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Schaefer–Krasnoselskii fixed point theorems using a usual measure of weak noncompactness

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ABSTRACT

We present some extension of a well-known fixed point theorem due to Burton and Kirk [T.A. Burton, C. Kirk, A fixed point theorem of Krasnoselskii–Schaefer type, Math. Nachr. 189 (1998) 423–431] for the sum of two nonlinear operators one of them compact and the other one a strict contraction. The novelty of our results is that the involved operators need not to be weakly continuous. Finally, an example is given to illustrate our results.

1. Introduction

Many nonlinear problems arising from the most areas of natural sciences can be modeled under the mathematical point of view and they involve the study of solutions of nonlinear equations of the form

$$Ax + Bx = x, \quad x \in K, \tag{1}$$

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where K is a closed convex subset of a Banach space X (see [17]). In 1958, Krasnoselskii [16] established one of the first results in this direction: the sum of two mappings $A+B$ has a fixed point in a nonempty closed convex subset C of a real Banach space $(X, \|\cdot\|)$ whenever A and B satisfy

(i) $A(C)+B(C) \subseteq C$,
(ii) A is continuous on C and $A(C)$ is a relatively compact subset of X,
(iii) B is a strict contraction on C, i.e., there exists $k \in (0,1)$ such that $\|B(x)-B(y)\| \leqslant k\|x-y\|$ for every $x, y \in C$.

Notice that the proof of Kransnoselskii's fixed point theorem combines the Banach contraction principle and Schauder's fixed point theorem (see [17,22]). There is a vast literature dealing with the improvement of such a result, we quote for example the papers [1,4,5,13,14,18,19,21,23] (also see the reference therein) and the list is still incomplete. For example in [9] Burton and Kirk proved the following generalization:

Theorem 1.1. *Let X be a Banach space, $A, B : X \to X$ are continuous mappings satisfying*:

(BK1) *A maps bounded subsets into compact sets,*
(BK2) *B is a strict contraction.*

Then either $A+B$ has a fixed point or the set $\{x \in X\colon\ x = \lambda B(\frac{x}{\lambda}) + \lambda A(x)\}$ is unbounded for each $\lambda \in (0,1)$.

The proof of Theorem 1.1 uses the Banach contraction principle and, in contrast to Krasnoselskii's fixed point theorem, requires Schaefer's theorem [22]. Recently, several papers give generalizations of both Krasnoselskii's theorem and Burton and Kirk's theorem using the weak topology (see [4,5,13,14,18,19,23]). The extension obtained in [13,18,19] rely on the concept of measure of weak noncompactness and, contrarily to those given in [4,5,23], the weak continuity of the operator A is not required.

The main goal of the present paper is establish new variants of Theorem 1.1 in the spirit of the works [11,13,18,19]. In particular, we prove that if A is continuous, weakly compact and it maps relatively weakly compact sets into relatively compact ones and B is an ω-condensing nonexpansive mapping, then the conclusion of Theorem 1.1 holds true. Evidently, our results do not require the weak continuity of the operator A. To justify our results we study the existence of solutions of a nonlinear integral equation in the context of L^1-spaces.

2. Preliminaries

Throughout this paper we suppose that $(X, \|\cdot\|)$ is a real Banach space. For any $r>0$, B_r denotes the closed ball in X centered in 0_X and with radius r. Here $\rightharpoonup$ denotes weak convergence and $\to$ denotes strong convergence in X, respectively.

$\mathcal{B}(X)$ means the collection of all nonempty bounded subsets of X, $\mathcal{W}(X)$ is the subset of $\mathcal{B}(X)$ consisting of all weakly compact subsets of X. Recall that the notion of the measure of weak noncompactness was introduced by De Blasi [10] and it is the map $w : \mathcal{B}(X) \to [0,\infty[$ defined by

$$\omega(M) := \inf\big\{r>0\colon \text{ there exists } W \in \mathcal{W}(X) \text{ with } M \subseteq W + B_r\big\},$$

for every $M \in \mathcal{B}(X)$. Now, we are going to recall some basic properties of $\omega(\cdot)$ needed later.

Let M_1, M_2 be two elements of $\mathcal{B}(X)$. The following properties hold (for instance see [2,10]):

1. If $M_1 \subseteq M_2$, then $\omega(M_1) \leqslant \omega(M_2)$,
2. $\omega(M_1) = 0$ if and only if, $\overline{M_1}^w \in \mathcal{W}(X)$ ($\overline{M_1}^w$ means the weak closure of M_1),
3. $\omega(\overline{M_1}^w) = \omega(M_1)$,
4. $\omega(M_1 \cup M_2) = \max\{\omega(M_1), \omega(M_2)\}$,

5. $\omega(\lambda M_1) = |\lambda|\omega(M_1)$ for all $\lambda \in \mathbb{R}$,
6. $\omega(co(M_1)) = \omega(M_1)$,
7. $\omega(M_1 + M_2) \leqslant \omega(M_1) + \omega(M_2)$.

Apell and De Pascale in [2] proved that in L^1-spaces the map $\omega(\cdot)$ can be expressed as

$$\omega(M) = \limsup_{\epsilon \to 0} \left\{ \sup_{\psi \in M} \left[\int_D \|\psi(t)\|_X \, dt \colon |D| \leqslant \epsilon \right] \right\}, \tag{2}$$

for every bounded subset M of $L^1(\Omega; X)$ where X is a finite dimensional Banach space and $|D|$ is the Lebesgue measure of the set D.

A mapping $T : C \subseteq X \to X$ is said to be ω-condensing if T is continuous and $\omega(T(A)) < \omega(A)$ for every bounded set $A \subseteq C$ with $\omega(A) > 0$.

On the other hand, T is said to be a ϕ-contraction if there exists a continuous nondecreasing function $\phi : [0, +\infty) \to [0, +\infty)$ such that $\phi(0) = 0$, $\phi(r) < r$ for any $r > 0$ and $\|Tx - Ty\| \leqslant \phi(\|x - y\|)$.

Let X be Banach space and $T : D(T) \subseteq X \to X$ a mapping. In what follows, we will use the following conditions:

($\mathcal{A}1$) If $(x_n)_{n\in\mathbb{N}} \subseteq D(T)$ is a weakly convergent sequence in X, then $(Tx_n)_{n\in\mathbb{N}}$ has a strongly convergent subsequence in X.
($\mathcal{A}2$) If $(x_n)_{n\in\mathbb{N}} \subseteq D(T)$ is a weakly convergent sequence in X, then $(Tx_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence in X.

The conditions ($\mathcal{A}1$) and ($\mathcal{A}2$) were already considered in the papers [13,14,18,19].

Remark 2.1.

1. Let us first observe that the hypothesis ($\mathcal{A}1$) does not imply the compactness of T even if T is a linear mapping. It is well known that a compact linear mapping from a Banach space X into a Banach space Y maps weakly convergent sequences onto norm convergent ones. The converse is true if X is reflexive. If X is not reflexive, the converse of the preceding assertion need not be true even when Y is reflexive. To see this, let T be the identity map injecting l_1 into l_2. It is clear that T is not compact. However, if (x_n) is a sequence in l_1 which converges weakly to x, then, by Corollary 14 in [12], (x_n) converges to x in norm in l_1. Using the continuity of T one sees that $(T(x_n))$ converges strongly to Tx in l_2.
2. The condition ($\mathcal{A}1$) holds also true for the class of weakly compact operators acting on Banach spaces with the Dunford–Pettis property. (A Banach space X has the Dunford–Pettis property if every weakly compact linear operator defined on X takes weakly compact sets into norm compact ones.) Indeed, if X is a Banach space with the Dunford–Pettis property, then every weakly compact linear operator from X into an arbitrary Banach space Y maps weakly convergent sequences in X onto norm convergent sequences in Y.
3. Operators satisfying ($\mathcal{A}1$) or ($\mathcal{A}2$) are not necessarily weakly continuous (see Remark 4.1 below).
4. Condition ($\mathcal{A}2$) holds for every bounded linear operator.

3. Fixed point theorems

Our first purpose here is to establish a sharpening of Lemma 2.2 in [11] giving a relationship between ϕ-contraction mappings satisfying condition ($\mathcal{A}2$) and ω-condensing mappings.

Lemma 3.1. *If $T : X \to X$ is a ϕ-contraction satisfying ($\mathcal{A}2$), then T is ω-condensing.*

Proof. Since T is a ϕ-contraction mapping it is clear that T is continuous. Thus, we only have to prove that if $S \in \mathcal{B}(X)$ such that $\omega(S) > 0$, then $\omega(T(S)) < \omega(S)$. To this end, let us consider $S \in \mathcal{B}(X)$ such that

$$\omega(S) = \inf\bigl\{r > 0\colon\ S \subseteq W + B_r(0),\ W \in \mathcal{W}(X)\bigr\} > 0,$$

taking $\varepsilon > 0$ there exist $r_\varepsilon = \omega(S) + \varepsilon$ and $W_0 \in \mathcal{W}(X)$ such that $S \subseteq W_0 + B_{r_\varepsilon}(0)$. Let $y \in T(W_0 + B_{r_\varepsilon}(0))$, then there exists $x \in W_0 + B_{r_\varepsilon}(0)$ such that $y = Tx$.

Since $x \in W_0 + B_{r_\varepsilon}(0)$, there are $w \in W_0$ and $b \in B_{r_\varepsilon}(0)$ such that $x = w + b$. Hence,

$$\|y - Tw\| = \|Tx - Tw\| \leqslant \phi\bigl(\|x - w\|\bigr) = \phi\bigl(\|b\|\bigr) \leqslant \sup_{0 \leqslant t \leqslant r_\varepsilon} \phi(t),$$

since ϕ is a continuous function, there exists $t_\varepsilon \in [0, r_\varepsilon]$ such that

$$\|y - Tw\| \leqslant \sup_{0 \leqslant t \leqslant r_\varepsilon} \phi(t) = \phi(t_\varepsilon),$$

that is, $y \in T(W_0) + B_{\phi(t_\varepsilon)}(0)$ and consequently $T(W_0 + B_{r_\varepsilon}(0)) \subseteq T(W_0) + B_{\phi(t_\varepsilon)}(0)$.

Since $S \subseteq W_0 + B_{r_\varepsilon}(0)$,

$$T(S) \subseteq T\bigl(W_0 + B_{r_\varepsilon}(0)\bigr) \subseteq T(W_0) + B_{\phi(t_\varepsilon)}(0)$$

and since T satisfies condition $(\mathcal{A}2)$ we have that $\overline{T(W_0)}^w \in \mathcal{W}(X)$,

$$\omega\bigl(T(S)\bigr) = \inf\bigl\{r > 0\colon\ T(S) \subseteq W + B_r(0),\ W \in \mathcal{W}(X)\bigr\} \leqslant \phi(t_\varepsilon).$$

Now, we argue as follows:

If there exists $\varepsilon > 0$ such that $\phi(t_\varepsilon) < \omega(S)$ we have arrived to the conclusion.

Otherwise, for every $\varepsilon > 0$ we have $\phi(t_\varepsilon) \geqslant \omega(S)$. In this case, the properties of ϕ yield

$$\omega(S) \leqslant \phi(t_\varepsilon) < t_\varepsilon \leqslant \omega(S) + \varepsilon.$$

Consequently, $\phi(t_\varepsilon)$, $t_\varepsilon \to \omega(S)$ as $\varepsilon \to 0$. Bearing in mind that ϕ is a continuous function,

$$\omega(S) = \lim_{\varepsilon \to 0} \phi(t_\varepsilon) = \phi\bigl(\omega(S)\bigr).$$

Hence, $\omega(S) = 0$, which is a contradiction. □

Remark 3.1. It should be noticed that the proof of Lemma 3.1 works without assuming that ϕ is nondecreasing. Therefore from now on, we say that $T : X \to X$ is a ϕ-contraction mapping if there exists a continuous function $\phi : [0, +\infty) \to [0, +\infty)$ such that $\phi(0) = 0$, $\phi(r) < r$ for any $r > 0$ and $\|Tx - Ty\| \leqslant \phi(\|x - y\|)$.

Now, we face the concept of separate contraction mapping which was introduced in [20].

Definition 3.1. Let X be a Banach space and $f : X \to X$ is said to be a separate contraction mapping if there exist two functions $\phi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying:

(a) $\psi(0) = 0$, ψ is strictly increasing,

(b) ϕ is continuous,
(c) $\|f(x) - f(y)\| \leqslant \phi(\|x - y\|)$,
(d) $\psi(r) \leqslant r - \phi(r)$ for $r > 0$.

By the above remark, we may consider that the class of separate contraction mappings is a subclass of the class of ϕ-contraction mappings. Moreover, it is easy to see that every strict contraction is in fact a separate contraction. In [8, Example 2] there is an example of a separate contraction mapping which is not a strict contraction. Anyway, if we consider the mapping $T : [0, \frac{1}{\sqrt{2}}] \to [0, \frac{1}{\sqrt{2}}]$ defined by $T(x) = x - x^3$, it is not difficult to prove that T is not a strict contraction but it is a separate contraction, taking

$$\phi(r) = \begin{cases} r(1 - \frac{r^2}{4}), & r \leqslant 1, \\ \frac{3}{4}r, & r \geqslant 1, \end{cases}$$

and $\psi(r) = r - \phi(r)$.

Now, we are going to introduce a concept of mapping which will be essential for our arguments (see [13,15]).

Definition 3.2. A mapping $T : D(T) \subseteq X \to X$ is said to be Φ-expansive if there exists a function $\Phi : [0, +\infty) \to [0, +\infty)$ satisfying

1. $\Phi(0) = 0$,
2. $\Phi(r) > 0$ for $r > 0$,
3. Φ is either continuous or nondecreasing,

such that for every $x, y \in D(T)$ the inequality $\|T(x) - T(y)\| \geqslant \Phi(\|x - y\|)$ holds.

Remark 3.2. It is clear that if $T : D(T) \subseteq X \to X$ is a ϕ-contraction, then $B := I - T : D(T) \subseteq X \to X$ is a Φ-expansive mapping, where $\Phi(r) = r - \phi(r)$. Indeed,

$$\big\|B(x) - B(y)\big\| \geqslant \|x - y\| - \big\|T(x) - T(y)\big\| \geqslant \|x - y\| - \phi\big(\|x - y\|\big) = \Phi\big(\|x - y\|\big).$$

When T is a separate contraction, it is not difficult to see that $I - T$ is ψ-expansive with ψ strictly increasing.

On the other hand, it is easy to see that the mapping $B : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $B(x, y) = (y, -x)$ is a nonexpansive mapping which is not ϕ-contraction for any ϕ but nevertheless $T = I - B$ is Φ-expansive where $\Phi(t) = \sqrt{2}t$. (This argument can be found in [13].)

3.1. Bounded domains

In the next results we will use the following well-known theorem:

Theorem 3.1. *(See [19, Theorem 2.1].) Let M be a nonempty closed convex subset of a Banach space X. Assume that $T : M \to M$ is a continuous map satisfying condition $(\mathcal{A}1)$. If $T(M)$ is relatively weakly compact, then there exists $x \in M$ such that $T(x) = x$.*

Theorem 3.2. *Let X be a Banach space. Let M be a nonempty closed convex and bounded subset of X and let $A, B : M \to X$ be two continuous mappings. If A, B satisfy the following conditions,*

(i) *$A(M)$ is relatively weakly compact,*
(ii) *A satisfies $(\mathcal{A}1)$,*

(iii) *B is nonexpansive and ω-condensing,*
(iv) *$I - B$ is ψ-expansive,*
(v) *$A(M) + B(M) \subseteq M$.*

Then, the equation $x = A(x) + B(x)$ has a solution.

Proof. It is easily checked that $x \in M$ is a solution for the equation $x = B(x) + A(x)$ if and only if x is a fixed point for the operator $(I - B)^{-1} \circ A$, whenever it is well defined. In order to prove the latter we have to check:

1. $(I - B)$ has an inverse over $R(I - B) := (I - B)(M)$.

This is equivalent to see that $I - B$ is injective. Consider $x, y \in M$, $x \neq y$. Since $I - B$ is ψ-expansive,

$$\left\|(I - B)x - (I - B)(y)\right\| \geqslant \psi\left(\|x - y\|\right) > 0,$$

and hence different points apply into different images.

2. The domain of $(I - B)^{-1}$ contains the range of A.

Take $y \in M$ and consider $A(y)$. We have to check if there exists some $x \in M$ such that $(I - B)(x) = A(y)$, which is equivalent to proving if $x = Bx + Ay$. If we define $T : M \to M$ such that for any $x \in M$, $Tx = Bx + Ay$, let us prove that such mapping has a fixed point. From the nonexpansiveness of B, T is also nonexpansive. Since $I - B$ is ψ-expansive

$$\begin{aligned}\left\|(I - T)(x) - (I - T)(z)\right\| &= \|x - z + Tz - Tx\| \\ &= \|x - z + Bz - Bx\| \\ &= \left\|(I - B)(x) - (I - B)(z)\right\| \\ &\geqslant \psi\left(\|x - z\|\right).\end{aligned}$$

Hence $I - T$ is ψ-expansive and, by [13, Proposition 3.4], we infer that T has a unique fixed point $x \in M$, that is, $A(y) = (I - B)(x)$ and so $R(A) \subseteq R(I - B) = D((I - B)^{-1})$.

Consequently $(I - B)^{-1} \circ A : M \to M$ is well defined. Let us prove now that this operator is under the conditions of [19, Theorem 2.1], that is, $(I - B)^{-1} \circ A$ is continuous weakly compact and satisfies $(\mathcal{A}1)$.

(I) $(I - B)^{-1} \circ A$ is continuous.

Consider a sequence (x_n) in $R(I - B)$ converging to some $x_0 \in R(I - B)$. Let $y_n = (I - B)^{-1}(x_n)$ and $y_0 = (I - B)^{-1}(x_0)$. Hence $(I - B)y_n = x_n$ and $(I - B)y_0 = x_0$. Since $I - B$ is ψ-expansive

$$\psi\left(\|y_n - y_0\|\right) \leqslant \left\|(I - B)(y_n) - (I - B)(y_0)\right\| = \|x_n - x_0\|.$$

Consequently

$$\lim_{n\to\infty} \psi\left(\|y_n - y_0\|\right) \leqslant \lim_{n\to\infty} \|x_n - x_0\| = 0. \tag{3}$$

If we assume that $(\|y_n - y_0\|)$ is a non-null sequence, then there exists $(\|y_{n_s} - y_0\|)$ subsequence of $(\|y_n - y_0\|)$ such that $\|y_{n_s} - y_0\| \to r > 0$. Now, if ψ is a continuous function, we obtain that

$$\lim_{s\to\infty} \psi\left(\|y_{n_s} - y_0\|\right) = \psi(r) > 0.$$

Otherwise, ψ will be nondecreasing and then

$$0 < \psi\left(\frac{r}{2}\right) \leqslant \lim_{s\to\infty} \psi\big(\|y_{n_s} - y_0\|\big).$$

In both cases, by (3) we have that $\lim_{s\to\infty} \psi(\|y_{n_s} - y_0\|) = 0$, which means that $\|y_n - y_0\| \to 0$, that is

$$\big\|(I-B)^{-1}(x_n) - (I-B)^{-1}(x_0)\big\| \to 0,$$

and $(I-B)^{-1}$ is continuous. Since A is also continuous by hypothesis, $(I-B)^{-1} \circ A$ is continuous.

(II) $(I-B)^{-1} \circ A = A + B \circ (I-B)^{-1} \circ A$.

Let $y \in M$ and let $z = (I-B)^{-1} \circ A(y)$. Hence, $A(y) = z - B(z)$ and

$$\begin{aligned}(I-B)^{-1} \circ A(y) &= z = z - B(z) + B(z) = A(y) + B(z)\\ &= A(y) + B\big((I-B)^{-1} \circ A(y)\big)\\ &= \big(A + B \circ (I-B)^{-1} \circ A\big)(y).\end{aligned}$$

Hence $(I-B)^{-1} \circ A = A + B \circ (I-B)^{-1} \circ A$.

(III) $(I-B)^{-1} \circ A(M)$ is relatively weakly compact.

Since $R((I-B)^{-1} \circ A) \subseteq M$, then $((I-B)^{-1} \circ A)(M)$ is a bounded subset. Suppose that such subset is not relatively weakly compact. By using the properties of $\omega(\cdot)$ and by assumptions (i) and (iii) we obtain

$$\begin{aligned}\omega\big((I-B)^{-1} \circ A(M)\big) &= \omega\big(\big(A + B \circ (I-B)^{-1} \circ A\big)(M)\big)\\ &\leqslant \omega\big(A(M)\big) + \omega\big(B \circ (I-B)^{-1} \circ A(M)\big)\\ &= \omega\big(B \circ (I-B)^{-1} \circ A(M)\big)\\ &< \omega\big((I-B)^{-1} \circ A(M)\big),\end{aligned}$$

which is a contradiction. So $(I-B)^{-1} \circ A(M)$ is relatively weakly compact.

(IV) $(I-B)^{-1} \circ A$ satisfies condition $(\mathcal{A}1)$.

Since A satisfies $(\mathcal{A}1)$ if (x_n) is weakly convergent in X, then $(A(x_n))$ has a strongly convergent subsequence. By the continuity of $(I-B)^{-1}$, $((I-B)^{-1} \circ A(x_n))$ has also a strongly convergent subsequence, that is, $(I-B)^{-1} \circ A$ satisfies condition $(\mathcal{A}1)$.

Consequently, $(I-B)^{-1} \circ A$ satisfies the hypothesis of [19, Theorem 2.1] as we claimed, and hence such operator has a fixed point. □

Next result is based on [21, Theorem 2.1]. In order to present such a result we need to recall the concept of demiclosedness. A mapping $T : \Omega \subseteq X \to X$ is said to be demiclosed at x if given a sequence (x_n) in Ω weakly convergent to $x_0 \in \Omega$ such that the sequence $(T(x_n))$ is strongly convergent to x, then $T(x_0) = x$. When a mapping is demiclosed at every point we say that it is demiclosed.

Theorem 3.3. *Let X be a Banach space and M a closed convex and bounded subset of X such that $0 \in M$. Let $A : M \to M$ and $B : X \to X$ be two continuous mappings satisfying the following conditions*

(i) *A maps bounded subsets into relatively weakly compact subsets,*
(ii) *A satisfies $(\mathcal{A}1)$,*
(iii) *B is nonexpansive and ω-condensing,*
(iv) *$A(M) + B(M) \subseteq M$,*
(v) *$I - (A + B) : M \to X$ is demiclosed at 0.*

Then, $A + B$ has a fixed point in M.

Remark 3.3. It is interesting to notice that condition (i) in Theorem 3.3 does not necessarily yield that A becomes weakly completely continuous. To see this, consider the classical Banach space $(l_2, \|\cdot\|_2)$ and define $A : l_2 \to l_2$ by

$$A(x) = \begin{cases} \frac{x}{\|x\|_2}, & \|x\|_2 \geqslant 1, \\ x, & \|x\|_2 \leqslant 1. \end{cases}$$

It is clear that A is continuous, A maps bounded set into relatively weakly compact sets ($A(X) \subseteq B_{l_2}$). However, A fails to be a weakly continuous mapping.

Indeed, let (e_n) be the classical Schauder basis of l_2, then $e_n + e_1 \rightharpoonup e_1$, moreover $\|e_n + e_1\|_2 = \sqrt{2}$ whenever $n \geqslant 2$. Therefore,

$$A(e_n + e_1) = \frac{e_n + e_1}{\sqrt{2}} \rightharpoonup \frac{e_1}{\sqrt{2}} \neq e_1 = A(e_1),$$

which means that A cannot be a weakly continuous mapping.

Proof of Theorem 3.3. Let us consider $A_n = (1 - \frac{1}{n})A : M \to M$ and $B_n = (1 - \frac{1}{n})B : X \to X$. Since B is nonexpansive and ω-condensing, B_n is a ω-contraction and (see Remark 3.2) therefore $I - B_n$ is Φ-expansive.

On the other hand, A_n is continuous, weakly compact and satisfies $(\mathcal{A}1)$. Moreover since $0 \in M$ we have that $A_n(M) + B_n(M) \subseteq M$. Applying Theorem 3.2, $A_n + B_n$ has a fixed point $u_n \in M$ for any $n \in \mathbb{N}$.

We claim that (u_n) is weakly convergent: (u_n) is a bounded sequence. Let us suppose that (u_n) is not weakly convergent, and consequently $\{(u_n)\colon n \in \mathbb{N}\}$ is not relatively weakly compact. Then, since B is ω-condensing,

$$\begin{aligned}
\omega\big(\{u_n\colon n \in \mathbb{N}\}\big) &= \omega\left(\left\{\left(1 - \frac{1}{n}\right)A(u_n) + \left(1 - \frac{1}{n}\right)B(u_n)\colon n \in \mathbb{N}\right\}\right) \\
&\leqslant \omega\left(\left\{\left(1 - \frac{1}{n}\right)A(u_n)\colon n \in \mathbb{N}\right\}\right) + \omega\left(\left\{\left(1 - \frac{1}{n}\right)B(u_n)\colon n \in \mathbb{N}\right\}\right) \\
&\leqslant \omega\big(co\big(\big\{A(u_n)\colon n \in \mathbb{N}\big\} \cup \{0\}\big)\big) + \omega\big(co\big(\big\{B(u_n)\colon n \in \mathbb{N}\big\} \cup \{0\}\big)\big) \\
&\leqslant \omega\big(\big\{A(u_n)\colon n \in \mathbb{N}\big\}\big) + \omega\big(\big\{B(u_n)\colon n \in \mathbb{N}\big\}\big) \\
&= \omega\big(B\big(\{u_n\colon n \in \mathbb{N}\}\big)\big) \\
&< \omega\big(\{u_n\colon n \in \mathbb{N}\}\big),
\end{aligned}$$

which is a contradiction. Therefore, without loss of generality, we can assume that $u_n \rightharpoonup u \in M$.

Since $u_n = (1 - \frac{1}{n})(A+B)(u_n) = A(u_n) + B(u_n) - \frac{1}{n}A(u_n) - \frac{1}{n}B(u_n)$ then,

$$\begin{aligned} \left\| \big(I - (A+B)\big)(u_n) \right\| &= \left\| u_n - \big(A(u_n) + B(u_n)\big) \right\| = \frac{1}{n} \left\| A(u_n) + B(u_n) \right\| \\ &\leqslant \frac{1}{n} \big(\left\| A(u_n) \right\| + \left\| B(u_n) - B(0) \right\| + \left\| B(0) \right\| \big) \\ &\leqslant \frac{1}{n} \big(\left\| A(u_n) \right\| + \| u_n \| + \left\| B(0) \right\| \big), \end{aligned}$$

and since (u_n) is a sequence in M, which is bounded, and $A(M)$ is relatively weakly compact, hence $(\|A(u_n)\|)$ is bounded. Consequently

$$\lim_{n \to \infty} \big(I - (A+B)\big)(u_n) = 0,$$

and since $I - (A+B)$ is demiclosed at 0, $(I - (A+B))(u) = 0$, that is, u is a fixed point for $A+B$. □

Corollary 3.1. *Let M be a nonempty bounded closed convex subset of a Banach space X with $0 \in M$. Let $A : M \to M$ and $B : X \to X$ be two continuous mappings satisfying the following conditions*

(i) *A is weakly–strongly continuous and AM is relatively weakly compact,*
(ii) *B is nonexpansive and ω-condensing,*
(iii) *$A(M) + B(M) \subseteq M$,*
(iv) *$I - B : M \to X$ is demiclosed.*

Then, $A + B$ has a fixed point in M.

Remark 3.4. It is well know that a mapping $B : X \to X$ satisfies condition $(\mathcal{A}_2)$ if and only if it maps relatively weakly compact sets into relatively weakly compact sets, therefore every ω-condensing mapping enjoys condition $(\mathcal{A}_2)$. This fact means that condition (ii) of [1, Theorem 2.1] is more general than condition (ii) of the above corollary. However, in the above corollary, we do not have to assume condition (iii) of [1, Theorem 2.1].

Proof of Corollary 3.1. Condition on mapping A implies that (i) and (ii) of Theorem 3.3 are satisfied. Thus, we only have to see that $I - (A+B)$ is demiclosed at zero. Indeed, consider (x_n) a sequence in M such that $x_n \rightharpoonup x$ and suppose that $x_n - (Ax_n + Bx_n) \to 0$. Since A is weakly–strongly continuous, clearly $Ax_n \to Ax$ and therefore $x_n - Bx_n \to Ax$. Finally, since $I - B$ is demiclosed, we derive that $x - Bx = Ax$ and this yields that $x - (Ax + Bx) = 0$. □

If in the above corollary X is assumed to be reflexive, then the mapping B is always ω-condensing. If, in addition, we suppose that X is a uniformly convex Banach space, then $I - B : M \to X$ is demiclosed. In the light of the aforementioned comments we obtain the following consequence (see [1]).

Corollary 3.2. *Let M be a nonempty bounded closed convex subset of a uniformly convex Banach space X with $0 \in M$. Let $A : M \to M$ and $B : X \to X$ be two continuous mappings satisfying the following conditions*

(i) *A is weakly–strongly continuous,*
(ii) *B is nonexpansive,*
(iii) *$A(M) + B(M) \subseteq M$.*

Then, $A + B$ has a fixed point in M.

3.2. The whole space

In order to present the next fixed point results we need the following result.

Lemma 3.2. *(See [14, Corollary 10].) Let M be a nonempty closed convex subset of a Banach space X such that $0 \in M$. Assume that $T : M \to M$ is a continuous map satisfying condition $(\mathcal{A}1)$. If $T(C)$ is relatively weakly compact whenever C is a bounded subset of M and there exists $R > 0$ such that $T(x) \neq \lambda x$, for every $\lambda > 1$ and for every $x \in M \cap S_R$. Then there exists $x \in M$ such that $T(x) = x$.*

It is easy to prove that the following result is a consequence of the above lemma.

Corollary 3.3. *Let X be a Banach space. If $A : X \to X$ is a continuous, weakly compact map and satisfies condition $(\mathcal{A}1)$, then either*

(a) *the equation $x = \lambda A(x)$ has a solution for $\lambda = 1$, or*
(b) *the set of all such solutions x, for $\lambda \in (0, 1)$, is unbounded.*

Indeed, take $M = X$, hence $0 \in M$, if there does not exist $R > 0$ under the conditions of the lemma, then the set $\{x \in X\colon\ x = \lambda T(x) \text{ for some } \lambda \in (0, 1)\}$ is unbounded. Otherwise, the equation $x = T(x)$ has a solution.

Theorem 3.4. *Let X be a Banach space and let $A, B : X \to X$ be two continuous mapping. If A, B satisfy the following conditions,*

(i) *A maps bounded sets into relatively weakly compact ones,*
(ii) *A satisfies $(\mathcal{A}1)$,*
(iii) *B is nonexpansive and ω-condensing,*
(iv) *$I - B$ is ψ-expansive where ψ is either strictly increasing or $\lim_{r\to\infty} \psi(r) = \infty$,*

then, either

(a) *the equation $x = B(x) + A(x)$ has a solution, or*
(b) *the set $\{x \in X\colon\ x = \lambda B(\frac{x}{\lambda}) + \lambda A(x)\}$ is unbounded for $\lambda \in (0, 1)$.*

Proof. As in Theorem 3.2, we need to prove the existence of a fixed point for the operator $(I - B)^{-1} \circ A$. We first check that this operator is well defined. In the same fashion as in Theorem 3.2, it can be seen that $I - B$ has an inverse.

To see that the domain of $(I - B)^{-1}$ contains the range of A, we show that $D((I - B)^{-1}) = X$. This is equivalent to $R(I - B) = X$. Since B is nonexpansive, $I - B$ is accretive and continuous, and its domain is X. By Corollary 3.2 in [3], $I - B$ is m-accretive. $I - B$ is by hypothesis ψ-expansive, and hence, by [15, Theorem 8] along with [13, Remark 3.8] we can conclude that $I - B$ is surjective and hence $R(I - B) = X$.

Consequently $(I - B)^{-1} \circ A : X \to X$ is well defined. Let us prove now that this operator is under the conditions of Lemma 3.2, that is, $(I - B)^{-1} \circ A$ is continuous weakly compact and satisfies $(\mathcal{A}1)$.

(I) $(I - B)^{-1} \circ A$ is continuous.

The proof is similar to that given in the proof of Theorem 3.2.

(II) $(I - B)^{-1} \circ A$ is relatively weakly compact.

As seen before, $(I - B)^{-1} \circ A = A + B \circ (I - B)^{-1} \circ A$. Let S be a bounded set and let us prove that $(I - B)^{-1} \circ A(S)$ is a bounded set. Let $x, y \in (I - B)^{-1} \circ A(S)$. Hence, there exist $z_1, z_2 \in S$ such that $x = (I - B)^{-1} \circ A(z_1)$, $y = (I - B)^{-1} \circ A(z_2)$. Then, $x - B(x) = A(z_1)$, $y - B(y) = A(z_2)$.

Since $I - B$ is ψ-expansive, we can write

$$\begin{aligned}\psi\big(\|x-y\|\big) &\leqslant \big\|x - B(x) - \big(y - B(y)\big)\big\| \\ &= \big\|A(z_1) - A(z_2)\big\| \\ &\leqslant \operatorname{diam}\big(A(S)\big) < +\infty.\end{aligned}$$

If $(I-B)^{-1} \circ A(S)$ is not a bounded set, then there exist $x_n, y_n \in (I-B)^{-1} \circ A(S)$ such that $\|x_n - y_n\| \to +\infty$. Hence,

$$\psi\big(\|x_n - y_n\|\big) \leqslant \operatorname{diam}\big(A(S)\big).$$

If ψ is such that $\lim_{r\to\infty} \psi(r) = \infty$, then necessarily $\operatorname{diam}(A(S)) = +\infty$, which is a contradiction.

Else, if ψ is strictly increasing, then ψ has an inverse on $[0, +\infty)$, which is strictly increasing as well. Then

$$\|x_n - y_n\| \leqslant \psi^{-1}\big(\operatorname{diam}\big(A(S)\big)\big) < +\infty,$$

which gives another contradiction. Hence, in any case, $(I-B)^{-1} \circ A(S)$ is a bounded set.

Suppose $(I-B)^{-1} \circ A(S)$ is not relatively weakly compact.

$$\begin{aligned}\omega\big((I-B)^{-1} \circ A(S)\big) &= \omega\big(A + B \circ (I-B)^{-1} \circ A(S)\big) \\ &\leqslant \omega\big(A(S)\big) + \omega\big(B \circ (I-B)^{-1} \circ A(S)\big) \\ &= \omega\big(B \circ (I-B)^{-1} \circ A(S)\big) \\ &< \omega\big((I-B)^{-1} \circ A(S)\big)\end{aligned}$$

which is a contradiction. So $(I-B)^{-1} \circ A$ maps bounded sets into relatively weakly compact set, that is $(I-B)^{-1} \circ A$ is weakly compact.

(III) $(I-B)^{-1} \circ A$ satisfies condition $(\mathcal{A}1)$. The proof is similar to that of Theorem 3.2.

By Corollary 3.3, then either

(a) the equation $x = A(x) + B(x)$ has a solution in X, or
(b) the set of all solutions $\{x \in X\colon\ x = \lambda(I-B)^{-1} \circ A(x)\} = \{x \in X\colon\ x = \lambda B(\frac{x}{\lambda}) + \lambda A(x)\}$ is unbounded. □

Remark 3.5. In assumption (iv) of Theorem 3.4 we have imposed that ψ is either strictly increasing or that $\lim_{r\to\infty} \psi(r) = \infty$ because otherwise we cannot guarantee that, if S is a bounded subset of X, then $(I-B)^{-1} \circ A(S)$ becomes bounded.

Corollary 3.4.
Let X be a Banach space and let $A, B : X \to X$ be two continuous mappings. If A, B satisfy the following conditions,

(i) *A maps bounded sets into relatively weakly compact ones,*
(ii) *A satisfies $(\mathcal{A}1)$,*
(iii) *B is a separate contraction satisfying condition $(\mathcal{A}2)$.*

Then, either

(a) *the equation $x = B(x) + A(x)$ has a solution, or*
(b) *the set $\{x \in X\colon\ x = \lambda B(\frac{x}{\lambda}) + \lambda A(x)\}$ is unbounded for $\lambda \in (0,1)$.*

Remark 3.6. The proof of this corollary is a consequence of Theorem 3.4, since it is clear that if $B : X \to X$ is a separate contraction satisfying condition $(\mathcal{A}2)$, then, by Lemma 3.1, B is nonexpansive and ω-condensing. Moreover, by Remark 3.2, $I - B$ is a ψ-expansive mapping with ψ strictly increasing.

Making use of [6, Theorem 4.1] (see also [11, Theorem 2.2]), Remark 3.6 and proof of Theorem 3.4 we infer the following alternative of Krasnoselskii fixed point theorem, which is a generalization of [11, Theorem 2.5].

Corollary 3.5. *Let X be a Banach space and U an open subset of X with $0 \in U$, denote by $\overline{U}$ its closure. Let $A : \overline{U} \to X$ and $B : X \to X$ be two continuous mappings satisfying*:

(i) *$A(\overline{U})$ is relatively weakly compact,*
(ii) *A satisfies $(\mathcal{A}1)$,*
(iii) *B is a separate contraction satisfying condition $(\mathcal{A}2)$.*

Then, either

(a) *the equation $x = B(x) + A(x)$ has a solution in $\overline{U}$, or*
(b) *there exists an element in $x \in \partial U$ such that $x = \lambda B(\frac{x}{\lambda}) + \lambda A(x)$ for some $\lambda \in (0,1)$.*

Proposition 3.1. *Let X be a Banach space and let $A, B : X \to X$ be two maps on X. Assume that A satisfies conditions* (i) *and* (ii) *of Theorem* 3.4 *and B is a bounded linear mapping such that, for some $p \in \mathbb{N}$, B^p is nonexpansive and $I - B^p$ is ψ-expansive. Then the conclusion of Theorem* 3.4 *holds.*

Proof. Following the arguments of the proof of Theorem 3.4 we infer that there exists $(I - B^p)^{-1} : X \to X$ and it is continuous and so $(I - B)^{-1} = (I - B^p)^{-1} \sum_{k=0}^{p-1} B^k$. This fact means that $(I - B)^{-1}$ is a bounded linear operator and thus it is both continuous and weakly continuous (cf. [7, p. 39]). Now, since A satisfies conditions (i) and (ii), it is easy to see that the mapping $T := (I - B)^{-1} \circ A$ is under the conditions of Corollary 3.3 which allows us to achieve the proof. □

Remark 3.7. As we have seen in Remark 3.2 there exists a linear mapping $B : X \to X$ with $\|B\| = 1$ and such that $I - B$ is ψ-expansive. For this class of mappings [4, Theorem 2.1] does not work.

As a consequence of the above proposition, we can also state the following corollary (see [4, Theorem 2.1]).

Corollary 3.6. *Let X be a Banach space and let $A, B : X \to X$ be two maps on X. If A satisfies conditions* (i) *and* (ii) *of Theorem* 3.4 *and B is a bounded linear mapping such that, for some $p \in \mathbb{N}$, $\|B^p\| < 1$. Then the conclusion of Theorem* 3.4 *holds.*

4. Application

Let $m(\Omega)$ be the set of all measurable functions $\psi : \Omega \to \mathbb{R}$. If f is a Carathéodory function, then f defines a mapping $N_f : m(\Omega) \to m(\Omega)$ by $N_f(\psi)(t) := f(t, \psi(t))$. This mapping is called the superposition (or Nemytskii) operator generated by f. The next two lemmas are of foremost importance for our subsequent analysis.

Lemma 4.1. *(See [16,17].) Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Then the superposition operator N_f maps $L^1(\Omega)$ into $L^1(\Omega)$ if and only if there exist a constant $b \geqslant 0$ and a function $a(\cdot) \in L^1_+(\Omega)$ such that*

$$\left|f(t,x)\right| \leqslant a(t) + b|x|,$$

where $L^1_+(\Omega)$ denotes the positive cone of the space $L^1(\Omega)$.

Lemma 4.2. *(See [19].) Let Ω be a bounded domain in $\mathbb{R}^N$. If $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and N_f maps $L^1(\Omega)$ into itself, then N_f satisfies $(\mathcal{A}2)$.*

Remark 4.1. Although the superposition operator N_f satisfies the condition $(\mathcal{A}_2)$, generally it is not weakly continuous. In fact, only linear functions generate weakly continuous Nemytskii operators in L^1 spaces (see, for instance, [2, Theorem 2.6]).

Next we give an example of application for Theorem 3.4 in the Banach space of integrable function $L^1(0,1)$.

Example 4.1. We will study now the existence of solutions for the integral equation

$$\psi(t) = \eta \int_0^t \varsigma(t,s) g\bigl(\psi(s)\bigr)\,ds + \int_0^1 \nu(t,s) f\bigl(s,\psi(s)\bigr)\,ds,$$

on $L^1(0,1)$, the space of real Lebesgue integrable functions on the interval $[0,1]$ where the functions ς, f, g and ν satisfy the following conditions:

1. The function $\varsigma(\cdot,\cdot)$ is essentially bounded on $[0,1]$ and $\|\varsigma\|_\infty$ is its essential bound.
2. The function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exist a constant $\rho > 0$ and a function $\gamma(\cdot) \in L_1^+(0,1)$ such that $|f(t,x)| \leqslant \gamma(t) + \rho|x|$.
3. The function $g : \mathbb{R} \to \mathbb{R}$ is nonexpansive.
4. The function $\nu : [-1,1] \times [0,1] \to \mathbb{R}$ is strongly measurable and $\int_0^1 \nu(\cdot,s)\psi(s)\,ds \in L^1(0,1)$ whenever $\psi \in L^1(0,1)$ and there exists a function $\mu : [0,1] \to \mathbb{R}$, belonging to $L^1(0,1)$ such that $|\nu(t,s)| \leqslant \mu(t)$ for all $(t,s) \in [0,1] \times [0,1]$.
5. $\rho\|\nu\| < 1$.
6. $0 < \eta\|\varsigma\|_\infty + \rho\|\mu\| < 1$.

For this purpose we define

$$A : L^1(0,1) \to L^1(0,1),$$
$$\psi \mapsto A(\psi)(t) = \int_0^1 \nu(t,s) f\bigl(s,\psi(s)\bigr)\,ds$$

and

$$B : L^1(0,1) \to L^1(0,1),$$
$$\psi \mapsto B(\psi)(t) = \eta \int_0^t \varsigma(t,s) g\bigl(\psi(s)\bigr)\,ds.$$

A is well defined because of condition 4. The operator A can be seen as the composition $K \circ N_f$ where N_f is the superposition operator and K is defined as follows,

$$K : L^1(0,1) \to L^1(0,1),$$
$$u \mapsto K(u)(t) := \int_0^1 \nu(t,s)u(s)\,ds.$$

A is continuous: since f satisfies conditions of Lemma 4.1, N_f maps continuously $L^1(0,1)$ into itself. Besides, K is continuous by condition 4, so the composition $A = K \circ N_f$ is continuous.

Let S be a bounded subset of $L^1(0,1)$ and let $M > 0$ such that $\|\psi\| \leqslant M$ for all $\psi \in S$. For $\psi \in S$ we have

$$\begin{aligned} \big|A(\psi)(t)\big| &\leqslant \int_0^1 \big|\nu(t,s)\big|\big|f\big(s,\psi(s)\big)\big|\,ds \\ &\leqslant \int_0^1 \big|\mu(t)\big|\big(a(s) + b\big|\psi(s)\big|\big)\,ds \\ &\leqslant \mu(t)\big(\|a\| + bM\big). \end{aligned} \tag{4}$$

A satisfies $(\mathcal{A}1)$: let (ρ_n) be a weakly convergent sequence of $L^1(0,1)$. By Lemma 4.2, the sequence $(N_f(\rho_n))$ has a weakly convergent subsequence, say $(N_f(\rho_{n_k}))$. Let ρ be the weak limit of $(N_f(\rho_{n_k}))$. Accordingly, bearing in mind the boundedness of the mapping $\nu(t,\cdot)$ (see assumption (4)) we get

$$A(\rho_{n_k})(t) = \int_0^1 \nu(t,s) f\big(s, \rho_{n_k}(s)\big)\,ds \to \int_0^1 \nu(t,s)\rho(s)\,ds. \tag{5}$$

Inequality (4) along with (5) allow us to apply the dominated convergence theorem to conclude that the sequence $(A\rho_{n_k})$ converges in $L^1(0,1)$.

A maps bounded sets of $L^1(0,1)$ into weakly compact sets. To see this, let S be a bounded subset of $L^1(0,1)$ and let $M > 0$ such that $\|\psi\| \leqslant M$ for all $\psi \in S$. From inequality (4) we have

$$\int_E \big|A(\psi)(t)\big|\,dt \leqslant \big(\|a\| + bM\big) \int_E \mu(t)\,dt,$$

for all measurable subsets E of $[0,1]$. Since $\mu(\cdot) \in L^1[0,1]$ it is well known that

$$\lim_{|E| \to 0} \int_E \mu(t)\,dt = 0,$$

therefore $\omega(A(S)) = 0$.

Now we need to check that the operator B is well defined and is under the conditions of Corollary 3.4. Since

$$\begin{aligned}\left|\varsigma(t,s)g\big(\psi(s)\big)\right| &= \left|\varsigma(t,s)\right|\left|g\big(\psi(s)\big)-g(0)+g(0)\right| \\ &\leqslant \left|\varsigma(t,s)\right|\big(\left|\psi(s)\right|+\left|g(0)\right|\big) \\ &\leqslant \|\varsigma\|_\infty\big(\left|\psi(s)\right|+\left|g(0)\right|\big),\end{aligned}$$

by the dominated convergence theorem, B is well defined.

B is a separate contraction:

$$\begin{aligned}\left|B(\psi)(t)-B(\phi)(t)\right| &= \left|\eta\int_0^t \varsigma(t,s)\big(g(\psi)(s)-g(\phi)(s)\big)\,ds\right| \\ &\leqslant \eta\int_0^t \left|\varsigma(t,s)\right|\left|g(\psi)(s)-g(\phi)(s)\right|ds \\ &\leqslant \eta\|\varsigma\|_\infty\int_0^t \left|\psi(s)-\phi(s)\right|ds \\ &\leqslant \eta\|\varsigma\|_\infty\int_0^1 \left|\psi(s)-\phi(s)\right|ds \\ &= \eta\|\varsigma\|_\infty\|\psi-\phi\|_1.\end{aligned}$$

Integrating now with respect to t,

$$\begin{aligned}\left\|B(\psi)-B(\phi)\right\|_1 &= \int_0^1 \left|B(\psi)(t)-B(\phi)(t)\right|dt \\ &\leqslant \int_0^1 \eta\|\varsigma\|_\infty\|\psi-\phi\|_1\,dt \\ &= \eta\|\varsigma\|_\infty\|\psi-\phi\|_1.\end{aligned}$$

Hence, B is $\eta\|\varsigma\|_\infty$-Lipschitzian, since by assumption (5), $\eta\|\varsigma\|_\infty < 1$, then B is a separate contraction.

B satisfies ($\mathcal{A}2$): let M be a relatively compact subset of $L^1(0,1)$. Since $M \in \mathcal{B}(L^1(0,1))$, there exists $k > 0$ such that $\|\chi\|_1 \leqslant k$ for each $\chi \in M$. Then,

$$\begin{aligned}\left|B(\chi)(t)\right| &= \left|\eta\int_0^t \varsigma(t,s)g\big(\chi(s)\big)\,ds\right| \\ &\leqslant \eta\int_0^t \left|\varsigma(t,s)\right|\left|g\big(\chi(s)\big)\right|dt\end{aligned}$$

$$\leqslant \eta\|\varsigma\|_\infty \int_0^t \big|g\big(\chi(s)\big)\big|\,ds$$
$$= \eta\|\varsigma\|_\infty \int_0^t \big|g\big(\chi(s)\big) - g(0) + g(0)\big|\,ds$$
$$\leqslant \eta\|\varsigma\|_\infty \int_0^t \big(\big|\chi(s)\big| + \big|g(0)\big|\big)\,ds$$
$$\leqslant \eta\|\varsigma\|_\infty\big(\|\chi\|_1 + \big|g(0)\big|\big).$$

To calculate $\omega(B(M))$ we need to consider $D \in \mathcal{M}([0,1])$ such that $|D| < \varepsilon$,

$$\begin{aligned}\sup_{\chi\in M}\int_D \big|B(\chi)(t)\big|\,dt &\leqslant \sup_{\chi\in M}\int_D \eta\|\varsigma\|_\infty\big(\|\chi\|_1 + \big|g(0)\big|\big)\\ &\leqslant \eta\|\varsigma\|_\infty\big(k + \big|g(0)\big|\big)\mu(D)\\ &\leqslant \eta\|\varsigma\|_\infty\big(k + \big|g(0)\big|\big)\varepsilon,\end{aligned}$$

which tends to 0 when $\varepsilon \to 0$. Consequently, $\omega(B(M)) = 0$, which means that B is relatively weakly compact. So, B enjoys $(\mathcal{A}2)$.

The above steps show that the mappings A and B satisfy assumptions of Corollary 3.4. Then, in order to see that $A + B$ has a fixed point we only have to prove that for $\lambda \in (0,1)$ the set

$$C_\lambda := \left\{\psi \in L^1(0,1)\colon\ \psi = \lambda B\left(\frac{\psi}{\lambda}\right) + \lambda A(\psi)\right\}$$

is bounded.

Indeed, let ψ be an element of C_λ, we obtain

$$\begin{aligned}\big|\psi(t)\big| &\leqslant \lambda\eta\int_0^1 \left|\varsigma(t,s)g\left(\frac{\psi(s)}{\lambda}\right)\right|ds + \lambda\int_0^1 \big|\nu(t,s)f\big(s,\psi(s)\big)\big|\,ds\\ &\leqslant \lambda\eta\|\varsigma\|_\infty\left(\frac{\|\psi\|}{\lambda} + \big|g(0)\big|\right) + \lambda\big|\mu(t)\big|\int_0^1 \big(\gamma(s) + \rho\big|\psi(s)\big|\big)\,ds\\ &\leqslant \eta\|\varsigma\|_\infty\big(\|\psi\| + \big|g(0)\big|\big) + \big|\mu(t)\big|\big(\|\gamma\| + \rho\|\psi\|\big).\end{aligned}$$

Integrating in t one gets

$$\|\psi\| \leqslant \big(\eta\|\varsigma\|_\infty + \rho\|\mu\|\big)\|\psi\| + \|\gamma\|\|\mu\| + \eta\|\varsigma\|_\infty\big|g(0)\big|.$$

Consequently, using assumption (5), i.e. $0 < \eta\|\varsigma\|_\infty + \rho\|\mu\| < 1$, we get

$$\|\psi\| \leqslant \frac{\|\gamma\|\|\mu\| + |g(0)|}{1 - (\eta\|\varsigma\|_\infty + \rho\|\mu\|)},$$

which proves the boundedness of C_λ.

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