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STABILITY AND FIXED POINTS FOR NONEXPANSIVE MAPPINGS

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ABSTRACT. We use a coefficient to give fixed point theorems in the setting of Banach spaces with the weak Opial condition

1. Introduction.

Let X be a Banach space. A self-mapping T of a closed convex subset C of X is said to be a nonexpansive mapping if $\| Tx - Ty \| \leq \| x - y \|$ for all x, y in C.

We will say that X has the weak fixed point property (f.p.p.) if every nonexpansive mapping defined on a nonempty weakly compact convex subset of X has a fixed point.

Since 1965, F. Browder, D. Gohde, W.A. Kirk [10-12], and other authors have established that, under several conditions of geometric type on the norm of X, the f.p.p. can be guaranteed. Uniform convexity and normal structure are examples of such conditions.

Until D.E. Alspach [2] gave an example of a fixed point free nonexpansive mapping on a weakly compact convex subset of $L_1[0,1]$ it remained open whether or not every Banach space posessed the f.p.p. Later B. Maurey, using nonstandard methods [1], [16], proved that every reflexive subspace of $L_1[0,1]$ and also the space of sequences c_0 have the f.p.p.

In order to generalize Maurey's ideas on the f.p.p. for c_0 to a larger class of Banach lattices, J. Borwein and B. Sims [3] introduced the notion

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of a weakly orthogonal Banach lattice. We say a Banach lattice is weakly orthogonal if $\lim_n \||x_n| \wedge |x|\| = 0$ for all x in X, whenever (x_n) is a weakly null sequence. Thus it is shown that a Banach space X has the fixed point property if there exists a weakly orthogonal Banach lattice Y such that $d(X,Y)\alpha(Y) < 2$, where $d(X,Y)$ is the Banach-Mazur distance between X and Y, and $\alpha(Y)$ is the Riesz angle of Y. $(\alpha(Y) := \sup\{\||x| \vee |y|\| : x, y \in B_X\})$.

On the other hand, R. Huff in [11] introduced the following definition: a Banach space X is UKK if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\|x\| \leq 1 - \delta$ whenever x is a weak limit of some sequence (x_n) in B_X with $\text{sep}(x_n) := \inf\{\|x_n - x_m\| : n \neq m\} > \epsilon$. A Banach space X is said to be NUC if it is reflexive and has the UKK property. D. van Dulst and B. Sims [6] have shown a fixed point theorem for a weakening of the UKK property (i.e. a Banach space is called WUKK if it is satisfies the condition from the UKK definition replacing "for every $\epsilon > 0$" by "for some ϵ in (0,1) ". A reflexive Banach space satisfying the WUKK is called WNUC.

In [17] S. Prus gave a characterization of Banach spaces which are dual to NUC. He called these spaces nearly uniformly smooth (NUS). A Banach space X is NUS provided that for every $\epsilon > 0$ there exists $\mu > 0$ such that if $0 < t < \mu$ and (x_n) is a basic sequence in the unit ball of X, then there is $k > 1$ so that $\|x_1 + x_k\| \leq 1 + \epsilon t$. A natural generalization of this notion is (WNUS). A Banach space is WNUS when it satisfies the condition obtained from the above definition by replacing "for every $\epsilon > 0$" by "for some $\epsilon \in (0,1)$".

Kutzarova, Prus and Sims in [14] have proved that if X is a WNUS Banach space with the weak Opial property (a Banach space X has the weak Opial property provided that for every weakly null sequence (x_n) in X and every $x \in X$ $\liminf_n \| x_n \| \leq \liminf_n \| x + x_n \|$) then X has the f.p.p..

In this paper, we define a coefficient $R(X)$ of the Banach space X which allows us to give a more general fixed point theorem than both the Borwein-Sims theorem and Kutzarova-Prus-Sims result. Moreover we characterize the Banach spaces with $R(X) < 2$ and we show that such spaces have the weak Banach-Saks property.

2 A Fixed Point Theorem.

Let X be a Banach space and $[X]$ be the quotient space $l_\infty(X)/c_0(X)$ endowed with the quotient norm given by $\| [z_n] \| := \limsup_n \|z_n\|$, where

$[z_n]$ denotes the equivalence class of $(z_n) \in l_\infty(X)$. When we write $x \in [X](M \in [X])$ it means $x = [(x, x, x, \dots)](M = \{x \in [X] : x \in M\})$. When K is a subset of X, we can consider the set $[K] := \{[z_n] \in [X] : z_n \in K,\ n = 1, 2, \dots\}$. If K is a closed bounded convex subset of X, then $[K]$ is also a closed bounded convex subset of $[X]$.

We suppose that C is a weakly compact convex subset of a Banach space X, and $T : C \to C$ is a nonexpansive mapping. The set C contains a weakly compact convex subset K which is minimal for T. That means $T(K)$ is contained into K and no strictly smaller weakly compact convex subset of K is invariant under T. If K contains only one point then T has a fixed point in K. Otherwise we can assume that diam(K) > 0, it is easy to see that K contains sequences (x_n) with $\lim_n \| Tx_n - x_n \| = 0$(we call such a sequence an approximate fixed point sequence for T). We can define the mapping $[T] : [K] \to [K]$ by $[T][z_n] := [Tz_n]$. Clearly $[T]$ is a nonexpansive self-mapping of $[K]$. Moreover it is known that

Lemma 2.1. *(Lin): Let $[W]$ be any nonempty closed convex subset of $[K]$ which is invariant under $[T]$, then $\sup\{\| [w_n] - x \| : [w_n] \in [W]\} = diam$ (K) for every x in K.[see 1-3-9-15]*

Definition 2.2. Let X be a Banach space, then $R(X) := \sup\{\liminf_n \|x_n + x\|\}$ where the supremum is taken over all weakly null sequences (x_n) in the unit ball and over all points x of the unit ball.

Later we will determine this coefficient for some special Banach spaces.

Theorem 2.3. *A Banach space X has the fixed point property if there exists a Banach space Y with the weak Opial condition such that $d(X,Y)R(Y) < 2$.*

Proof. Let us suppose that X does not have the f.p.p. Then there is a weakly compact convex subset K of X with diam(K) $= 1$, which is minimal for a nonexpansive mapping $T : K \to$K. Thus we can assume that there exists an a.f.p. weakly null sequence (x_n) for T in K.

We will consider the following subset of $[X]$:

$$[W] := \{[z_n] \in [K] : \| [z_n] - [x_n] \| \leq 1/2 \text{ and } \exists x \in K : \| [z_n] - x \| \leq 1/2\}$$

Clearly $[W]$ is a $[T]$-invariant, closed and convex subset of $[K]$ and moreover $[W]$ is nonempty ($[x_n]/2 \in [W]$, since $\| [x_n] - 0 \| \leq 1$). Thus by the Lemma 2.1 we know that

$$\sup\{\| [w_n] - x \| : [w_n] \in [W]\} = 1 \text{ for all } x \text{ in K}.$$

Let $[z_n]$ be an element of $[W]$, $\| [z_n] \| = \limsup_n \| z_n \| = \lim_k \| z_{n_k} \|$, where (z_{n_k}) is some subsequence of (z_n). As K is a weakly compact subset of X we can assume that $\{z_{n_k}\}$ is weakly convergent to $y \in$ K. Let U be a linear isomorphism from X onto Y with $\| U \| . \| U^{-1} \| .R(Y) < 2$. Since U is a linear isomorphism and $\{z_{n_k}\}$ is weakly convergent to y, we have that $\{Uz_{n_k}\}$ is weakly convergent to Uy.

Then, passing to subsequences, we can suppose that, for each $k \in$ N,

$$(1 - 1/k) \| Uz_{n_k} - Uy \| \leq \liminf_k \| Uz_{n_k} - Uy \| .$$

Hence, for all $k \geq 1$

$$(1 - 1/k)(Uz_{n_k} - Uy).(\max\{\liminf_k \| Uz_{n_k} - Uy \|, \| Uy \|\})^{-1} \in B_Y$$

and

$$\text{Uy}.(\max\{\liminf_k \| Uz_{n_k} - Uy \|, \| Uy \|\})^{-1} \in B_Y.$$

Then, by definition of the coefficient $R(Y)$, we derive

$$(1) \qquad \liminf_k \| Uz_{n_k} \| \leq R(Y). \max\{\liminf_k \| Uz_{n_k} - Uy \|, \| Uy \|\}.$$

On the other hand, since $[z_n] \in [W]$ there exists $x_0 \in K$ such that $\| [z_n] - x_0 \| \leq 1/2$ and moreover $\| [z_n] - [x_n] \| \leq 1/2$. Hence

$$(2) \qquad \| y \| \leq \liminf_k \| z_{n_k} - x_{n_k} \| \leq \| [z_n] - [x_n] \| \leq 1/2.$$

Since Y satisfies the weak Opial condition, we have

$$(3) \qquad \liminf_k \| Uz_{n_k} - Uy \| \leq \liminf_k \| Uz_{n_k} - Ux_0 \| .$$

Then, by using (1), (2) and (3)

$$\begin{aligned} \liminf_k \|Uz_{n_k}\| &\leq R(Y). \|U\| . \max\{\liminf_k \|z_{n_k} - x_o\| , \|y\|\} \\ &\leq R(Y). \|U\| /2 \end{aligned}$$

The above conditions imply

$$\begin{aligned} \| [z_n] - 0 \| = \lim_k \| z_{n_k} \| = \lim_k \| U^{-1}Uz_{n_k} \| &\leq \| U^{-1} \| . \lim_k \| Uz_{n_k} \| \\ &\leq \| U^{-1} \| . \| U \| .R(Y)/2 < 1. \end{aligned}$$

This contradicts the Lemma 2.1. □

Corollary 2.4. *Let X be a Banach space with the weak Opial condition such that $R(X) < 2$, then X has the fixed point property.*

3. Relation to Other Fixed Point Theorems.

Proposition 3.1. *Let X be a weakly orthogonal Banach lattice, then $R(X) \leq \alpha(X)$, where $\alpha(X)$ is the Riesz angle.*

Proof. Let (x_n) be a weakly null sequence in the unit ball of X, and let x be an element of the unit ball of X.

In all Banach lattices, it is known that

$$|x_n| + |x| = |x_n| \vee |x| + |x_n| \wedge |x| .$$

Then
$$\| |x_n| + |x| \| \leq \| |x_n| \vee |x| \| + \| |x_n| \wedge |x| \| .$$

Therefore, by the definition of Riesz angle

$$\| |x_n| + |x| \| \leq \alpha(X) + \| |x_n| \wedge |x| \|$$

Since X is a weakly orthogonal Banach lattice, we obtain

$$\liminf_n \| |x_n| + |x| \| \leq \alpha(X), \quad \text{and hence,}$$
$$\liminf_n \|x_n + x\| = \liminf_n \| |x_n + x| \| \leq \liminf_n \| |x_n| + |x| \| \leq \alpha(X).$$

Thus, we can conclude that $R(X) \leq \alpha(X)$. □

The next result is a consequence of the previous proposition and Proposition 4.1 of [3].

Corollary 3.2. *For any set η, we have*

a) $R(c_0(\eta)) = 1$,
b) $R(l_p(\eta)) \leq 2^{1/p}$ *for* $1 \leq p < \infty$, *and*
c) $R(c(\eta)) = 1$.

The following natural generalization of weak orthogonality is investigated in [18].

Definition 3.3. Suppose that whenever (x_n) is a weakly null sequence in a Banach space X we have $\lim_n |\|x_n - x\| - \|x_n + x\|| = 0$ for all x in X. Such a Banach space is said to have the property WORTH.

It is shown in [18] that the weak Opial condition is implied by WORTH, thus we obtain the Borwein-Sims result as a consequence of Proposition 3.1 and Theorem 2.3.

On the other hand, W.L.Bynum [4] has proved that the fixed point property is inherited by Banach spaces whose Banach-Mazur distance from a uniformly convex space is not too large. In particular, he proves that if $d(l_p, X) \leq 2^{1/p}(1 < p < \infty)$ then X has the f.p.p.. He also proved that the space $l_{p,\infty}$(which is l_p renormed by $|x| := \max\{\|x^+\|_p, \|x^-\|_p\}$), does not have asymptotically normal structure but has the f.p.p.. The result of Borwein-Sims does not recapture this for $p \leq 2$; moreover $l_{p,\infty}$ is not a Banach lattice.

Proposition 3.4. *A Banach space X has the fixed point property whenever there exists $1 < p < \infty$ such that $d(X, l_{p,\infty}) < 2^{1/q}$ where $1/p+1/q = 1$.*

Proof. Clearly $l_{p,\infty}$ is a Banach space with a suppression unconditional Schauder basis (e_n), (i.e. the projections have norm one), hence $l_{p,\infty}$ satisfies the weak Opial condition, and then it is sufficient, by Theorem 2.3, to see that $R(l_{p,\infty}) \leq 2^{1/p}$.

Let (x_n) be a weakly null sequence in the unit ball of $l_{p,\infty}$, and let x be also an element of the unit ball.

$$|x_n + x| := \max\{\|(x_n + x)^+\|_p, \|(x_n + x)^-\|_p\}.$$

Since (x_n) is a weakly null sequence there exists (x_{n_k}) subsequence of (x_n) such that

$$\lim_k \|x_{n_k} - P_k x_{n_k}\| = 0 \text{ and } \lim_k \|x_{n_k} - P_k x_{n_k}\| = 0$$

where we denote by P_k the projection $P_{[a_k,b_k]}$, being $[a_k, b_k] := \{s \in \mathbb{N} : a_k \leq s \leq b_k\}$ with $b_k < a_{k+1}$ and $\lim_k a_k = \infty$.

Moreover, we can assume that x has finite support (i.e. $x = P_{[a,b]}x$).

Therefore, there exists $k_0 \in \mathbb{N}$ such that $a_k > b$ for all $k \geq k_0$.

Consequently, for each $k \geq k_0$ we have

$$\|(P_k x_{n_k} + x)^+\|_p = \|P_k x_{n_k}^+ + x^+\|_p \text{ and } \|(P_k x_{n_k} + x)^-\|_p = \|P_k x_{n_k}^- + x^-\|_p$$

and hence the conclusion follows from $\liminf_n |x_n + x| \leq \liminf_k |x_{n_k} + x| \leq \liminf_k |x_{n_k} - P_k x_{n_k}| + \liminf_k |P_k x_{n_k} + x| = \liminf_k |P_k x_{n_k} + x| \leq \max\{\liminf_n \| x_{n_k}^+ + x^+ \|_p, \liminf_n \|x_{n_k}^- + x^-\|_p\} \leq 2^{1/p}$. □

Remark 3.5. In [18], B.Sims proved that all weakly orthogonal Banach lattices have the f.p.p. and he asked whether the f.p.p. holds for Banach spaces with the property WORTH. In [8] the author has shown that spaces with WORTH which are uniformly nonsquare have the f.p.p.. Now, we will see that this result is a particular case of Corollary 2.4.

In an attempt to simplify Schafer's notion of girth and perimeter, Gao and Lau [7] studied the parameter

$$J(X) := \sup\{\|x + y\| \wedge \|x - y\| : x, y \in S(X)\}$$

They showed [7,p. 51] that $J(X) < 2$ if and only if X is uniformly non-square.

Proposition 3.6. *Let X be a Banach space with the property WORTH, then $R(X) \leq J(X)$.*

Proof. Adapting a proof of [5, p. 126] it is not difficult to see that

$$J(X) = \sup\{\|x + y\| \wedge \|x - y\| : x, y \in B_X\}.$$

Let (x_n) be a weakly null sequence in B_X. Since X has the property WORTH, fixing $x \in B_X$ we have that for each $1/k$ there exists $n_k \in \mathbb{N}$ such that $|\|x_n - x\| - \|x_n + x\|| \leq 1/k$ for all $n \geq n_k$. Hence for $k \geq 1$,

$$-1/k + \|x_{n_k} + x\| \leq \|x_{n_k} - x\| \leq \|x_{n_k} + x\| + 1/k.$$

Consequently

$$-1/k + \|x_{n_k} + x\| \leq \|x_{n_k} - x\| \wedge \|x_{n_k} + x\| \leq \|x_{n_k} + x\|.$$

Therefore

$$\liminf_k \|x_{n_k} + x\| \leq \liminf_k (\|x_{n_k} - x\| \wedge \|x_{n_k} + x\|) \leq \liminf_k \| x_{n_k} + x \|$$

Thus, using the coefficient $J(X)$, we obtain

$$\liminf_n \|x_n + x\| \leq \liminf_k \|x_{n_k} + x\| = \liminf_k (\|x_{n_k} + x\| \wedge \|x_{n_k} - x\|) \leq J(X)$$

and consequently $R(X) \leq J(X)$. □

This result allows us to rederive the result that every uniformly non-square Banach space with the WORTH property has the f.p.p..

4. Banach Spaces with R(X) < 2.

Definition 4.1.. A Banach space is said to have the (KK)-property if whenever $x = \lim_n x_n$ and $\|x\| = \lim_n \|x_n\|$ we have $\lim_n \|x_n - x\| = 0$.

See [11] for this. Now, we shall give a characterization of Banach spaces X with $R(X) = 1$ and with the (KK)-property.

Proposition 4.2. *Let X be a Banach space, then the following conditions are equivalent*

(a) X is a Schur space.

(b) X has the (KK)-property and $R(X) = 1$.

Proof. (a)⇒ (b) is evident.

$(b) \Rightarrow (a)$ Otherwise, we can suppose that there exists a weakly null sequence (x_n) in B_X such that $\lim_n \| x_n \| \neq 0$. Hence, there exist $\epsilon > 0$ and a subsequence (x_{n_k}) of (x_n) such that for all $k \in \mathbb{N}$ $\| x_{n_k} \| > \epsilon$. Consider $x \in X$ with $\| x \| = 1$. Since $R(X) = 1$, then $\liminf_k \| x_{n_k} + x \| \leq 1$

On the other hand, since $(x_{k_n} + x)$ is weakly convergent to x and $\| x \| = 1$, we have $1 = \| x \| \leq \liminf_k \| x_{n_k} + x \| \leq 1$. Since X has the (KK)-property, we obtain $\lim_k \| x_{n_k} \| = 0$ which is a contradiction. □

We continue with a characterization of Banach spaces X obeying $R(X) < 2$.

Theorem 4.3. *Let X be a Banach space. The following conditions are equivalent:*

(a) *There exists $\epsilon \in (0,1)$ and $\eta > 0$ such that for all $t \in [0,\eta)$ and for every weakly null sequence (x_n) in B_X there is $k > 1$ with $\| x_1 + tx_k \| \leq 1 + t\epsilon$.*

(b) *There exists $c \in (0,1)$ such that for every weakly null sequence (x_n) in B_X there is $k > 1$ with $\| x_1 + x_k \| \leq 2 - c$.*

(c) *$R(X) < 2$.*

Proof. $(b) \Rightarrow (c)$ Let (x_n) be a weakly null sequence in the unit ball of X and let x be an element of B_X. We consider the sequence (y_n), where $y_1 := x$ and $y_{n+1} := x_n n = 1, 2, \ldots$. Then (y_n) is a weakly null sequence in the unit ball of X.

Hence, since X satisfies (b), we obtain that there exists $k_1 > 1$ such that $\| x + x_{k_1} \| \leq 2 - c$.

Now, we take the sequence, $z_1 := x$ and $z_n := x_{k_1+n}n = 1, 2, \ldots$. It is easy to see that (z_n) is a weakly null sequence in the unit ball of X, and then there exists $k_2 > k_1$ such that $\| x + x_{k_2} \| \leq 2 - c$.

Thus by a recurrence argument, we obtain $\liminf_n \| x + x_n \| \leq 2 - c$ which means $R(X) < 2$.

(c)$\Rightarrow$ (b) Suppose that (x_n) is a weakly null sequence of B_X.

Since $R(X) < 2$ then there is $c \in (0, 1)$ such that $R(X) < 2 - c$.

Therefore $\liminf_n \| x_1 + x_n \| \leq R(X) < 2 - c$, and so, we have that there exists $k > 1$ satisfying $\| x_1 + x_k \| \leq 2 - c$.

$(a) \Rightarrow (b)$ There exist $\epsilon > 0$ and $\eta > 0$ such that for all $t \in]0, \eta[$ and for every weakly null sequence (x_n) in the unit ball of X there is $k > 1$ with $\| x_1 + x_k \| \leq 1 + t\epsilon$. Consider $\delta := \min\{1, \eta\}$, then if $t < \delta$, we have

$$\| x_1 + x_k \| \leq \| x_1 + tx_k \| + (1 - t) \| x_k \| \leq 1 + \epsilon t + 1 - t = 2 - t(1 - \epsilon).$$

$(b) \Rightarrow (a)$ Since there exists $c \in]0, 1[$ such that for every weakly null sequence (x_n) in the unit ball of X, there is $k > 1$ with $\| x_1 + x_k \| \leq 2 - c$, then, for all $t \in]0, 1[$

$$\| x_1 + tx_k \| \leq t \| x_1 + x_k \| + (1 - t) \| x_1 \| \leq 1 + t(1 - c). \quad \square$$

The following results are a consequence of Theorem 4.3 and of the definition of WNUS.

Corollary 4.4. *Let X be a Banach space. The following conditions are equivalent (a) X is WNUS*

(b) X is reflexive and $R(X) < 2$.

In [17], S.Prus gave a characterization of Banach spaces which are dual to NUC. As a consequence of Corollary 4.4 we will show that whenever X is a WNUC Banach space then X^* is a WNUS Banach space.

Definition 4.5. A Banach space X is said to have the property WUKK' if there exist $\epsilon \in (0, 1)$ and $\delta > 0$ such that $\| x \| \leq 1 - \delta$ *whenever* x is a weak limit of some sequence (x_n) in B_X with $\liminf_n \| x_n - x \| \geq \epsilon$.

See [13]. It is not difficult to prove that a Banach space X is WUKK' whenever X is WUKK.

Proposition 4.6. *If X is a reflexive WUKK'-space, then X^* is WNUS-space*

Proof. By Corollary 4.4 it is sufficient to see that $R(X^*) < 2$. Consider $x^* \in B_{X^*}$ and (x_n^*), *a* weakly null sequence in the unit ball of X^*.

Since X is reflexive, we can choose elements y_n in the unit ball of X such that for every $n \in \mathbb{N}$, $(x_n^* + x^*)(y_n) = \|x_n^* + x^*\|$. Moreover, passing to subsequences if it is necessary, we can suppose that (y_n) is weakly convergent to $y \in B_X$.

Hence given $\eta > 0$ such that $\epsilon + \eta < 1$ there exists $k_0 \in \mathbb{N}$ such that for $n \geq k_0$, $|x_n^*(y)| < \eta/2$, and $|x^*(y_n - y)| < \eta/2$.

Therefore, for every $n \geq k_0$, we have

$$\begin{aligned} \|x_n^* + x^*\| &= x^*(y_n) + x_n^*(y_n) \\ &= x^*(y) + x_n^*(y_n - y) + x^*(y_n - y) + x_n^*(y) \\ &\leq \|y\| + \|y_n - y\| + \eta \qquad (1) \end{aligned}$$

Let us consider two cases.

(a) There exists a strictly increasing sequence (n_k) of positive integers with $n_1 > k_0$ so that $\|y_{n_k} - y\| < \epsilon$. Then by (1), $\left\|x_{n_k}^* + x^*\right\| \leq 1 + \epsilon + \eta$. Consequently

$$\liminf_n \|x_n^* + x^*\| \leq 1 + \epsilon + \eta. \qquad (2)$$

(b) There exists $k_1 > k_0$ so that, for every $n \geq k_1$, $\|y_n - y\| > \epsilon$. Therefore $\liminf_n \|y_n - y\| \geq \epsilon$ and then

$$\|x^* + x_n^*\| \leq |x^*(y_n)| + |x_n^*(y_n)| \leq |x^*(y_n)| + 1$$

since X is WUKK' and (y_n) is weakly convergent to y we have

$$\liminf_n \|x^* + x_n^*\| \leq |x^*(y)| + 1 \leq \|y\| + 1 \leq 2 - \delta \qquad (3)$$

and so, by (2) and (3)

$$R(X^*) \leq \max\{1 + \epsilon + \eta, 2 - \delta\} < 2. \quad \square$$

Remark 4.7.. Adapting a theorem of [17,p. 513] it is not difficult to see that a Banach space X which has $R(X) < 2$ enjoys the weak-Banach-Saks property. On the other hand, using Corollary 4.4 and Corollary 2.4 we obtain that if X is a WNUS Banach space and has the weak Opial condition then X has the f.p.p.

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