solutions to a class of nonlocal evolution equations

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Alicante, mayo 2011

Let X be a real Banach space. Our goal in this paper is to obtain existence results regarding integral solutions to a class of nonlocal evolution equations of the form

$$\begin{cases} u'(t) + A(u(t)) \ni f(t, u(t)), \ t \in (0, T), \\ u(0) = g(u), \end{cases} \tag{1}$$

where $A : D(A) \subseteq X \to X$ is an m-accretive operator on X, and $f : [0, T] \times X \to X$ and $g : C(0, T; X) \to X$ are given functions.

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where $A : D(A) \subseteq X \to X$ is an m-accretive operator on X, and $f : [0, T] \times X \to X$ and $g : C(0, T; X) \to X$ are given functions. The nonlocal initial condition, which is a generalization of the classical Cauchy initial condition, was motivated by problems which arise in the natural sciences. For example, in 1993, K. Deng used this type of problems to describe the diffusion phenomenon of a small amount of gas in a transparent tube. The study of abstract nonlocal nonlinear initial value problems was initiated by Aizicovici and Gao in 1997.

Results on the existence of periodic solutions to Problem (1), *i.e.,* the particular case where $g(u) = u(T)$, have been obtained under several assumptions on A and f by many authors. (we can mention, the authors Bostan, Cascaval, Vrabie, Cwiszewski, Paicu).

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One of the usual methods for proving the existence of T-periodic solutions is to show that the corresponding Poincaré map, *i.e.*, the map $P : \overline{D(A)} \to 2^{\overline{D(A)}}$, which assigns to each $x \in \overline{D(A)}$ the values at T of all integral solutions satisfying $u(0) = x$, has at least one fixed point.

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However, in the nonlocal case, the most usual technique used to obtain a solution is to use some fixed point theorem in $C(0, T; X)$ in order to see that the mapping $\mathcal{S} : C(0, T; X) \to C(0, T; X)$, defined by letting $\mathcal{S}(v), \text{where } v \in C(0, T; X)$, be the solution of the problem

$$\begin{cases} u'(t) + A(u(t)) \ni f(t, v(t)), \ t \in (0, T), \\ u(0) = g(v), \end{cases}$$

has a fixed point.

In this talk we show that under some additional assumptions on X, A and f, it is still possible to establish an existence result regarding integral solutions to Problem (1), which does allow us to include functions $g : C(0, T; X) \to X$ of the form

- $g(u) = u(T)$;
- $g(u) = \frac{1}{T}\int_{\delta}^{T} u(\tau)d\tau$;
- $g(u) = \sum_{i=1}^{n} \alpha_i u(t_i)$, where $\sum_{i=1}^{n} |\alpha_i| \leq 1$ and $0 < t_1 < t_2, ..., < t_n \leq T$.

An operator A on X is said to be *accretive* if the inequality $\|x - y + \lambda(z - w)\| \geq \|x - y\|$ holds for all $\lambda \geq 0$, $(x, z); (y, w) \in A$.

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Those accretive operators which are m-accretive play an important role in the study of nonlinear partial differential equations.

Consider the Cauchy problem

$$\begin{cases} u'(t) + A(u(t)) \ni f(t), \ t \in (0, T), \\ u(0) = x_0 \in \overline{D(A)}, \end{cases} \tag{2}$$

where A is m-accretive on X and $f \in L^1(0, T, X)$.

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$$\begin{cases} u'(t) + A(u(t)) \ni f(t), & t \in (0,T), \\ u(0) = x_0 \in \overline{D(A)}, \end{cases} \tag{3}$$

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There exists a unique continuous function $u : [0,T] \to \overline{D(A)}$ such that $u(0) = x_0$, and moreover, for each $(x,y) \in A$ and $0 \le s \le t \le T$, we have

$$\|u(t) - x\|^2 - \|u(s) - x\|^2 \le 2 \int_s^t \langle f(\tau) - y, u(\tau) - x \rangle_+ d\tau. \tag{4}$$

Here the function $\langle \cdot,\cdot \rangle_+ : X \times X \to \mathbb{R}$ is defined by $\langle y, x \rangle_+ = \sup\{x^*(y) \ : \ x^* \in J(x)\}$, where $J : X \to 2^{X^*}$ is the duality mapping on X, *i.e.*, $J(x) = \{x^* \in X^* \ : \ x^*(x) = \|x\|^2, \|x^*\| = \|x\|\}$.

A *strong solution* of Problem (3) is a function $u \in W^{1,\infty}(0, T; X)$, *i.e.*, u is locally absolutely continuous and almost differentiable everywhere, $u' \in L^{\infty}(0, T; X)$, and $u'(t) + A(u(t)) \ni f(t)$ for almost all $t \in [0, T]$.

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Theorem

If X is a Banach space with the Radon-Nikodym property, $A : D(A) \subseteq X \to 2^X$ is an m-accretive operator, and $f \in BV(0,T;X)$, i.e., f is a function of bounded variation on $[0,T]$, then Problem (3) has a unique strong solution whenever $x_0 \in D(A)$.

we say that $u \in C(0, T; X)$ is a *weak solution* of Problem (3) if there are sequences $(u_n) \subseteq W^{1,\infty}(0, T; X)$ and $(f_n) \subseteq L^1(0, T; X)$ satisfying the following four conditions:

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1. $u_n'(t) + Au_n(t) \ni f_n(t)$ for almost all $t \in [0, T]$, $n = 1, 2, \dots;$
2. $\lim_{n\to\infty} \|u_n - u\|_\infty = 0;$
3. $u(0) = x_0;$
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Theorem

Let X be a Banach space with the Radon-Nikodym property. Then Problem (3) admits a unique weak solution which is the unique integral solution of this problem.

Consider the nonlocal Cauchy problem

$$\begin{cases} u'(t) + A(u(t)) \ni f(t, u(t)), \ t \in (0, T), \\ u(0) = g(u), \end{cases} \tag{5}$$

(C1) $A : D(A) \subseteq X \to 2^X$ is an m-accretive operator such that $-A$ generates a compact semigroup $\{S(t) : \overline{D(A)} \to \overline{D(A)}\}, \ t \geq 0$;

(C2) There exists $r > 0$ such that for each $(x, y) \in A$ with $\|x\| \geq r$, $z \in B_r(0)$ and for each $t \in [0, T]$, $\langle y - f(t, z), x \rangle_s \geq 0$;

(C4) $g : C(0, T; X) \to \overline{D(A)}$ is a continuous mapping such that $\sup_{u \in W_r} \|S(\frac{1}{n})g(u)\| \leq r$ for all $n \geq 1$, where $r > 0$ is given by (C2), and there is a $\delta \in (0, T)$ such that $g(u) = g(v)$ whenever $u(s) = v(s)$ for all $s \in [\delta, T]$;

(F4) $f(\cdot, \cdot)$ is a Carathéodoric function and for any $l > 0$, there exists a function $k_l \in L^1(0, T; \mathbb{R})$ such that $\|f(t, x)\| \leq k_l(t)$ for a.e. $t \in [0, T]$ and every $x \in B_l(0)$.

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Let X be a real Banach space with the Radon-Nikodym property. Under the above assumptions the problem

$$\begin{cases} u'(t) + A(u(t)) \ni f(t, u(t)), \ t \in (0, T), \\ u(0) = g(u), \end{cases} \tag{6}$$

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For a fixed $n \geq 1$, we consider the problem

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Lemma

Assume that the hypotheses of Theorem 3 hold. Then for any $n \geq 1$, Equation (7) has an integral solution.

Proof

Given $v \in W_r$, let $\mathcal{S}_n(v)$ be the unique integral solution of the problem

$$\begin{cases} u'(t) + A(u(t)) \ni f(t, v(t)), t \in (0, T), \\ u(0) = S(\frac{1}{n})g(v) \end{cases}$$

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By Condition (C4):
$g : C(0, T; X) \to \overline{D(A)}$ is a continuous mapping such that $\sup_{u \in W_r} \|S(\frac{1}{n})g(u)\| \leq r$ for all $n \geq 1$, where $r > 0$ is given by (C2), and there is a $\delta \in (0, T)$ such that $g(u) = g(v)$ whenever $u(s) = v(s)$ for all $s \in [\delta, T]$;

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we have $S(\frac{1}{n})g(v) \in \overline{D(A)} \cap B_r(0)$.

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Now, let us see that W_r is a S_n-invariant set.

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If $\mathcal{S}_n(v) \notin W_r$, then since $\mathcal{S}_n(v)$ is a continuous function and $\mathcal{S}_n(v)(0) \in W_r$, there exists $0 \leq t_0 < T$ such that $\|\mathcal{S}_n(v)(t)\| > r$ for every $t \in (t_0, t_0 + \delta)$ and $\|\mathcal{S}_n(v)(t_0)\| \leq r$.

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Since $\mathcal{S}_n(v)$ is a weak solution, there exist sequences $(u_m) \subseteq W^{1,\infty}(0, T; X)$ and $(f_m) \subseteq L^1(0, T; X)$ such that

(a) $u'_m(t) + Au_m(t) \ni f_m(t)$ a.e. $t \in [0, T]$, $m = 1, 2, \ldots$;

(b) $\lim_{m\to\infty} \|u_m - \mathcal{S}_n(v)\|_\infty = 0$;

(c) $\lim_{m\to\infty} \|f_m - f(\cdot, \mathcal{S}_n(v)(\cdot))\|_1 = 0$.

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By assumption (a), we have $f_m(t) - u_m'(t) \in Au_m(t)$ for almost all t.
Now, it follows from Hypothesis (C2):

There exists $r > 0$ such that for each $(x, y) \in A$ with $\|x\| \geq r$, $z \in B_r(0)$ and for each $t \in [0, T]$, $\langle y - f(t, z), x\rangle_s \geq 0$;

that the inequality

$$\langle f_m(t) - u_m'(t) - f(t, u_m(t)), u_m(t)\rangle_s \geq 0, \quad \text{a.e. on } (t_0, t_0 + \delta),$$

holds. This inequality, when combined with Kato's differential rule, yields

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Hence we conclude that for all $t \in (t_0, t_0 + \delta)$,

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Second, since (u_n) is a bounded sequence in $C(0, T; X)$, there exists $l > 0$ such that for all $n \in \mathbb{N}$, $\|u_m\|_\infty \leq l$ and then, by Hypothesis (F4), there exists $k_l \in L^1(0, T; \mathbb{R})$ such that $\|f(t, u_m(t))\| \leq k_l(t)$.

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Consequently, we may apply the dominated convergence theorem and conclude that $\|f(\cdot, u_m(\cdot)) - f(\cdot, \mathcal{S}_n(v)(\cdot))\|_1 \to 0$.
Finally, taking limits as $n \to \infty$ in (8), we obtain

$$\|\mathcal{S}_n(v)(t)\| \leq \|\mathcal{S}_n(v)(t_0)\| \leq r,$$

which is a contradiction. Therefore we conclude that $\|\mathcal{S}_n(v)\|_\infty \leq r$.

S_n is a continuous function

Suppose that (v_m) is a sequence in W_r such that $v_m \longrightarrow v \in W_r$. Since $\mathcal{S}_n(v_m)$ and $\mathcal{S}_n(v)$ are integral solutions and $S(\frac{1}{n})$ is a nonexpansive mapping, we have, for each $t \in [0, T]$,

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$$\|\mathcal{S}_n(v_m)(t) - \mathcal{S}_n(v)(t)\| \leq \|g(v_m) - g(v)\| + \int_0^T \|f(\tau, v_m(\tau)) - f(\tau, v(\tau))\| d\tau.$$

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Now, since g is a continuous function and since by condition (F4) we may invoke the dominated convergence theorem to obtain that $f(\cdot, v_m(\cdot) \longrightarrow f(\cdot, v(\cdot))$ in $L^1(0, T; X)$, we conclude that $\mathcal{S}_n$ is indeed a continuous mapping.

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Schauder's fixed point theorem now yields the result.
If we consider

$$Q := \{u_n \in W_r : u_n \text{ is a solution of problem } (7),\ n \in \mathbb{N}\},$$

then Q is a relatively compact subset of $C(0, T; X)$ and therefore we may assume that $u_n \to u$ as $n \to \infty$ and it is easy to see that u is an integral solution of problem (1).

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary $\partial\Omega$ and let $\beta : D(\beta) \subseteq \mathbb{R} \to 2^{\mathbb{R}}$ be m-accretive with $0 \in \beta(0)$. Let $p \in (2, \infty[$ be given. Consider the $p-$Laplacian operator

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$$\triangle_p^\lambda u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) - \lambda u |u|^{p-2},$$

where the partial derivatives are taken in the sense of distributions over Ω and $\lambda > 0$.

Let $L_p^\lambda : D(L_p^\lambda) \to L^2(\Omega)$, defined by

$$L_p^\lambda = -\triangle_p^\lambda u, \ \forall u \in D(L_p^\lambda),$$

be the subdifferential of the convex function Φ_p^λ defined by

$$\Phi_p^\lambda(u) := \begin{cases} \frac{1}{p}\sum_{i=1}^n \int_\Omega |\frac{\partial u}{\partial x_i}|^p dx + \frac{\lambda}{p}\int_\Omega |u|^p dx + \int_{\partial\Omega} j(u) d\sigma, \\ \qquad \text{if } u \in W^{1,p}(\Omega) \text{ and } j(u) \in L^1(\partial\Omega) \\ +\infty, \ \text{otherwise,} \end{cases}$$

where $j : \mathbb{R} \to \mathbb{R}^+ \cup \{+\infty\}$ is a proper lower semicontinuous convex function such that its subdifferential $\partial j = \beta$ (see, for instance, Example 1.6.3 in Barbu, 2010).

It is well known that L_p^λ is an m-accretive operator on $L^2(\Omega)$ and $-L_p^\lambda$ generates a nonexpansive compact semigroup on $\overline{D(L_p^\lambda)} = L^2(\Omega)$

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Consider the nonlocal initial boundary value problem

$$\begin{cases} u_t(t,x) - \triangle_p^\lambda(u(t,x)) = h(t,x,u(t,x)), \ (t,x) \in (0,T) \times \Omega, \\ u(0,x) = \sum_{i=1}^k c_i u(t_i,x), \ \text{a.e on } \Omega, \\ -\frac{\partial u}{\partial \nu_p} \in \beta(u(t,x)), \ \text{a.e. on } (0,T) \times \partial\Omega, \end{cases} \tag{9}$$

where $h : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist $m, b \geq 0$ such that $|h(t,x,u)| \leq m|u| + b$ for each $(t,x,u) \in [0,T] \times \overline{\Omega} \times \mathbb{R}$, $0 < t_1 < \ldots < t_p \leq T$ and $\sum_{i=1}^p |c_i| \leq 1$.

Consider $X = L^2(\Omega)$ and $A = L_p^\lambda$, $u(t) := u(t, .) \in X$. Define $f : [0, T] \times X \to X$ by $f(t, v)(x) = h(t, x, v(x))$ for all $t \in (0, T]$ and $g : C(0, T; X) \to X$ by $g(u)(x) = \sum_{i=1}^{p} c_i u(t_i)(x)$. We interpret and rewrite Problem (9) as follows:

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In order to show that Problem (10) has at least one integral solution, we are going to check whether the conditions of the previous theorem are met.

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(e) Since $0 \in A(0)$, it is not difficult to prove that g satisfies condition (C4) of Theorem 3.

Finally, we show that Condition (C2)

There exists $r > 0$ such that for each $(x, y) \in A$ with $\|x\| \geq r$, $z \in B_r(0)$ and for each $t \in [0, T]$, $\langle y - f(t, z), x \rangle_s \geq 0$;

is satisfied. This will be done in several steps.

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Proposition

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Recall that

Definition

Let X be a Banach space and let $\gamma : [0, +\infty[\to [0, +\infty[$ be a function with $\lim_{t\to\infty} \gamma(t) = \infty$. An accretive operator $A \subset X \times X$ is said to be γ-coercive if for each $(x, y) \in A$, we have $\langle y, x \rangle_s \geq \gamma(\|x\|)\|x\|$.

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On the other hand,

Proposition

Let X be a Banach space and A an m-accretive and γ-coercive operator on X. If $f : [0, T] \times X \to X$ satisfies

$$\lim_{r\to\infty} \frac{\sup\{\|f(t, v)\| : \; t \in [0, T], \|v\| = r\}}{\gamma(r)} < 1, \tag{11}$$

then there exists $r > 0$ such that $\langle y - f(t, x), x\rangle_s \geq 0$ for all $t \in [0, T]$ and for all $(x, y) \in A$ with $\|x\| \geq r$.

Now it is not difficult to see that

$$\lim_{r\to\infty} \frac{\sup\{\|f(t,u)\| : \ t \in [0,T], \ \|u\| \leq r\}}{Mr^{p-1}} < 1.$$

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Finally, by a result due to Brezis we can conclude that the integral solutions obtained are strong solutions.