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ZEROES OF ACCRETIVE OPERATORS AND THE ASYMPTOTIC BEHAVIOR OF NONLINEAR SEMIGROUPS

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Abstract. We study necessary and sufficient conditions for an accretive operator which satisfies the range condition to have a zero. We obtain, in particular, a characterization of this property for m-accretive operators in L^1. We also study the asymptotic behavior of nonexpansive semigroups in L^1 and then apply our results to certain initial value problems.

1. Introduction

Let X be a real Banach space. A mapping $A : X \to 2^X$ will be called an operator on X. The domain of A is denoted by $D(A)$ and its range by $R(A)$. An operator A on X is said to be *accretive* if the inequality $\|x-y+\lambda(z-w)\| \geq \|x-y\|$ holds for all $\lambda \geq 0$, $z \in Ax$, and $w \in Ay$. If, in addition, $R(I+\lambda A)$ is for one, hence for all $\lambda > 0$, precisely X, then A is called m-accretive. We say that A satisfies the range condition if $\overline{D(A)} \subset R(I+\lambda A)$ for all $\lambda > 0$. (See, for instance, [28] and [11] to find hypotheses which imply the range condition.) Accretive operators were introduced by F.E. Browder [9] and T. Kato [21] independently. Those accretive operators which are m-accretive or satisfy the range condition play an important role in the study of nonlinear semigroups, differential equations in Banach spaces, and fully nonlinear partial differential equations. For example, it is well known that if X is a Banach space and $A : D(A) \to 2^X$ is an accretive operator which

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satisfies the range condition, then the initial value problem

$$u'(t) + A(u(t)) \ni 0, \quad u(0) = x_0, \tag{1.1}$$

has a unique mild solution for each $x_0 \in \overline{D(A)}$, which is given by the Crandall-Liggett exponential formula [13]:

$$u(t) := \lim_{n\to\infty} (I + \frac{t}{n}A)^{-n}(x_0).$$

Moreover, the family

$$\mathcal{F} := \{S(t) : \overline{D(A)} \to \overline{D(A)} : \ t \geq 0\},$$

where $S(t)x = \lim_{n\to\infty}(I + \frac{t}{n}A)^{-n}(x)$, is a nonexpansive semigroup.

Among the problems treated by accretive operator theory, one of the most studied is that of determining when A has a zero (i.e., $0 \in R(A)$). Note that if $0 \in R(A)$, then the problem $u'(t) + A(u(t)) \ni 0$ has stationary (i.e., constant) solutions.

Recall that a Banach space X is said to have the fixed point property (FPP) if for each nonempty, bounded, closed and convex subset D of X, every nonexpansive self-mapping T has a fixed point (see [18]). It is of interest to note the usefulness of fixed point theory for nonexpansive mappings in studying the problem of the existence of zeroes of m-accretive operators in Banach spaces. In fact, Reich-Torrejón [38] and Morales [29] gave a characterization of such m-accretive operators when the Banach space X has the FPP. (The reader is also referred to [20] for a discussion of this type of results; see also [17] and [30]). However, $L_1([0,1])$ is an important example of a Banach space which is outside the scope of the results of [38], [29], [30], [20] and [17]. This is because $L_1([0,1])$ fails to have the FPP even for nonempty, weakly compact and convex subsets (see [1]).

On the other hand, the problem of convergence of continuous semigroups was initiated by Brezis [7] who studied, among other topics, the behavior as $t \to \infty$ of the solutions of

$$\frac{du}{dt} \in -A(u(t)) \ \text{ for a.e. } \ t \geq 0,$$

where A is a maximal monotone operator in a Hilbert space. Pazy [33] showed that when the operator A is maximal monotone, the weak asymptotic convergence of a solution $u(t)$ is equivalent to the condition $\omega_w(x) \subset A^{-1}(0)$, where $u(0) = x$ and $\omega_w(x)$ denotes the set of weak subsequential limits of $u(t)$ as $t \to \infty$. Since then, many efforts have been devoted to the study of the asymptotic behavior of nonexpansive semigroups in say, uniformly convex Banach spaces. (The reader is referred to [16] for a discussion of this type of results.)

In connection with his study of the strong convergence of semigroups, Pazy introduced in [34] the convergence condition in a Hilbert space H, and showed that if A is a maximal monotone operator which satisfies the convergence condition and $f \in L^1(0, \infty; H)$, then the solution of the problem

$$\begin{cases} u'(t) + A(u(t)) \ni f(t) \\ \\ u(0) = x_0 \end{cases} \tag{1.2}$$

converges strongly to a point in $A^{-1}(0)$.

This convergence condition was subsequently extended in 1979 by Nevanlinna and Reich [31] to a Banach space setting. Let X be a uniformly convex and uniformly smooth Banach space, and let A be an m-accretive operator in X. Then A is said to satisfy the convergence condition if

(a) $F = A^{-1}(0)$ is nonempty.

(b) If $(x_n, y_n) \in A$, $\|x_n\| \le K$, $\|y_n\| \le K$, and $\lim_{n\to\infty}\langle y_n, J(x_n - Px_n)\rangle = 0$, then $\liminf_{n\to\infty} \|x_n - Px_n\| = 0$.

Here $P : X \to F$ is the nearest point projection which is well-defined because X is reflexive and F is nonempty, closed and convex, and $J : X \to 2^{X^*}$ is the normalized duality map (see the definition below).

Nevanlinna and Reich showed that if X and A satisfy the above conditions, then the unique mild solution of the homogeneous problem (1.1) converges strongly as $t \to \infty$ to a zero of A. In fact, the same conclusion holds when X is merely a reflexive, strictly convex, and smooth Banach space.

Xu [40] has recently shown that if X is a uniformly convex and uniformly smooth Banach space and A satisfies the convergence condition, then the same result holds for almost-orbits of the nonexpansive semigroup generated by $-A$. In particular, he has proved that the solution of the inhomogeneous problem (1.2) converges as $t \to \infty$ to a zero of A.

In the present paper, we study conditions for an accretive operator A which satisfies the range condition to have a zero. We obtain, in particular, a characterization of this property (i.e., $0 \in R(A)$), for m-accretive operators in L_1. Another goal of our paper is to show that Xu's result also holds outside the class of uniformly convex and uniformly smooth Banach spaces. Thus we establish an improvement of Xu's result by showing that the same conclusion holds in the framework of reflexive, strictly convex, and smooth Banach spaces. Finally, we study the asymptotic behavior of nonexpansive semigroups in L_1. In this way we obtain several new results on the asymptotic behavior of solutions to the initial value problem (1.1).

2. Preliminaries

Throughout this paper we assume that X is a real Banach space and denote by $X^\star$ the dual space of X. We define the normalized duality mapping by

$$J(x) := \{j \in X^\star : \langle x, j\rangle = \|x\|^2, \|j\| = \|x\|\}.$$

Let $\langle y, x\rangle_+ := \max\{\langle y, j\rangle : j \in J(x)\}$.

Given an operator $A : D(A) \to 2^X$, we define

$$J_\lambda := (I + \lambda A)^{-1}, \quad A_\lambda := \frac{I - J_\lambda}{\lambda}, \quad |Ax| := \inf\{\|u\| : \ u \in Ax\}.$$

The operators J_λ and A_λ are the resolvent and the Yosida approximant of A, respectively.

We now recall some important facts regarding accretive operators which will be used in our paper (see, for instance, [12]).

Proposition 2.1. *(i).- The operator $A \subset X \times X$ is accretive if and only if $\langle u - v, x - y\rangle_+ \geq 0$ for all $x, y \in D(A)$, and for each $u \in Ax$ and $v \in Ay$.*

(ii).- The operator $A \subset X \times X$ is accretive if and only if for each $\lambda > 0$, the resolvent J_λ is a single-valued nonexpansive mapping.

When A is an accretive operator which satisfies the range condition we have

(iii).- For all $x \in D(A)$ and for each $\lambda > 0$, $A_\lambda x \in AJ_\lambda x$.

(iv).- For all $x \in D(A)$ and for each $\lambda > 0$, $\|A_\lambda x\| \leq |Ax|$.

(v).- Given $x \in D(A)$, the mapping $\lambda \to J_\lambda x$ is continuous for $\lambda \geq 0$, with $J_0 x = x$.

As already mentioned in the Introduction, when the Banach space X has the FPP, Reich-Torrejón [38] and Morales [29] gave a characterization of those m-accretive operators $A \subset X \times X$ which have zeroes. We pick up those results in the next two theorems.

Theorem 2.2. *([29],[38]) Let X be a Banach space with the FPP, and let $A : D(A) \to 2^X$ be m-accretive. Then the following are equivalent:*

(a).- $0 \in R(A)$.

(b).- $E := \{x \in D(A) \ : \ tx \in Ax, \ t < 0\}$ is bounded.

(c).- There exist $x_0 \in D(A)$ and a bounded open neighborhood $\mathcal{U}$ of x_0 such that for each $x \in \partial\mathcal{U} \cap D(A)$, $\langle y, x - x_0\rangle_+ \geq 0$ for all $y \in Ax$.

Theorem 2.3. [38] *Let X be a Banach space such that its unit ball has the fixed point property for nonexpansive mappings, and let $A \subset X \times X$ be m-accretive.*

Then $0 \in R(A)$ *if and only if there exist* $R > 0$ *and a point* $x_0 \in \overline{D(A)}$ *such that* $\langle y, x - x_0 \rangle_+ \geq 0$ *for all* $y \in Ax$ *with* $\|x - x_0\| = R$.

To test whether a mapping satisfies the range condition is not an easy task. Now we introduce some other conditions which can be easily checked in some significant cases.

A mapping $U : D \to X$ is said to satisfy the *weak range condition* if

$$\lim_{r \to 0^+} \frac{1}{r} dist(x, R(I + rU)) = 0$$

for each $x \in D$.

We say that U is subtangential on D if $\lim_{r \to 0^+} \frac{1}{r} dist(x - rUx, D) = 0$ for each $x \in D$.

Let us recall the following characterization of those continuous single-valued accretive operators which satisfy the range condition. It is due to Martin [28]. In this connection see also [37].

Theorem 2.4. [28] *Let* $A : D(A) \to X$ *be a single-valued continuous mapping with* $D(A)$ *a closed subset of a Banach space* X.

(i).- If A *is subtangential on* $D(A)$, *then* A *satisfies the weak range condition.*

(ii).- If A *is an accretive mapping satisfying the weak range condition, then* A *is subtangential on* $D(A)$.

(iii).- If $D(A)$ *is also convex and* A *is an accretive mapping satisfying the weak range condition, then* A *satisfies, in fact, the range condition.*

When D is a convex subset of a Banach space X and $U : D \to X$ is subtangential on D, we will say that $I - U$ is weakly inward on D (see, for example, [10] and [36]). This condition is weaker than the assumption that $I - U$ map the boundary of D into D.

Let X be a real Banach space and let C be a nonempty subset of X.

Let $\mathcal{F} = \{T(t) : C \to C : \ t \geq 0\}$ be a family of self-mappings of C. We recall that $\mathcal{F}$ is said to be a nonexpansive semigroup acting on C if the following conditions are satisfied:

(a).- $T(0) = I$, where I is the identity mapping on C.

(b).- $T(s + t)x = T(s)T(t)x$ for all $s, t \in [0, \infty[$ and $x \in C$.

(c).- $\|T(t)x - T(t)y\| \leq \|x - y\|$ for all $x, y \in C$ and $t \in [0, \infty[$.

(d).- $t \to T(t)x$ is continuous in $t \in [0, \infty[$ for each $x \in C$.

The set of common fixed points of $\mathcal{F}$ is denoted by $Fix(\mathcal{F})$.

Given $x \in C$, the orbit of x under $\mathcal{F}$ is the function

$$\gamma : [0, \infty[\to C \quad \text{defined by } \gamma(t) := T(t)x.$$

Sometimes the set $\gamma(x) := \{T(t)x : t \in [0, \infty[\}$ is also called the orbit of x under $\mathcal{F}$.

The following facts about nonexpansive semigroups can be found in [25].

A continuous function $u : [0, \infty[\to C$ is called an almost-orbit of $\mathcal{F}$ if

$$\lim_{s \to \infty} (\sup_{t \in [0, \infty[} \|u(t+s) - T(t)u(s)\|) = 0.$$

Of course, every orbit is an almost-orbit. The notion of an almost-orbit is useful because if A is an m-accretive operator on X, then the integral solutions (see [3]) of the initial value problem

$$u'(t) + Au(t) \ni f(t), \quad t \geq 0, \quad u(0) = x \in \overline{D(A)},$$

with $f(\cdot) \in L^1(0, \infty, X)$ are almost-orbits of the nonexpansive semigroup generated by $-A$.

Lemma 2.5 ([25])**.** *Let X be a Banach space and let $\mathcal{F}$ be a nonexpansive semigroup on a subset C of x. If u, v are almost-orbits of $\mathcal{F}$, then we have: (a) $\lim_{t \to \infty} \|u(t) - v(t)\|$ exists; (b) for every $h \geq 0$, $\|u(t+h) - u(t)\|$ converges as $t \to \infty$; (c) if $Fix(\mathcal{F}) \neq \emptyset$, then $u([0, \infty[)$ is bounded.*

3. Zeroes of accretive operators satisfying the range condition

In order to proceed, we shall first give the following definitions.

Definition 1. [14] Let X be a Banach space and let τ be a topology on X which is weaker than the norm topology. We say that X has the τ-fixed point property (τ-FPP) if every nonexpansive self-mapping defined on a bounded, convex and τ-sequentially compact subset C of X has a fixed point.

Let (x_n) be a τ-null sequence which is norm bounded. The function

$$\phi_{(x_n)}(x) = \limsup_{n \to \infty} \|x - x_n\|$$

will be called a function of τ-null type.

Definition 2. A subset K of a Banach space X is said to be "locally" τ-sequentially compact if the intersection of K with any closed ball in X is τ-sequentially compact.

Definition 3. Let X be a Banach space and let τ be a topology on X weaker than the norm topology. We say that an operator $A \subset X \times X$ satisfies the τ-range condition whenever $\overline{D(A)}^{\tau} \subset R(I + \lambda A)$ for every $\lambda > 0$.

Theorem 3.1. *Let X be a Banach space with the τ-FPP property and assume that the functions of τ-null type defined on it are τ-slsc, (i.e., τ-sequentially lower semi-continuous). Let $A \subset X \times X$ be an accretive operator satisfying the τ-range condition. If $\overline{D(A)}^{\tau}$ is convex and "locally" τ-sequentially compact, then the following are equivalent:*

(a).- $0 \in R(A)$.

(b).- There exist $x_0 \in D(A)$ and a bounded open neighborhood $\mathcal{U}$ of x_0 such that for each $x \in \partial\mathcal{U} \cap D(A)$, $\langle y, x - x_0\rangle_+ \geq 0$ for all $y \in Ax$.

(c).- There exists $x_0 \in \overline{D(A)}$ such that $E := \{x \in D(A) \ : \ t(x - x_0) \in Ax, \ t < 0\}$ is bounded.

Proof. $(a) \implies (b)$. By hypothesis, $0 \in R(A)$, which means that there exists $x_0 \in D(A)$ such that $0 \in Ax_0$.

Let $\mathcal{U}$ be a bounded open neighborhood of x_0. Given $x \in \partial\mathcal{U} \cap D(A)$, we have $\langle y, x - x_0\rangle_+ \geq 0$ for all $y \in Ax$ because A is an accretive operator and $0 \in Ax_0$.

$(b) \implies (c)$. Suppose, to get a contradiction, that E is not a bounded set. Then we can assume that $\{J_\lambda x_0 : \lambda \geq 0\}$ is unbounded.

Therefore there exists $\lambda_0 > 0$ such that $J_{\lambda_0} x_0 \notin \mathcal{U}$, and by Proposition 2.1 (v) we have

$$\lim_{\lambda \to 0^+} J_\lambda x_0 = x_0.$$

Now, by connectedness arguments, there exists $\lambda_1 \in]0, \lambda_0[$ such that $J_{\lambda_1} x_0 \in \partial\mathcal{U} \cap D(A)$. However, by Proposition 2.1 (iii) we know that $A_{\lambda_1} x_0 \in AJ_{\lambda_1} x_0$ and consequently,

$$\langle A_{\lambda_1} x_0, J_{\lambda_1} x_0 - x_0\rangle_+ \geq 0,$$

which is indeed a contradiction.

$(c) \implies (a)$. Given $n \in \mathbb{N}$, consider $x_n \in E$ such that $\frac{-1}{n}(x_n - x_0) \in Ax_n$. By hypothesis, E is a bounded set and $\overline{D(A)}^{\tau}$ is locally τ-sequentially compact. Hence we may assume that (x_n) is τ-convergent to $y_0 \in \overline{D(A)}^{\tau}$.

We denote

$$R := \limsup_{n \to \infty} \|x_n - y_0\|$$

and consider the following nonempty set:

$$K := \{y \in \overline{D(A)}^{\tau} \cap B(y_0, 2R) : \phi_{(x_n)}(y) \leq R\}.$$

Now, the τ-*slsc* of the functions of τ-null type ensures that K is a nonempty τ-sequentially compact convex subset. Thus, by the definition of τ-FPP, we only need to check that K is J_1-invariant.

Consider $y \in K$. Since

$$\phi_{(x_n)}(J_1 y) \leq \limsup_{n\to\infty} \|J_1(x_n) - J_1 y\| + \limsup_{n\to\infty} \|J_1(x_n) - x_n\|,$$

it follows from the definition of the Yosida approximant and the fact that J_1 is a nonexpansive mapping that

$$\phi_{(x_n)}(J_1 y) \leq \phi_{(x_n)}(y) + \limsup_{n\to\infty} \|A_1 x_n\|.$$

Furthermore, since by the construction of (x_n) we know that $|Ax_n| \to 0$ and by Proposition 2.1(iv), $\|A_1 x_n\| \leq |Ax_n|$, we obtain

$$\phi_{(x_n)}(J_1 y) \leq R.$$

Now, since $\|J_1 y - y_0\| \leq \phi_{(x_n)}(y) + \phi_{(x_n)}(y_0) \leq 2R$, we conclude that K is indeed J_1-invariant. $\square$

Remark. From the Eberlein-Smulian Theorem, it is clear that if X is a Banach space with the weak FPP, then in Theorem 3.1 it is sufficient to assume that $\overline{D(A)}^{w}$ is convex and "locally" weakly compact. On the other hand, if we assume that X has the FPP and A is an m-accretive operator, then from Theorem 3.1 we can deduce Theorem 2.2.

To see other types of topologies to which Theorem 3.1 can be applied, the reader is referred to [23] and [15].

The next example shows that in Theorem 3.1 we cannot remove the condition that X has the τ-FPP.

Example.- Let X be the Banach space $(L^1[0,1], \|\cdot\|_1)$ and consider the Alspach mapping [1], i.e., first let

$$K := \{f \in L^1[0,1] : \ 0 \leq f \leq 2, \ \ \|f\|_1 = 1\}.$$

Clearly, K is a weakly compact convex subset of $L^1[0,1]$.

Now consider $T : K \to K$ such that for each $f \in K$, $T(f)$ is defined by

$$\begin{array}{llll} T(f): & [0,1] & \longrightarrow & \mathbf{R} \\ & t & \mapsto & T(f(t)) = \begin{cases} \min\{2f(2t), 2\}, & t \in [0, \frac{1}{2}] \\ \max\{2f(2(t-1)) - 2, 0\}, & t \in (\frac{1}{2}, 1]. \end{cases} \end{array}$$

Since it is proved in [1] that T is a fixed-point-free nonexpansive mapping, $A = I - T$ is a continuous accretive mapping such that $0 \notin R(A)$. On the other hand, since K is a weakly compact convex subset, we may use Theorem 2.4 to obtain that A is an accretive mapping satisfying the range condition. Moreover, it can be easily seen that A satisfies conditions (b) and (c) of Theorem 3.1.

If X is a Banach space with a closed unit ball which has the fixed point property for nonexpansive mappings, then Theorem 2.3 allows us to give a characterization of those m-accretive operators in X which have a zero. In particular, if X is a dual Banach space with the weak-star FPP, i.e., for each nonempty convex weak-star-compact subset of X, every nonexpansive self-mapping has a fixed point, then it satisfies the above condition regarding its closed unit ball and therefore we can use Theorem 2.3 to get a characterization of such m-accretive operators. However, it is well known that there exist dual Banach spaces without the FPP which do have the weak-star-FPP and therefore such spaces are outside the scope of Theorem 2.2.

Definition 4. [14] A Banach space X is said to have property $M(\tau)$ if the functions of τ-null type are constant on its spheres.

In [14] the authors showed that if X is a Banach space with property $M(\tau)$, where τ is a linear topology, then the norm and the functions of τ-null type are τ-*slsc* (see [14] to find examples with this property). Khamsi [20] and Sims [39] showed that under certain conditions on a dual Banach space X, it is possible to guarantee that the weak-star null type functions are w^*-slsc, and moreover, X has the w^*-FPP. The following result is a consequence of the above comment and of Theorem 3.1.

Corollary 3.2. *Let X be a dual separable Banach space which either has property $M(w^*)$, or satisfies either the conditions of* [20] *or the condition of* [39]. *Let $A \subset X \times X$ be an m-accretive operator. Then the following are equivalent:*

(a).- $0 \in R(A)$.

(b).- There exist $x_0 \in D(A)$ and a bounded open neighborhood $\mathcal{U}$ of x_0 such that for each $x \in \partial\mathcal{U} \cap D(A)$, $\langle y, x - x_0\rangle_+ \geq 0$ for all $y \in Ax$.

(c).- $E := \{x \in D(A) \ : \ tx \in Ax, \ t < 0\}$ is bounded.

(d).-There exist $R > 0$ and a point $x_0 \in \overline{D(A)}$ such that $\langle y, x - x_0\rangle_+ \geq 0$ for all $y \in Ax$ with $\|x - x_0\| = R$.

Let X be a Banach space and let τ be a vector space topology on X which is weaker than the norm topology. We say that X satisfies the τ-Opial condition (see [32] and [24]) if given a norm bounded sequence (x_n) of X which is τ-null, we have that

$$\limsup_{n\to\infty} \|x_n\| < \limsup_{n\to\infty} \|x_n - x\|$$

whenever $x \in X$ and $x \neq 0$.

It is well known (see [18]) that every Hilbert space, as well as the classical Banach spaces $(l_p, \|\cdot\|_p)$ with $1 \leq p < \infty$ satisfy the weak-Opial condition, while all the L_p spaces with $p \neq 2$ do not satisfy the weak-Opial condition. In general, the τ-Opial condition for a Banach space X does not imply the FPP for this space, even when τ is the weak topology. We can, however, obtain the following result.

Theorem 3.3. *Let X be a Banach space and let τ be a vector space topology weaker than the norm topology. Assume that X satisfies the τ-Opial condition and let $A \subset X \times X$ be an accretive operator which satisfies the τ-range condition. Then the following are equivalent.*

(a).- $0 \in R(A)$.

(b).- There exist $x_0 \in \overline{D(A)}$ and a norm bounded τ-sequentially compact subset $\mathcal{U}$ with $x_0 \in \mathcal{U}$ such that for all $x \in D(A) \cap (X \backslash \mathcal{U})$, $\langle y, x - x_0\rangle_+ \geq 0$ whenever $y \in Ax$.

(c).- There exist $x_0 \in \overline{D(A)}$ and a norm bounded τ-sequentially compact subset $\mathcal{U}$ with $x_0 \in \mathcal{U}$ such that for all $x \in D(A) \cap (X \backslash \mathcal{U})$, $t(x - x_0) \notin Ax$ for $t < 0$.

(d).- There exists $x_0 \in \overline{D(A)}$ such that $(J_{t_n} x_0)$ is a norm bounded sequence which is τ-convergent as $t_n \to \infty$.

(e).- There exists a norm bounded sequence (x_n) in $D(A)$ such that $|Ax_n| \to 0$ and (x_n) is τ-convergent.

Proof. $(a) \Longrightarrow (b)$. If $0 \in R(A)$, then there exists $x_0 \in D(A)$ such that $0 \in Ax_0$, and this means that $J_\lambda x_0 = x_0$ for all $\lambda \geq 0$. Hence, by the definition of an accretive operator, it is sufficient to take $\mathcal{U} := \{x_0\}$.

$(b) \Longrightarrow (c)$. Consider $x \in D(A) \cap (X \backslash \mathcal{U})$ and suppose that $t < 0$ and $t(x - x_0) \in Ax$. Then, by hypothesis, it is clear that

$$\langle t(x - x_0), x - x_0\rangle_+ \geq 0,$$

which means that $0 \leq t\|x - x_0\|^2$. But this is a contradiction, since $t < 0$ and $x \neq x_0$. Therefore, if $x \in D(A) \cap (X \backslash \mathcal{U})$ and $t < 0$, then we can conclude that $t(x - x_0) \notin Ax$.

$(c) \Longrightarrow (d)$. Let

$$E := \{x \in D(A) : \exists t < 0 \text{ such that } t(x - x_0) \in Ax\}.$$

It is not difficult to see that $E = \{J_\lambda x_0 : \lambda > 0\}$. By (c), it is clear that $E \subset \mathcal{U}$ and that $\mathcal{U}$ is τ-sequentially compact and norm bounded. Therefore any sequence of the form $(J_{t_n} x_0)$ admits a τ-convergent subsequence $(J_{t_{n_k}} x_0)$.

$(d) \Longrightarrow (e)$. It is sufficient to show that $|AJ_{t_n} x_0| \to 0$.

The sequence $(J_{t_n} x_0)$ is norm bounded. Therefore, if we consider the Yosida approximant

$$A_{t_n} x_0 := \frac{x_0 - J_{t_n} x_0}{t_n},$$

then we see that $\|A_{t_n} x_0\| \to 0$.

Now, by Proposition 2.1 (iii), we know that $A_{t_n} x_0 \in AJ_{t_n} x_0$ and this implies that $|AJ_{t_n} x_0| \to 0$.

$(e) \Longrightarrow (a)$. Since (x_n) is τ-convergent and norm bounded, we may denote

$$y := \tau - \lim_{n\to\infty} x_n$$

and

$$R := \limsup_{n\to\infty} \|x_n - y\| < \infty.$$

Define the following nonempty set:

$$B := \{z \in \overline{D(A)}^{\tau} : \limsup_{n\to\infty} \|x_n - z\| \le R\}.$$

We claim that B is J_1-invariant. Indeed, suppose that $z \in B$. Then

$$\|J_1 z - x_n\| \le \|J_1 z - J_1 x_n\| + \|J_1 x_n - x_n\|.$$

Since J_1 is a nonexpansive mapping, we have

$$\|J_1 z - x_n\| \le \|z - x_n\| + \|A_1 x_n\|.$$

Further, since $|Ax_n| \to 0$, we can apply Proposition 2.1 (iv) to obtain

$$\limsup_{n\to\infty} \|J_1 z - x_n\| \le R.$$

Now we can finish the proof because X satisfies the τ-Opial condition, and hence $B = \{y\}$, which means that $J_1 y = y$, and thus $0 \in Ay$. $\square$

Remark. When we consider in Theorem 3.3 a Banach space satisfying the weak-Opial condition, the hypothesis of such a theorem can be relaxed in the following way: Let $\mathcal{U}$ be weakly compact, and $(J_{t_n} x_0)$ and (x_n) weakly convergent.

4. Zeroes of accretive operators in L^1

Let (Ω, Σ, μ) be a positive σ-finite measure space. The measure μ will always be assumed to be separable. The space $L_1(\Omega)$ is the Banach space of all (equivalence classes of) Σ-measurable functions f for which $\|f\|_1 < \infty$, where

$$\|f\|_1 = \int_\Omega |f| d\mu.$$

Denote by $L_0(\Omega)$ the topological vector space of all (equivalence classes) of Σ-measurable functions with the topology generated by the translation-invariant metric determined by

$$\|f\|_0 := \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{1}{\mu(E_n)} \int_{E_n} \frac{|f|}{1+|f|} d\mu.$$

Here (E_n) is a Σ-partition of Ω into sets with $0 < \mu(E_n) < \infty$ for each n. Whenever $\mu(\Omega) < \infty$, we use the following simpler definition:

$$\|f\|_0 := \int_\Omega \frac{|f|}{1+|f|} d\mu.$$

The $L_0(\Omega)$-topology restricted to $L_1(\Omega)$ will be called the topology of convergence locally in measure (**clm**); or the topology of convergence in measure (**cm**) when $\mu(\Omega) < \infty$.

It is well known that any sequence in L_0 that converges almost everywhere to $f \in L_0$ must converge to f locally in measure. On the other hand, every **clm**-convergent sequence of scalar-valued Σ-measurable functions has a subsequence which converges almost everywhere to the same limit function.

It is clear that $L_1(\Omega)$ endowed with the **clm** topology is a topological vector space and that this topology is weaker than the norm topology. Moreover, it follows from [8] and [6] that $L_1(\Omega)$ satisfies the **clm**-Opial condition. In fact, in [8] the authors show that if (f_n) is a bounded sequence in $L_1(\Omega)$ which **clm**-converges to $f \in L_1(\Omega)$, then for each $g \in L_1(\Omega)$,

$$\limsup_{n\to\infty} \|f_n - g\|_1 = \|f - g\|_1 + \limsup_{n\to\infty} \|f_n - f\|_1.$$

Hence we may say that $L_1(\Omega)$ has property $M(\mathbf{clm})$.

On the other hand, since μ is assumed to be separable, $(L_0, \|.\|_0)$ is a separable F-space and therefore a subset of $L_0(\Omega)$ is **clm**-compact if and only if it is **clm**-sequentially compact.

Corollary 4.1. *Let $A \subset L_1(\Omega) \times L_1(\Omega)$ be an accretive operator satisfying the* **clm**-*range condition. Then the following conditions are equivalent.*

(a).- $0 \in R(A)$.

(b).- There exist $x_0 \in \overline{D(A)}$ and a norm bounded **clm**-*compact subset $\mathcal{U}$ with $x_0 \in \mathcal{U}$ such that for all $x \in D(A) \cap (X \backslash \mathcal{U})$, $\langle y, x - x_0 \rangle_+ \geq 0$ whenever $y \in Ax$.*

(c).- There exist $x_0 \in \overline{D(A)}$ and a norm-bounded **clm**-*compact subset $\mathcal{U}$ with $x_0 \in \mathcal{U}$ such that for all $x \in D(A) \cap (X \backslash \mathcal{U})$, $t(x - x_0) \notin Ax$ for $t < 0$.*

(d).- There exists $x_0 \in \overline{D(A)}$ such that $(J_{t_n} x_0)$ is a norm bounded sequence which is **clm**-*convergent as $t_n \to \infty$.*

(e).- There exists a norm bounded sequence (x_n) in $D(A)$ such that $|Ax_n| \to 0$ and (x_n) is **clm**-*convergent.*

(f).- There exists a $\|.\|_1$-bounded sequence (x_n) in $D(A)$ such that $|Ax_n| \to 0$ and (x_n) converges almost everywhere in Ω.

Proof. By Theorem 3.3 and the above comment about the **clm**-topology, it is sufficient to show the equivalence between (a) and (f).

$(a) \Longrightarrow (f)$. To see this, notice that, by [8] and [6], $L_1(\Omega)$ satisfies the **clm**-Opial condition and so, by Theorem 3.3 (e), there exists a $\|.\|_1$-bounded and **clm**-convergent sequence (y_n) in $D(A)$ with $|Ay_n| \to 0$. Such a sequence has a subsequence that converges almost everywhere in Ω.

$(f) \Longrightarrow (a)$. Suppose that (x_n) converges almost everywhere to $f \in L_0(\Omega)$. Then (x_n) **clm**-converges to f. Thus, by Theorem 3.3, we only have to show that $f \in L_1(\Omega)$.

Since (x_n) converges almost everywhere to f, the sequence $(|x_n|)$ converges almost everywhere to $|f|$. Moreover, since $x_n \in L_1(\Omega)$, clearly $|x_n| \in L_1(\Omega)$ and $\|x_n\|_1 = \||x_n|\|_1$.

Now, using the fact that (x_n) is a $\|.\|_1$-bounded sequence and Fatou's lemma, we obtain that $|f|$ belongs to $L_1(\Omega)$ and so does f. □

Corollary 4.2. *Let $A \subset L_1(\Omega) \times L_1(\Omega)$ be an m-accretive operator such that $\overline{D(A)}^{\mathbf{clm}}$ is locally* **clm**-*compact. Then the following conditions are equivalent.*

(a).- $0 \in R(A)$.

(b).- There exist $x_0 \in D(A)$ and a bounded open neighborhood $\mathcal{U}$ of x_0 such that for each $x \in \partial\mathcal{U} \cap D(A)$, $\langle y, x - x_0 \rangle_+ \geq 0$ for all $y \in Ax$.

(c).- $E := \{x \in D(A) \ : \ tx \in Ax, \ t < 0\}$ is bounded.

5. Asymptotic behavior

The first result we present in this section improves upon the result of Xu [40] mentioned in the Introduction.

Theorem 5.1. *Let X be a reflexive, strictly convex and smooth Banach space. If A is an m-accretive operator in X satisfying the convergence condition, and $\mathcal{T} := \{S(t) : \ t \geq 0\}$ is the nonexpansive semigroup generated by $-A$ via the exponential formula, then every almost orbit of $\mathcal{T}$ is strongly convergent to a zero of A.*

Proof. By hypothesis, A is m-accretive and satisfies the convergence condition. The null point set $A^{-1}0$ is nonempty and closed, and since X is strictly convex, $A^{-1}0$ is also convex. Therefore, since X is both reflexive and strictly convex, the nearest point projection $P : X \to A^{-1}0$ is well-defined and single-valued.

Let $u : [0, \infty[\to X$ be an almost orbit of A and consider the following initial value problem:

$$\begin{cases} w_s'(t) + A(w_s(t)) \ni 0 \\ \\ w_s(0) = u(s). \end{cases} \tag{5.1}$$

We first assume that $u(s) \in D(A)$ for a fixed $s \geq 0$. Then the unique solution of Problem (5.1) is $w_s(t) = S(t)u(s)$. Since X is reflexive, this solution is a strong solution. Therefore the derivative $w_s'(t)$ exists a.e., and moreover satisfies $-w_s'(t) \in Aw_s(t)$ a.e. Thus

$$\langle -w_s'(t), j(t)\rangle = \frac{1}{h}\langle w_s(t-h) - w_s(t), j(t)\rangle + \langle \xi(t,h), j(t)\rangle,$$

where $\lim_{h\to 0} \xi(t,h) = 0$. Note that since X is smooth, the normalized duality map is single valued, and we may denote $j(t) := J(w_s(t) - Pw_s(t))$.

If we now follow step by step the proof of Theorem 1 of Nevanlinna and Reich [31], we arrive at

$$\lim_{t\to\infty} \|w_s(t) - Pw_s(t)\| = 0. \tag{5.2}$$

Now notice that $t \to \|w_s(t) - p\|$ is decreasing for any $p \in A^{-1}0$ and therefore we can write

$$\|w_s(t) - w_s(t+h)\| \leq \|w_s(t) - Pw_s(t)\| + \|Pw_s(t) - w_s(t+h)\| \leq$$

$$\|w_s(t) - Pw_s(t)\| + \|Pw_s(t) - w_s(t)\| = 2\|w_s(t) - Pw_s(t)\|.$$

This means that there exists $z \in X$ such that $\lim_{t\to\infty} w_s(t) = z$. Hence by (5.2) we have that

$$\lim_{t\to\infty} Pw_s(t) = z \in A^{-1}0.$$

Finally, suppose that $u(s) \in \overline{D(A)}$. Then there exists a sequence $(x_n) \subseteq D(A)$ such that $x_n \to u(s)$. If we denote $u_n(t) := S(t)x_n$, then by the above argument we have

$$\lim_{t\to\infty} u_n(t) = z_n \in A^{-1}0.$$

We claim that the sequence (z_n) is convergent to a zero of A.

Indeed, since the sequence (x_n) is convergent, given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for any $n, m \geq n_0$, $\|x_n - x_m\| \leq \frac{\epsilon}{3}$.

Consider $n, m \geq n_0$. Then there exists $t_0 > 0$ such that

$$\|z_m - u_m(t_0)\| \leq \frac{\epsilon}{3} \ \text{ and } \ \|z_n - u_n(t_0)\| \leq \frac{\epsilon}{3}.$$

Therefore

$$\|z_n - z_m\| \leq \|z_n - u_n(t_0)\| + \|u_n(t_0) - u_m(t_0)\| + \|u_m(t_0) - z_m\| <$$
$$< \frac{2\epsilon}{3} + \|x_n - x_m\| < \epsilon.$$

Since $A^{-1}0$ is closed, we conclude that $z_n \to z \in A^{-1}0$, as claimed. Now we assert that $\lim_{t\to\infty} \|w_s(t) - Pw_s(t)\| = 0$. Indeed, given $\epsilon > 0$, we know that there exists $n_1 \in \mathbb{N}$ such that

$$\|u(s) - x_{n_1}\| < \frac{\epsilon}{3} \ \text{ and } \ \|z - z_{n_1}\| < \frac{\epsilon}{3}.$$

Consequently, if t is sufficiently large, then we have

$$\|w_s(t) - Pw_s(t)\| \leq \|w_s(t) - z\| \leq \|w_s(t) - u_{n_1}(t)\| + \|u_{n_1}(t) - z\| \leq$$
$$\leq \|u(s) - x_{n_1}\| + \|u_{n_1}(t) - z_{n_1}\| + \|z_{n_1} - z\| \leq \epsilon.$$

The above argument shows that $\lim_{t\to\infty} \|w_s(t) - Pw_s(t)\| = 0$ for any fixed $s > 0$.

On the other hand, since u is an almost-orbit of $\mathcal{T}$, we know that

$$\|S(t)u(s) - u(s+t)\| \leq \varphi(s) \to 0 \ \text{ as } \ s \to \infty.$$

Hence

$$\|u(t+s) - Pu(t+s)\| \leq \|u(t+s) - Pu(s)\| \leq$$
$$\|u(t+s) - S(t)u(s)\| + \|S(t)u(s) - PS(t)u(s)\| \leq \varphi(s) + \|w_s(t) - Pw_s(t)\|.$$

Therefore

$$\limsup_{t\to\infty} \|u(t+s) - Pu(t+s)\| \leq \varphi(s) \to 0 \ \text{ as } \ s \to \infty. \tag{5.3}$$

Next, we show that $\lim_{t\to\infty}\|u(t)-Pu(t)\|$ exists. Indeed, since u is an almost-orbit of $\mathcal{T}$, letting

$$\varphi(t) := \sup_{s\geq 0}\|u(s+t)-S(s)u(t)\|,$$

we note that $\lim_{t\to\infty}\varphi(t)=0$.

Consequently,

$$\|u(t+s)-Pu(t+s)\| \leq \|u(t+s)-Pu(t)\| \leq$$

$$\|u(t+s)-S(s)u(t)\|+\|S(s)u(t)-Pu(t)\| \leq \varphi(t)+\|u(t)-Pu(t)\|,$$

which implies that $\lim_{t\to\infty}\|u(t)-Pu(t)\|$ exists.

Therefore, by using this fact and (5.3) we obtain that

$$\lim_{t\to\infty}\|u(t)-Pu(t)\|=0.$$

Finally, we have for all $t,s\geq 0$,

$$\|u(t)-u(t+s)\| \leq \|u(t)-Pu(t)\|+\|Pu(t)-S(s)u(t)\|+\|S(s)u(t)-u(t+s)\| \leq$$

$$\leq 2\|u(t)-Pu(t)\|+\varphi(t)\to 0 \;\text{ as }\; t\to\infty.$$

Thus $u(t)$ does indeed converge strongly to a zero of A, as asserted. The proof is complete. $\square$

Regarding the asymptotic behavior of nonexpansive semigroups on a Banach space, we now quote two results from [16].

Theorem 5.2. [16] *Let X be a uniformly convex Banach space such that $X^\star$ has the Kadec-Klee property, and let C be a nonempty, bounded and closed subset of X. If $\mathcal{F}$ is a nonexpansive semigroup acting on C and u is an almost orbit of $\mathcal{F}$, then the following are equivalent:*

(1). $\omega_w(u)\subset Fix\mathcal{F}$.

(2). $w-\lim_{t\to\infty}u(t)=x\in Fix(\mathcal{F})$.

(3). $w-\lim_{t\to\infty}(u(t)-u(t+s))=0$ for each $s\in[0,\infty[$.

Theorem 5.3. [16] *Let X be a reflexive locally uniformly convex Banach space and let C be a closed subset of X. Suppose $\mathcal{S}$ is a nonexpansive semigroup on C with $int(Fix(\mathcal{S}))\neq\emptyset$. Then for each almost orbit u, $u(t)$ converges in norm to a point of $Fix(\mathcal{S})$ as $t\to\infty$.*

Clearly, both theorems do apply when the Banach space X is an L^p-space with $1 < p < \infty$, but they do not apply in the case of L^1. However, using the idea developed in [26], we will now present several results for L^1.

Let C be a **clm**-compact subset of $L_0(\Omega)$ and let $\mathcal{F} = \{S(t) : C \cap L_1(\Omega) \to C \cap L_1(\Omega)\}$ be a nonexpansive semigroup. Given an almost orbit u of $\mathcal{F}$, we denote by $\omega_d(u)$ the set of all **clm**-subsequential limits of $\{u(t)\}_{t\in[0,\infty[}$ as $t \to \infty$.

Theorem 5.4. *Let C be a* **clm**-*compact subset of $L_0(\Omega)$ such that $K := C\cap L_1(\Omega)$ is $\|.\|_1$-bounded. If $\mathcal{F}$ is a nonexpansive semigroup on K, then the following are equivalent:*

(a).- $\omega_d(u) \subset Fix(\mathcal{F})$.

(b).- $\{u(t)\}$ is **clm**-*convergent as $t \to \infty$ to an element of $Fix(\mathcal{F})$.*

(c).- For any $h > 0$, **clm**-$\lim_{t\to\infty}(u(t+h) - u(t)) = 0$.

Proof. $(a) \Longrightarrow (b)$. Let $f \in Fix(\mathcal{F})$, and define $v(t) = f$ for all $t \geq 0$. According to the definition of an almost orbit, it is clear that v is an almost orbit and therefore there exists

$$\lim_{t\to\infty} \|u(t) - f\|_1.$$

Suppose that $f, g \in \omega_d(u) \subset Fix(\mathcal{F})$. We only have to show that $f = g$. By the definition of $\omega_d(u)$, there exist two sequences (t_n) and (s_n) such that

$$\mathbf{clm} - \lim_{n\to\infty} u(t_n) = f \text{ and } \mathbf{clm} - \lim_{n\to\infty} u(s_n) = g.$$

Since we may assume that f and g are almost-orbits of $\mathcal{F}$, we have

$$\lim_{t\to\infty} \|u(t) - f\|_1 = \lim_{n\to\infty} \|u(t_n) - f\|_1 = \lim_{n\to\infty} \|u(s_n) - f\|_1 \tag{5.4}$$

and

$$\lim_{t\to\infty} \|u(t) - g\|_1 = \lim_{n\to\infty} \|u(t_n) - g\|_1 = \lim_{n\to\infty} \|u(s_n) - g\|_1. \tag{5.5}$$

Now, assuming $f \neq g$, and noting that $L_1(\Omega)$ satisfies the **clm**-Opial condition, we see that

$$\lim_{n\to\infty} \|u(t_n) - f\|_1 < \lim_{n\to\infty} \|u(t_n) - g\|_1 \tag{5.6}$$

and

$$\lim_{n\to\infty} \|u(s_n) - g\|_1 < \lim_{n\to\infty} \|u(s_n) - f\|_1, \tag{5.7}$$

and therefore using (5.4), (5.5), (5.6) and (5.7), we conclude that

$$\lim_{t\to\infty} \|u(t) - g\|_1 < \lim_{t\to\infty} \|u(t) - f\|_1 < \lim_{t\to\infty} \|u(t) - g\|_1,$$

which is obviously a contradiction.

$(b) \Longrightarrow (c)$. This is evident.

$(c) \Longrightarrow (a)$. Let $f \in \omega_d(u)$. By definition, there is a sequence of positive numbers (t_n) such that $t_n \to \infty$ as $n \to \infty$ and

$$f = \mathbf{clm} - \lim_{n\to\infty} u(t_n).$$

There exists a subsequence (t_{n_k}) of (t_n) such that

$$u(t_{n_k}) \to f \quad a.e. \;\; in \;\; \Omega$$

and therefore

$$|u(t_{n_k})| \to |f| \quad a.e. \;\; in \;\; \Omega.$$

On the other hand, we know that $u(t_{n_k}) \in L_1(\Omega)$ and K is a $\|.\|_1$-bounded subset of $L_1(\Omega)$. Therefore it is clear by Fatou's lemma that $|f| \in L_1(\Omega)$. Since $f \in L_0(\Omega)$, we conclude that $f \in K$.

Define

$$r_m := \limsup_{n\to\infty} \|u(t_n + mt) - f\|_1.$$

We claim that (r_m) is a decreasing sequence.

Indeed, since, by hypothesis, for each $m \in \mathbb{N}$,

$$f = \mathbf{clm} - \lim_{n\to\infty} u(t_n + mt) \tag{5.8}$$

and $L_1(\Omega)$ satisfies the **clm**-Opial condition, given a fixed $t > 0$, we have

$$r_{m+1} \leq \limsup_{n\to\infty} \|u(t_n + (m+1)t) - S(t)f\|_1.$$

Hence,

$$r_{m+1} \leq \limsup_{n\to\infty} \|u(t_n + (m+1)t) - S(t)(u(t_n + mt))\|_1 + \limsup_{n\to\infty} \|S(t)(u(t_n + mt)) - S(t)f\|_1.$$

Now, using the facts that u is an almost orbit and $\mathcal{F}$ is a nonexpansive semigroup, we deduce that

$$r_{m+1} \leq r_m.$$

Since $\|u(t_n + mt) - f\|_0 \to 0$, there exists a subsequence (t_{n_m}) of (t_n) such that

$$\|u(t_{n_m}+mt)-f\|_0 \le \frac{1}{m}, \qquad \|u(t_{n_m}+mt)-f\|_1 \ge r_m - \frac{1}{m},$$
$$\|u(t_{n_m}+mt)-S(t)f\|_1 \le r_{m-1} + \frac{1}{m}.$$

Therefore,

$$\limsup_{m\to\infty} \|u(t_{n_m}+mt)-f\|_1 \ge \limsup_{m\to\infty} r_m \ge \limsup_{m\to\infty} \|u(t_{n_m}+mt)-S(t)f\|_1. \tag{5.9}$$

On the other hand, from (5.8) and [8] we know that

$$\limsup_{m\to\infty} \|u(t_{n_m}+mt)-S(t)f\|_1 = \limsup_{m\to\infty} \|u(t_{n_m}+mt)-f\|_1 + \|S(t)f-f\|_1. \tag{5.10}$$

Combining (5.9) and (5.10), we conclude that $S(t)f = f$. □

Here is a consequence of Theorem 5.4.

Corollary 5.5. *Let K be a* **clm***-closed subset of $L_1(\Omega)$, and let $\mathcal{F}$ be a nonexpansive semigroup on K. If $u : [0,\infty[\to K$ is a bounded almost orbit of $\mathcal{F}$ such that $u([0,\infty[)$ is relatively* **clm***-compact, then the following are equivalent:*

(a).- $\omega_d(u) \subset Fix(\mathcal{F})$.

(b).- $\{u(t)\}$ is **clm***-convergent as $t\to\infty$ to an element of $Fix(\mathcal{F})$.*

(c).- For any $h>0$, $\mathbf{clm}-\lim_{t\to\infty}(u(t+h)-u(t)) = 0$.

In our next results we assume that (Ω,Σ,μ) is a positive finite measure space with a separable μ. Moreover, we will use the following form of Vitali's convergence theorem.

Theorem 5.6. [19] *Let (Ω,Σ,μ) be a finite measure space and let $1 \le p < \infty$. Let (f_n) be a sequence in $L_p(\Omega)$ and let f be a Σ-measurable function such that f is finite $\mu-a.e.$ and $f_n \to f \quad \mu-a.e.$ Then $f \in L_p(\Omega)$ and $\|f_n - f\|_p \to 0$ if and only if for each $\epsilon > 0$, there is a $\delta > 0$ such that $E \in \Sigma$ and $\mu(E) < \delta$ imply $\int_E |f_n|^p d\mu < \epsilon$ for all $n \in \mathbb{N}$.*

Corollary 5.7. *Let K be a* **cm***-closed subset of $L_1(\Omega)$, and let $\mathcal{F}$ be a nonexpansive semigroup on K. If $u : [0,\infty[\to K$ is an almost orbit of $\mathcal{F}$ such that $u([0,\infty[)$ is relatively* **cm***-compact, there exists $1 < p < \infty$ such that $u(t) \in L_p(\Omega)$ for every $t \ge 0$, and, moreover, $u([0,\infty[)$ is $\|\cdot\|_p$-bounded, then the following are equivalent:*

(a).- Given $h>0$, $\mathbf{cm}-\lim_{t\to\infty}(u(t+h)-u(t)) = 0$.

(b).- $\{u(t)\}$ is $\|.\|_1$-convergent as $t\to\infty$ to $f \in Fix(\mathcal{F})$.

PROOF. $(b) \Longrightarrow (a)$. Evident.

$(a) \Longrightarrow (b)$. By Corollary 5.5, we know that $\{u(t)\}$ is **cm**-convergent as $t \to \infty$ to $f \in Fix(\mathcal{F})$.

Suppose, to get a contradiction, that

$$\lim_{t\to\infty} \|u(t) - f\|_1 \neq 0.$$

Then there exists $\epsilon_0 > 0$ such that for each $k \in \mathbb{N}$ we can find $n_k > n_{k-1}$ satisfying

$$\|u(t_{n_k}) - f\|_1 > \epsilon_0. \tag{5.11}$$

Since $\{u(t)\}$ is **cm**-convergent to f as $t \to \infty$, we know that

$$\lim_{k\to\infty} \|u(t_{n_k}) - f\|_0 = 0.$$

Hence there exists a subsequence $(t_{n_{k_s}})$ of (t_{n_k}) such that

$$u(t_{n_{k_s}}) \to f \quad a.e. \;\; in \;\; \Omega$$

as $s \to \infty$.

On the other hand, since there exists $M > 0$ such that $\|u(t)\|_p \leq M$ for all $t \geq 0$ and $\mu(\Omega) < \infty$, it follows from Hölder's inequality that if $E \in \Sigma$, $\frac{1}{p} + \frac{1}{q} = 1$, and ψ_E is the characteristic function of E, then

$$\int_E |u(t)| d\mu = \int_\Omega |u(t)|\psi_E d\mu \leq \|u(t)\|_p \mu(E)^{\frac{1}{q}}. \tag{5.12}$$

Therefore, given $\epsilon > 0$ and $E \in \Sigma$ such that $\mu(E) < \delta$ with $0 < \delta < (\frac{\epsilon}{M})^q$, we obtain from (5.12) that

$$\int_E |u(t)| d\mu < \epsilon. \tag{5.13}$$

If we now take the sequence $(u(t_{n_{k_s}}))$ and the function f, we may use Theorem 5.6 to conclude that

$$\lim_{s\to\infty} \|u(t_{n_{k_s}}) - f\|_1 = 0. \tag{5.14}$$

But then (5.11) and (5.14) yield a contradiction. $\square$

Let $A \subset L_1(\Omega) \times L_1(\Omega)$ be an accretive operator satisfying the range condition and let $\{S(t)\}$ be the semigroup generated by $-A$ on $\overline{D(A)}$. Given $h > 0$, define the function

$$f \in \overline{D(A)} \to \psi_h(f) := \|f - S(h)f\|_1.$$

It is clear that for each $h > 0$, the function ψ_h is lower semi-continuous and positive.

In the sequel we will use the following result from [13]:

Lemma 5.8 ([13]). *Let $A \subset X \times X$ be an accretive operator satisfying the range condition and let $\{S(t)\}$ be the semigroup generated by $-A$. If for some $t_0 > 0$, $S(t_0)u_0$ with $u_0 \in \overline{D(A)}$ is differentiable, then $S(t_0)u_0 \in D(A)$ and $S(t)u_0$ satisfies the differential equation at that point, i.e.,*

$$\frac{d}{dt}S(t_0)u_0 + AS(t_0)u_0 \ni 0.$$

A consequence of Lemma 5.8 is that if $u_0 \in \overline{D(A)}$ is a fixed point of the semigroup $\{S(t)\}$, then $u_0 \in D(A)$ and, moreover, $0 \in Au_0$, which implies, of course, that $0 \in R(A)$.

Corollary 5.9. *Let $A \subset L_1(\Omega) \times L_1(\Omega)$ be an accretive operator satisfying the range condition and let $\{S(t)\}$ be the semigroup generated by $-A$ on $\overline{D(A)}$. If*

(i).- For every $x \in \overline{D(A)}$, the orbit $\gamma(x) := \{S(t)x : t \geq 0\}$ is relatively **cm**-*compact and norm bounded.*

(ii).- Given $h > 0$, there exists a positive lower semi-continuous function ϕ_h such that

$$\limsup_{\lambda \to 0^+} \frac{1}{\lambda}(\phi_h(J_\lambda x) + \lambda\psi_h(J_\lambda x) - \phi_h(x)) \leq 0$$

holds uniformly on bounded subsets of $D_h := \overline{D(A)} \cap D(\phi_h)$.

Then $S(t)x_0$ **cm**-*converges to a zero of A as $t \to \infty$ whenever $x_0 \in \bigcap_{h>0} D_h$.*

If there exist $p > 1$ and $x_0 \in \bigcap_{h>0} D_h$ such that $(\|S(t)x_0\|_p : t \geq 0)$ is bounded, then $S(t)x_0$ $\|.\|_1$-converges to a zero of A as $t \to \infty$.

Proof. It is easy to see that, for each $h > 0$, ψ_h is a Lyapunov function for A. Therefore, by (ii) and Corollary 3.5 of [35], we have

$$\psi_h(S(t)x) \leq \frac{1}{t}\phi_h(x).$$

This means that if $x_0 \in \bigcap_{h>0} D_h$, then for each $h > 0$, we obtain

$$\lim_{t \to \infty} \|S(t+h)x_0 - S(t)x_0\|_1 = 0. \tag{5.15}$$

Now, by using (i), (5.15), Corollary 5.5 and Lemma 5.8 we see that $S(t)x_0$ **cm**-converges as $t \to \infty$ to an element of $A^{-1}0$.

If there exist $p > 1$ and $x_0 \in \bigcap_{h>0} D_h$ such that $(\|S(t)x_0\|_p : t \geq 0)$ is bounded, then our claim follows from Corollary 5.7. □

6. Applications

Sections 2 and 3 contain several necessary and sufficient conditions for an accretive operator satisfying the range condition to have a zero. All these results hold true when the underlying Banach space X has some kind of a fixed point property for nonexpansive mappings. In this section we present analogous results for a certain class of accretive operators without imposing any requirements on X.

Definition 5. Let X be a Banach space and let $A \subset X \times X$ be an operator. Assume that $rD(A) \subset D(A)$ for $r > 0$. Then A is said to be homogeneous of degree $\alpha \geq 0$ if $A(rx) = r^{\alpha} A(x)$ for all $r > 0$ and $x \in D(A)$.

Proposition 6.1. *Let $A \subset X \times X$ be an accretive operator which satisfies the range condition and is homogeneous of degree $\alpha \geq 0$. Then the following conditions are equivalent:*

(a).- $0 \in R(A)$.

(b).- There exist $x_0 \in D(A)$ and a bounded open neighborhood $\mathcal{U}$ of x_0 such that for each $x \in \partial\mathcal{U} \cap D(A)$, $\langle y, x - x_0 \rangle_+ \geq 0$ for all $y \in Ax$.

(c).- There exists $x_0 \in \overline{D(A)}$ such that $E := \{x \in D(A) \ : \ t(x - x_0) \in Ax, \ t < 0\}$ is bounded.

(d).- $0 \in A(0)$.

Proof. Following step by step the proof of Theorem 3.1 we obtain that $(a) \Rightarrow (b) \Rightarrow (c)$.

$(c) \Rightarrow (d)$. Since A is homogeneous of degree α, it is easy to see that

$$J_{\lambda}(ru) = rJ_{\lambda r^{\alpha-1}}(u) \tag{6.1}$$

for all $u \in D(A)$ and for all $r > 0$.

Therefore (6.1) implies that given $m \in \mathbb{N}$,

$$J_{\lambda}(\frac{1}{m}x_0) = \frac{1}{m}J_{\lambda(\frac{1}{m})^{\alpha-1}}(x_0). \tag{6.2}$$

On the other hand, it is not difficult to see that $E = \{J_{\lambda}(x_0) : \lambda > 0\}$. Hence, by (c), there exists $M > 0$ such that $\|J_{\lambda}(x_0)\| \leq M$ for all $\lambda > 0$.

Now we may use (6.2) and the fact that J_{λ} is a continuous mapping to obtain that

$$\lim_{m\to\infty} J_{\lambda}(\frac{1}{m}x_0) = J_{\lambda}(0) = 0.$$

This means that $0 \in A(0)$.

$(d) \Rightarrow (a)$. This is obvious. □

Remark. Under the hypotheses of Proposition 6.1, if A is an m-accretive (hence closed) homogeneous operator of degree $\alpha > 0$, then we may conclude that $0 \in R(A)$ if and only if $0 \in D(A)$. Also, in the setting of Proposition 6.1, if A is homogeneous of positive degree and satisfies the range condition, then $0 \in R(A)$. To see this, take a positive sequence $\{r_n\}$ which converges to zero and a point z in the domain of A. Then $A(r_n z)$ also converges to zero and so $(0,0)$ belongs to the graph of the closure $\overline{A}$ of A. But the domain of the closure of A is contained in the closure of the domain of A which is contained in the range of $I + A$ by the range condition. Thus there is a point $x \in D(A)$ such that $0 \in x + Ax$. But we already know that the origin is a solution of the inclusion $0 \in y + \overline{A}y$ and that this solution is unique. Therefore the origin coincides with x and this means that $0 \in R(A)$, as claimed.

Theorem 6.2. *Let A be an m-accretive operator in $L^1(\Omega)$. Assume that it is homogeneous of degree $0 \le \alpha \ne 1$ and that $0 \in R(A)$. Then the following assertion holds:*

If $u : [0,\infty[\to \overline{D(A)}$ is an almost-orbit of $\mathcal{F} := \{S(t) : t \ge 0\}$, where $\mathcal{F}$ is the semigroup generated by $-A$ and $u([0,\infty[)$ is relatively **clm***-compact, then $u(t)$* **clm***-converges to a zero of A as $t \to \infty$.*

Proof. We argue as follows.

First, we note that since $0 \in R(A)$, we know that $Fix(\mathcal{F}) \neq \emptyset$ and thus we may apply Lemma 2.5 to conclude that $u([0,\infty[)$ is a bounded set.

Second, we know by (6.1) that for each $x \in \overline{D(A)}$, $J_\mu(rx) = rJ_{\mu r^{\alpha-1}}(x)$. Therefore, for $r = (1 + \frac{2h}{t})^{\frac{1}{\alpha-1}}$, we have

$$S(\frac{t}{2} + h)x = \frac{1}{r} S(\frac{t}{2})(rx). \tag{6.3}$$

Moreover, since it is assumed that $0 \in R(A)$, Proposition 6.1 implies that $0 \in A(0)$ and that $S(t)0 = 0$ for all $t \ge 0$. Consequently, (6.3) yields

$$\|u(t+h) - u(t)\|_1 \le \|S(\frac{t}{2}+h)u(\frac{t}{2}) - S(\frac{t}{2})u(\frac{t}{2})\|_1 + 2\sup_{k\ge 0}\|u(\frac{t}{2}+k) - S(k)u(\frac{t}{2})\|_1 \le$$

$$\|(1-r)S(\frac{t}{2}+h)u(\frac{t}{2}) - S(\frac{t}{2})u(\frac{t}{2}) - S(\frac{t}{2})(ru(\frac{t}{2}))\|_1 + 2\sup_{k\ge 0}\|u(\frac{t}{2}+k) - S(k)u(\frac{t}{2})\|_1 \le$$

$$|1-r|\|S(\frac{t}{2}+h)u(\frac{t}{2}) - S(\frac{t}{2}+h)0\|_1 + \|S(\frac{t}{2})u(\frac{t}{2}) - S(\frac{t}{2})(ru(\frac{t}{2}))\|_1$$

$$+2\sup_{k\ge 0}\|u(\frac{t}{2}+k) - S(k)u(\frac{t}{2})\|_1 \le$$

$$2|1-r|\|u(\frac{t}{2})\|_1 + 2\sup_{k\geq 0}\|u(\frac{t}{2}+k) - S(k)u(\frac{t}{2})\|_1.$$

Since $r = (1+\frac{2h}{t})^{\frac{1}{\alpha-1}}$ and u is a bounded almost-orbit, it follows that

$$\lim_{t\to\infty}\|u(t+h)-u(t)\|_1 = 0. \tag{6.4}$$

Finally, since $u([0,\infty[)$ is assumed to be relatively **clm**-compact, Corollary 5.5 and Lemma 5.8 yield our assertion. □

Ph. Bénilan and M. Crandall have introduced in [5] the concept of a completely accretive operator. This kind of operators, in the particular case of $L^1(\Omega)$ with a bounded $\Omega \subset \mathbb{R}^n$, can be defined in the following way. An operator A in $L^1(\Omega)$ is said to be *completely accretive* if one of the following (equivalent) conditions is satisfied:

1. For $\lambda > 0$, $(u,v),(\hat{u},\hat{v}) \in A$ and $j \in J_0$,

$$\int_\Omega j(u-\hat{u}) \leq \int_\Omega j(u-\hat{u}+\lambda(v-\hat{v}), \tag{6.5}$$

where

$$J_0 = \{j : \mathbb{R} \to [0,\infty] \ : j \ is\ convex\ and\ lowersemi{-}continuous,\ and\ j(0)=0\}.$$

2. For $(u,v),(\hat{u},\hat{v}) \in A$ and $p \in P_0$,

$$\int_\Omega p(u-\hat{u})(v-\hat{v}) \geq 0, \tag{6.6}$$

where

$$P_0 = \{p \in C^\infty(\mathbb{R}) \ : \ 0 \leq p' \leq 1, \quad supp(p') \ \ is \ \ compact, \quad 0 \notin supp(p)\}.$$

In the sequel, we will use two important facts about m-completely accretive operators in $L^1(\Omega)$ which have been presented in [5].

(*). If $\{S(t)\}$ is the semigroup generated by $-A$, then $\{S(t)\}$ is order preserving, that is, if $x,y \in \overline{D(A)}$ and $x \leq y$, then $S(t)x \leq S(t)y$.

(**). For each $t \geq 0$, the inequality $\|S(t)x\|_p \leq \|x\|_p$ holds whenever $S(t)0 = 0$ and $x \in \overline{D(A)} \cap L^p$.

Given $\alpha > 0$, consider the three evolution equations

$$(1)_\alpha \qquad \frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}(|u|^{\alpha-1}u), \qquad t > 0$$

$$(2)_\alpha \qquad \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(|u|^{\alpha-1}u), \qquad t > 0$$

$$(3)_\alpha \qquad \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(|\frac{\partial u}{\partial x}|^{\alpha-1}\frac{\partial u}{\partial x}), \qquad t>0,$$

as well as the equation

$$(4)_\alpha \qquad \frac{\partial u}{\partial t} = div(Du|Du|^{\alpha-1}), \qquad t>0.$$

Associated with each of the above equations is a densely defined m-accretive operator in L^1. Therefore they may be studied as initial value problems of the form (see [4])

$$\frac{du}{dt} + A(u) = 0, \qquad u(0) = x.$$

For example, for equation $(1)_\alpha$ we may consider the operator A_α in L^1 defined by

$$D(A_\alpha) = \{v \in L^1 : |v|^{\alpha-1}v \in L^1_{loc}, \quad (|v|^{\alpha-1}v)'' \in L^1\}$$
$$A_\alpha(v) = -(|v|^{\alpha-1}v)'' \quad \forall v \in D(A_\alpha).$$

This operator is m-accretive and homogeneous of degree α. Moreover, the mild solutions provided by this A_α are uniquely characterized as solution of $(1)_\alpha$ in the sense of distributions. Problems $(2)_\alpha$, $\alpha \neq 1$, correspond to the m-accretive operators

$$D(B_\alpha) := \{v \in L^\infty : \ (|v|^{\alpha-1}v)' \in L^1\}$$
$$B_\alpha(v) = -(|v|^{\alpha-1}v)'$$

in L^1. The operators of the form A_α and B_α are completely accretive (see [5]).

As for equation $(4)_\alpha$ with $\alpha = 0$, we note that the following equation has been studied in [2] in the context of image denoising and reconstruction by using a densely defined m-completely accretive operator which is homogeneous of degree zero. We let Ω be a bounded set in $\mathbb{R}^n$ with a Lipschitz continuous boundary $\partial\Omega$.

(6.7)

$$\frac{\partial u}{\partial t} = div(\frac{Du}{|Du|}) \text{ in } (0,\infty)\times\Omega. \ \ \frac{\partial u}{\partial \eta} = 0 \text{ on } (0,\infty)\times\partial\Omega. \ \ u(0,x) = u_0(x) \text{ in } x \in \Omega$$

Thus the following results can be applied to the above equations.

Corollary 6.3. *Let (Ω, Σ, μ) be a positive finite measure space with a separable measure μ. Let A be an m-completely accretive operator in $L^1(\Omega)$ which is homogeneous of degree $0 \leq \alpha \neq 1$. Assume that $0 \in R(A)$ and consider the semigroup $\mathcal{F} := \{S(t) : L^1(\Omega) \to L^1(\Omega)\}$ generated by $-A$. Then the following assertions hold true:*

(a). If $x \in L^1(\Omega)$ and the orbit of x (that is, $\gamma(x) = \{S(t)x \ : \ t \geq 0\}$) is relatively **cm***-compact, then $S(t)x$* **cm***-converges to a zero of A as $t \to \infty$.*

(b). If, moreover, there exists $p > 1$ such that $x \in L^p(\Omega)$, then $S(t)x$ $\|.\|_1$-converges to a zero of A as $t \to \infty$.

Proof. (a). Since $0 \in R(A)$, this assertion follows from Theorem 6.2.

(b).- This is a consequence of (a), (**) and Corollary 5.7. $\square$

A Lebesgue measurable function $f : [0,1] \to \mathbb{R}$ is said to be decreasing almost everywhere (d.a.e.) if there exists a decreasing function $f : [0,1] \to \mathbb{R}$ such that $f(x) = g(x)$ almost everywhere.

Proposition 6.4. *Let $A \subset L^1([0,1]) \times L^1([0,1])$ be a densely defined m-completely accretive operator which is homogeneous of degree $0 \leq \alpha \neq 1$. Assume that $0 \in R(A)$ and let $\{S(t)\}$ be the semigroup generated by $-A$. If $x \in L^1([0,1])$ is positive and d.a.e., then $S(t)x$ $\|.\|_1$-converges to a zero of A as $t \to \infty$.*

Proof. Since x a positive measurable function which is decreasing almost everywhere, it is clear that $x \in L^p[0,1]$ for every $p \geq 1$, and thus, by Corollary 6.3, it suffices to show that the orbit $\gamma(x)$ is relatively **cm**-compact.

To see this, we argue as follows.

First, we will show that $S(t)x$ is positive a.e. By hypothesis, x is a positive function. Using (*), we see that $S(t)x \geq S(t)0$. Since A is homogeneous and m-accretive, and $0 \in R(A)$, we also know that $0 \in A(0)$. Hence $S(t)0 = 0$ and $S(t)x$ is positive.

Now, we intend to prove that $S(t)x$ is decreasing almost everywhere. Indeed, since x is decreasing a.e., there exists a decreasing function y such that $x = y$ almost everywhere.

Given $s, r \in [0,1],\ s < r$, we define the following functions:

$$z(t) = y(s) \quad \forall t \in [0,1]$$

$$v(t) = y(r) \quad \forall t \in [0,1].$$

It is clear that $v, z \in L^1[0,1]$ and, moreover, $v \leq z$. Applying (*) again, we see that $S(t)v \leq S(t)z$, which implies that

$$S(t)y(r) \leq S(t)y(s). \tag{6.8}$$

Inequality (6.8) means that $S(t)y$ is a decreasing function. Hence we conclude that $S(t)x$ is decreasing a.e., as asserted.

The inequality $\|S(t)x\|_1 \leq \|x\|_1$ holds by (**). Consequently,

$$\gamma(x) \subset \{f \in L^1[0,1] : f \geq 0 \text{ a.e.}, f \ \text{ is } \ \text{d.a.e. and } \|f\|_1 \leq \|x\|_1\}.$$

It is proved in [27] that the above set is **cm**-compact (but not $\|.\|_1$-compact). Thus we have indeed shown that the orbit $\gamma(x)$ is relatively **cm**-compact, as claimed. □

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