log X_0)/W_q\Omega_X^{p+q+1}(\log X_0).$$

This weight filtration is also called the monodromy weight filtration, we will see later why. These two define a mixed Hodge structure on $A^\bullet$. Moreover via the map $\Omega^q_{X/\Delta}(\log X_0)\otimes O_{X_0} \to A^{0,q}$ defined via $\omega \mapsto (-1)^q(dt/t)\wedge\omega \mod W_0$ we get a bi-filtered quasi-isomorphism

$$\mu(\Omega^q_{X/\Delta}(\log X_0)\otimes O_{X_0}, W, F) \longrightarrow (A^\bullet, W, F).$$

Hence $\phi_f\underline{\mathbb{C}}_X \simeq A^\bullet$ in $D^+(X_0)$. We claim $(A^\bullet, W, F)$ is a mixed Hodge complex of sheaves so this is the Hodge-theoretic nearby cycle complex we were looking for (see Theorem 9).

How do the weight pieces of the monodromy weight filtration look like? Briefly recall that for $Y = \cup_{i\in I} E_i$ snc divisor we defined on a previous talk $E_J := \cap_{j\in J} E_j$, $E(n) := \coprod_{|J|=m} E_J$ and $a_n : E(n) \hookrightarrow Y$ $J \subseteq I$. Using this notation we can write the k-th graded piece of this complex as Moreover as the differential of degree $(0,1)$ satisfies that $dW_r \subset W_{r-1}$ we get:

$$Gr_k^W A^\bullet \simeq \bigoplus_{k\geq 0,-r} Gr^W_{r+2k+1}\Omega_X^\bullet(\log X_0)[1] \simeq \bigoplus_{k\geq 0,-r} (a_{r+2k+1})_*\Omega^\bullet_{E(r+2k+1)}[-r-2k]). \quad (3.2)$$

The second equality follows from the fact we saw a couple of talks ago that $Gr_q^W\Omega_X(\log X_0) \simeq (a_m)_*\Omega^\bullet_{E(q)}[-q]$. This description will be used later on for computations.

Remark 3. There is a rational analogue construction of a logarithmic de Rham complex, call it $K_p^\bullet$ satisfying $K_p^\bullet \otimes_{\mathbb{Q}} \mathbb{C} \simeq W_p\Omega_X^\bullet$, taking $K_\infty^\bullet = \lim_{\underset{p}{\to}} K_p^\bullet \simeq \psi_f\underline{\mathbb{Q}}_X$, with filtration

$$W_r K_\infty^\bullet = \operatorname{Im}(K_r^\bullet \hookrightarrow K_\infty^\bullet).$$

We get an induced filtered quasi-isomorphism between them. So we can repeat the construction of $A^{p,q}$, by setting:

$$C^{p,q} = (i^*K_\infty^{p+q+1}/i^*K_p^{p+q+1})(p+1), \quad p\geq 0,\ p+q\geq -1$$

the same tecniques as before allow us to define a weight filtration with very similar properties (cf. [PS08, 11.2.6]) Unfortunately we do not have enough time to discuss it.

Remark 4. We did not talk about the pseudo-morphism included in the definition of mixed Hodge complex of sheaves, simply because it is not very interesting, take the natural map $\underline{\mathbb{Q}}_X \hookrightarrow \underline{\mathbb{C}}_X$ and the apply the functor ψ_f, this induces the pseudo-morphism necessary to complete the Hodge structure.

4 The limit mixed Hodge structure and the monodromy

We have succesfully constructed the limit mixed Hodge structure for $\psi_f\underline{\mathbb{C}}_{X_0}$, unfortunately it does not have a simple description. To solve this problem we turn to study the action of the monodromy of the punctured disk on the fibres and see how it relates to the MHS. For this we go back to our original setting of a degeneration.

Notice that the map $h : X_\infty \to X_\infty$ given by $(x, u) \mapsto (x, u + 1)$ satisfies $k \circ h = k$, where $k : X_\infty \to X$, on the punctured disk. This amounts to circling around the center which defines an element of $\pi_1(\Delta^*)$. This defines an action of the fundamental group on the cohomology which translates into an automorphism $h^* : H^*(X_t) \to H^*(X_t)$. This construction extends to a automorphism on $\psi_f \underline{\mathbb{C}}_{X_0}$ and $\psi_f \underline{\mathbb{Q}}_{X_0}$. So define the **monodromy operator**

$$T := (h^*)^{-1} : \psi_f \mathcal{K}^\bullet \longrightarrow \psi_f \mathcal{K}^\bullet.$$

Example 5. Consider a complex torus and the cycle degenerating as in Figure 2, i.e. the so called Dehn twist. Then we get that X_0 only has one ordinary double point so there is exactly one vanishing cycle. The Picard-Lefschetz formula [PS08, Theorem C.20] gives that for the vanishing cycle δ_0[3] the action of the monodromy is:

$$T_0 : H^1(X_t; \mathbb{Q}) \longrightarrow H^1(X_t; \mathbb{Q}), \quad \alpha \longmapsto \alpha + (-1)^{\frac{1}{2}(1+1)(1+2)} \langle \alpha, \delta_0 \rangle \delta_0^\vee. \tag{4.1}$$

Then $\delta_0 \mapsto \delta_0$ and $\delta_0^\vee \mapsto \delta_0 - \delta_0^\vee$. Thus the monodromy operator acts on $H^1(X_t, \mathbb{Q})$ via the matrix

$$T_0 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

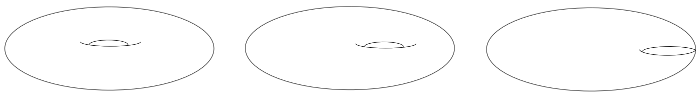


Figure 2: Degeneration of a torus.

Remark 6. We may relate T to the double complex we defined for $\psi_f \underline{\mathbb{C}}_{X_0}$. Define a morphism $\nu : A^{p,q} \to A^{p-1,q+1}$

$$\begin{array}{ccc} \Omega_X^{p+q+1}(\log X_0)/W_q \Omega_X^{p+q+1}(\log X_0) & \xrightarrow{\nu} & \Omega_X^{p+q+1}(\log X_0)/W_{q+1} \Omega_X^{p+q+1}(\log X_0) \\ \omega & \longmapsto & \omega \mod W_{q+1} \end{array}.$$

This morphism commutes with both differentials and so it induces a morphism on the total complex. Now one can show using the Gauss-Manin connection that this operator is related to the monodromy operator T via the equality:

$$\log T = 2\pi i \nu.$$

Now we turn to the study of our monodromy operator T.

Theorem 7 (Monodromy Theorem (weak version))**.** *[Pet10, Lemma 7.11] If T is quasi-unipotent, i.e. there exists $n, k \in \mathbb{N}$ s.t. $(T^n - I)^k = 0$ and all multiplicities of f along E_i are 1 then it is unipotent, i.e. $n = 1$*[4]

This implies then that using the Taylor expansion $N = \log T$ is well-defined and it is nilpotent.

[3] $\langle -, - \rangle$ is the Kronecker duality pairing.

[4] The statement of the Monodromy Theorem [PS08, Corollary 11.42] is: Suppose that for $k, l \in \mathbb{Z}$ one has that $H^{p,k-p}(X_t) = 0$ for all $p > k/2 + l$ then $N^{l+1} = 0$ on $H^k(X_\infty)$, where $N = \log T$. The weak version follows easily.

Lemma 8. *[Sch73, Lemma 6.4] Let N be a nilpotent endomorphism of a finite dimensional $\mathbb{Q}$-vector space H, there is a unique increasing filtration $W := W(N)$ on H such that $N(W_jH) \subset W_{j-2}H$ and $N^j : Gr_j^W \to Gr_{-j}^W H$ is an isomorphism for all $j \geq 0$.*

The shifted $W[k]$ filtration given by the lemma on $H^k(X_\infty)$ is called the monodromy weight filtration. This existence via linear algebra may not be very satisfying but there is nice geometric description to it. Pick $t \in \Delta$ non-zero, fix a choice of logarithm and set $z := 2\pi i \log(t)$. The generic fibre carries naturally a pure Hodge structure with Hodge filtration $F^\bullet$. Let $F^p(z) = F^pH^k(X_z) \subset H^k(X_\infty) \otimes \mathbb{C}$, observe that

$$exp(-(z+1)N) = exp(-zN)T^{-1}$$

so we have

$$exp(-(z+1)N)F^p(z+1) = exp(-zN)F^p(z) \subset H^k(X_\infty).$$

Set $F^p(t) := exp(-zN)$ as a subspace of $H^k(X_\infty) \otimes \mathbb{C}$, this subspace converges in the sense of points in a Grassmannian to a limit F^p_∞ when t approaches 0 along radii (cf. [Sch73, Theorem 6.16]).

Theorem 9 (Steenbrink-Schmid)**.** *[Pet10, 7.1.3] The Hodge-theoretic nearby cycle complex ψ_f^{Hdg} as described before is a mixed complex of sheaves on X_0. The Hodge-theoretic nearby cycle complex puts a mixed Hodge structure on the cohomology groups $H^k(X_\infty)$, the weight filtration is the monodromy filtration and the limit Hodge filtration coincides with the above limit on $F^\bullet H^k(X_t)$, in particular*

$$\dim F^pH^k(X_\infty) = \dim F^pH^k(X_t).$$

Corollary 10. *One has that $h^{p,q}(X_s) = \sum_{s\geq 0} h^{p,s}(H^{p+q}(X_\infty))$.*

Example 11. N puts a pure Hodge structure on the central fibre if and only if $N = 0$, for example if the fibre is smooth.

Example 12. Continuing Example 5 we observe that

$$N = \log(T) = \log\left(I + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \Longrightarrow N^2 = 0.$$

In particular notice that the invariant class for the N is δ_0, which is precisely what we expected from our geometric description. Out of the 2 homology 1-cycles of the torus one degenerates and the other is left invariant in this case, then we dualize and we get that δ_0 was left invariant.

Turns out that this is true in much generality and can be described with the following theorem:

Theorem 13 (Local invariant cycle theorem)**.** *[PS08, Theorem 11.43] Let $f : X \to \Delta$ be a projective one-parameter Kähler degeneration. For all $k \geq 0$ the sequence:*

$$H^k(X_0; \mathbb{Q}) \xrightarrow{sp} H^k(X_\infty; \mathbb{Q}) \xrightarrow{T-I} H^k(X_\infty; \mathbb{Q})$$

is exact. Here sp denotes the specialization map, i.e., the map induced by $r_t : X_0 \to X_t$. In particular, the invariant classes of X_∞ are the classes in the image of the specialization map, i.e., the classes which come from restricting classes in the total space.

Note that assuming that T is unipotent we get in particular that $\ker N = \ker(\log T) = \ker(T - I)$. So $\ker N$ consists exactly of the vanishing cycles.

The limit Hodge structure satisfies the following decomposition theorem.

Lemma 14. *[Sch73, Lemma 6.4] There is a Lefschetz type decomposition*

$$H^k(X_\infty) = \bigoplus_{l=0}^{k} \bigoplus_{r=0}^{l} N^r P_{k+l}$$

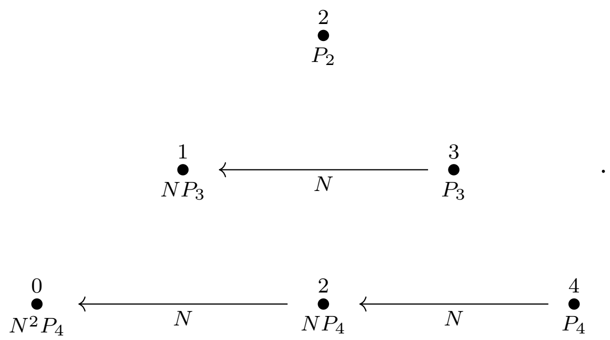
where P_{k+l} is of pure weight $k+l$ and N has $\dim P_{k+m-1}$ Jordan blocs of size m.

Example 15. Substituting in the formula we get for $H^1(X_\infty)$:

$$H^1(X_\infty) = \overbrace{P_1}^{1} \oplus \overbrace{P_2 \oplus NP_2}^{2}$$

there are $\dim Gr_2^W$ Jordan blocs of size 2 and $\dim Gr_1^W$ ones of size 1. So then we get $\ker N = NP_2 \oplus P_1$, and so $W_0 \cap \ker N = NP_2 = W_0 H^1(X_\infty) \simeq W_0 H^1(X_0)$.

For $H^2(X_\infty)$ we get a similar situation.



Here the numbers over the points denote in which $Gr_r PW$ they live. In particular notice that $Gr_1^W H^2(X_\infty) = NP_3 \oplus NP_4$ and so $Gr_1^W \ker N = NP_3 = Gr_1 H^2(X_0)$

5 The motivic nearby cycle

Now we want to compute $\chi_{Hdg}(\psi_f^{Hdg})$ and carry our construction to motives. Following [PS08, Corollary 11.23] we note that a decreasing finite filtration F on a complex $K^\bullet$ induces a spectral sequence satisfying

$$\begin{cases} E_0^{p,q} = & Gr_p^F(K^{p+q}) \\ E_1^{p,q} = & H^{p+q}(Gr_p^F(K^\bullet)). \end{cases}$$

Moreover one can show that:

$$E_r^{p,q} \Longrightarrow Gr_p^F H^{p+q}(K^\bullet).$$

We want to apply this to the Hodge filtration on $Gr_p^W(\psi_f^{Hdg})$. From this construction, equation 3.2 and Remark 3 we obtain

$$Gr_s^W(\psi_f^{Hdg}) \simeq \bigoplus_k \underline{\mathbb{Q}}_{E(2k+s+1)}[s+2k](-s-k)$$

and then the E_1 is the graded Hodge structure:

$$E_1^{-s,q+s} = \bigoplus_{k\geq 0,-s} H^{q-s-2k}(E(2k+s+1);\mathbb{Q})(-s-k) \Longrightarrow H^q(X_\infty).$$

Example 16. Suppose we have a plane curve degenerating to d $\mathbb{P}^1$'s intersecting at m points with no triple points we compute the spectral sequence of this degeneration. The only intersections that are non-trivial are the total space $E(1)$ and the intersection points $E(2)$.

$$\begin{cases} 2k+1+s=1 \Longrightarrow 2k+s=0 \overset{k\geq 0,-s}{\Longrightarrow} k=s=0 \\ 2k+s+1=2 \Longrightarrow 2k+s=1 \text{ then } s\in\{-1,1\}. \end{cases}$$

So then we get an E_1 page as

$$\begin{array}{cccc} & E_1^{-1,\bullet} & E_1^{0,\bullet} & E_1^{1,\bullet} \\ E_1^{\bullet,2} & H^0(E(2)) & H^2(E(1)) & H^2(E(2)) \\ E_1^{\bullet,1} & 0 & H^1(E(1)) & 0 \\ E_1^{\bullet,0} & 0 & H^0(E(1)) & H^0(E(2)) \end{array} \longrightarrow \begin{array}{cccc} & E_1^{-1,\bullet} & E_1^{0,\bullet} & E_1^{1,\bullet} \\ E_1^{\bullet,2} & \mathbb{Q}^m & \mathbb{Q}^d & 0 \\ E_1^{\bullet,1} & 0 & 0 & 0 \\ E_1^{\bullet,0} & 0 & \mathbb{Q}^d & \mathbb{Q}^m \end{array}$$

with maps $\mathbb{Q}^d \to \mathbb{Q}^m$ one induced by pushforward and the other by restrictions. Now we know the sequence converges in E_2 to $H^{p+q}(X_\infty)$

$$\begin{array}{cccc} & E_2^{-1,\bullet} & E_2^{0,\bullet} & E_2^{1,\bullet} \\ E_2^{\bullet,2} & ? & \mathbb{Q} & 0 \\ E_2^{\bullet,1} & 0 & 0 & 0 \\ E_2^{\bullet,0} & 0 & \mathbb{Q} & ?. \end{array}$$

Then as $\mathbb{Q}$ is the kernel of $\mathbb{Q}^d \to \mathbb{Q}^m$ and using symmetry we deduce that

$$\begin{array}{cccc} & E_2^{-1,\bullet} & E_2^{0,\bullet} & E_2^{1,\bullet} \\ E_2^{\bullet,2} & \mathbb{Q}^{m-(d-1)} & \mathbb{Q} & 0 \\ E_2^{\bullet,1} & 0 & 0 & 0 \\ E_2^{\bullet,0} & 0 & \mathbb{Q} & \mathbb{Q}^{m-(d-1)}. \end{array}$$

When we defined the Hodge characteristic on a previous talk we associated a variety to a mixed Hodge structure and then we took the associated class in $K_0(\mathfrak{hs})$, the Grothendieck ring over the category of Hodge structures $\mathfrak{hs}$. We could skip this step, by definition E_1-page is a complex with cohomology E_2-page, this spectral sequence abouts at E_2 so $[E_2] = [E_3] = ... = [E_\infty]$, here $[...]$ denotes the class in $K_0(\mathfrak{hs})$. So then we can compute the Hodge characteristic of ψ_f^{Hdg}, to do so first we set:

$$\begin{cases} a := s+k \\ b := 2+s+1 \\ c := q-s-2k \end{cases} \quad \text{from the MHS we deduce that } a,c\geq 0, b\geq 1.$$

Since $k = b-a-1$ we get that $0 \leq a \leq b-1$ so

$$\chi_{Hdg}(X_\infty) = \sum_{b\geq 1, c\geq 0}\sum_{a=0}^{b-1}(-1)^{c+b+1}[H^c(E(b))(-a)] = \sum_{b\geq 1}(-1)^{b+1}\chi_{Hdg}(E(b))\left[\sum_{a=0}^{b-1}\mathbb{L}^a\right]$$

here $\mathbb{L}$ denotes the Lefschetz motive, i.e. the class of $\mathbb{A}^1$. Now using as fact from an earlier talk that $[\mathbb{P}^k] = (1 + \mathbb{L} + ... + \mathbb{L}^k)$ we get that

$$\chi_{Hdg}(X_\infty) = \sum_{b \geq 1} (-1)^{b+1} \chi_{Hdg}(E(b) \times \mathbb{P}^{b-1}) \tag{5.1}$$

which suggests the following definition.

Definition 17. *Define the **motivic nearby fibre** of f as*

$$\psi_f^{mot} := \sum_{m \geq 1} (-1)^{m-1} [E(m) \times \mathbb{P}^{m-1}] \in K_0(\underline{Var}).$$

Proposition 18. *[Pet10, Lemma 7.2.2] Suppose that $\sigma : Y \to X$ is a bimeromorphic map which is an isomorphism over $X \backslash X_0$. Put $g = f \circ \sigma$ and assume that $g^{-1}(0)$ is a divisor with strict normal crossings. Then*

$$\psi_f = \psi_g$$

Proof. Using the weak factorization theorem [AKMW02, Theorem 0.1.1], it suffices to prove this for σ being the blow-up of X along a connected submanifold $Z \subset X_0$, with the following property. As X_0 is snc we write $X_0 = \cup_{i \in I} E_i$, we have $A \subset I$ be the set of indices satisfying that Z intersects the divisor $\cup_{i \notin A} E_i$ transversely. In particular $Z \cap \cup_{i \notin A} E_i$ is snc.

For simplicity assume that $|A| = 1$ and so Z is contained in a divisor E_1 and that E_2 is the only component of X_0 intersecting Z. Let $c = \operatorname{codim}_Z X$ and so $\operatorname{codim}_Z E_1 = \operatorname{codim}_{Z \cap E_2} E_{12} = c - 1$ and $\operatorname{codim}_{Z \cap E_2} E_2 = c$. Now $g^{-1}(0)$ has as extra component, the exceptional divisor E'_1. Similarly denote the proper transforms of E_j as E'_j. After blowing up we have two new double intersections E'_{01} and E'_{02} and a new triple intersection E'_{012}. Then

$$\begin{aligned}\psi_g - \psi_f = ([E'_1] - [E_1]) + ([E'_2] - [E_2]) + [E'_0] - ([E'_{12}] - [E_{12}]) \\ + ([E'_{01}] + [E'_{02}]) \times [\mathbb{P}^1] + [E'_{012}] \times [\mathbb{P}^2]\end{aligned}$$

now the full exceptional divisor E'_0 is a $\mathbb{P}^{c-1}$ bundle over Z; E'_{01} is a $\mathbb{P}^{c-2}$ bundle over Z; E'_{02} is a $\mathbb{P}^{c-1}$ bundle over $Z \cap E_2$ and E'_{012} is a $\mathbb{P}^{c-2}$ bundle over $Z \cap E_2$. So we can make use of the following fact:

Fact 19. *[Pet10, Example 2.1.7] Let $E \to Y$ be a $\mathbb{P}^k$-bundle over Y, a projective variety. The scissor relations imply that*

$$[E] = [Y] \cdot [\mathbb{P}^k] = [Y] \cdot (1 + \mathbb{L} + ... + \mathbb{L}^k).$$

From this we deduce that the $[Z]$ coefficient is

$$[\mathbb{P}^{c-1}] + ([\mathbb{P}^{c-2}] - 1) - [\mathbb{P}^{c-2}] \cdot [\mathbb{P}^1] = 0$$

and the coefficient of $[Z \cap E_2]$ is

$$([\mathbb{P}^{c-1}] - 1) + ([\mathbb{P}^{c-2}] - 1 + [\mathbb{P}^{c-1}]) \cdot [\mathbb{P}^1] + [\mathbb{P}^{c-2}] \cdot [\mathbb{P}^2] = 0.$$

□

Corollary 20.

$$\chi_{Hdg}(\psi_f^{mot}) = \chi_{Hdg}(X_\infty)$$

Example 21. Suppose that we have a planar curve degenerating to d $\mathbb{P}^1$'s only intersecting pairwise. So they intersect at $d(d-1)/2$ and there are no triple intersections. Then we can compute the Hodge characteristic of the motivic nearby fibre

$$\chi_{Hdg}(X_\infty) = d \cdot \chi_{Hdg}(\mathbb{P}^1) - \sum_i \chi_{Hdg}(E(i) \times \mathbb{P}^1) =$$

$$= d \cdot \chi_{Hdg}(\mathbb{P}^1) - \frac{d(d-1)}{2} \cdot \chi_{Hdg}(\mathbb{P}^1) = \frac{3d-d^2}{2} \cdot \chi_{Hdg}(\mathbb{P}^1).$$

Recall we had a commutative diagram:

$$\begin{array}{ccc} K_0(Var) & \xrightarrow{\chi_{Hdg}} & K_0(\mathfrak{hs}) \\ & {\scriptstyle \chi_c} \searrow \quad \swarrow {\scriptstyle dim} & \\ & \mathbb{Z} & \end{array}$$

thus we obtain

$$(3d-d^2) = \frac{3d-d^2}{2} \dim(\chi_{Hdg}(\mathbb{P}^1)) = \dim(\chi_{Hdg}(X_\infty)) = \chi_c(X_t) = 2-2g$$

Were g is the genus so we obtained $g = \frac{(d-1)(d-2)}{2}$. We were able to compute the genus of the curve only having information about its degeneration.

Example 22. Let $F, L_1, ..., L_d \in \mathbb{C}[X_0, X_1, X_2]$ be homogeneous forms with degree $\deg F = d$ and $\deg L_i = 1$, such that $F \cdot L_1 \cdots L_d = 0$ defines a reduced divisor with normal crossings on $\mathbb{P}^2$. Now consider

$$X = \left\{ ([x_0, x_1, x_2]) \in \mathbb{P}^2 \times \Delta : \prod_{i=1}^{d} L_i(x) + tF(x) = 0 \right\}$$

X is smooth and the projection on Δ has fibre over 0 $E_1 \cup ... \cup E_d$ of the lines such that $E_i : Li = 0$. So the Hodge characteristic of the motivic nearby fibre (5.1) is

$$\chi_{Hdg}(\psi_f^{mot}) = (1-g)(1+\mathbb{L})$$

where $g = \binom{d-1}{2}$ is the genus of the generic fibre, as it is a smooth projective curve of degree d.

The weights are shown in Figure 3. One sees that there are only even weight terms for

	H^0	H^1	H^2
weight 0	1	g	0
weight 2	0	$g(\mathbb{L})$	$\mathbb{L}$

Figure 3: Dimensions of limit mixed Hodge structure of Example 22

$H^1(X_\infty)$ and its only primitive subspace has weight 2 and dimension g, since $\dim H^1(X_\infty) = 2g$. In particular N has g blocs of size 2, so it is maximally unipotent. Moreover the monodromy diagram is $\underset{g}{\overset{0}{\bullet}} \overset{N}{\leftarrow} \underset{g\cdot\mathbb{L}}{\overset{2}{\bullet}}$.

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