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STRONG CONVERGENCE THEOREMS FOR RESOLVENTS OF ACCRETIVE OPERATORS

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ABSTRACT. In this note we study two different classes of accretive operators which have a unique zero. Moreover, we show that the resolvents J_r of such operators converge strongly to the zero of the operator, as $r \to \infty$.

1. INTRODUCTION

Let X be a real Banach space. An operator $A \subseteq X \times X$ with domain $D(A)$ and range $\mathcal{R}(A)$ is said to be accretive if the inequality $\|x - y\| \le \|x - y + r(z - u)\|$ holds for all $x, y \in D(A)$, $z \in Ax$, $u \in Ay$ and $r > 0$. If A is accretive, we may define, for each positive r, a single-valued mapping $J_r^A : \mathcal{R}(I + rA) \to D(A)$ by $J_r^A := (I + rA)^{-1}$, where I is the identity. J_r^A is called the resolvent of A. In [6] Reich showed the following result: Let X be a uniformly smooth Banach space, and let $A \subseteq X \times X$ be m-accretive (which means that A is accretive and moreover $\mathcal{R}(I + A) = X$). If $0 \in \mathcal{R}(A)$, then for each $x \in X$ the strong $\lim_{r\to\infty} J_r^A x$ exists and belongs to $A^{-1}0$. Later, in 1984 W. Takahashi and Y. Ueda [8] improved the Reich's theorem in the following sense: Let X be a reflexive space with a uniformly Gâteaux differentiable norm, and let $A \subseteq X \times X$ be an accretive operator that satisfies the range condition (which means that $\overline{D(A)} \subseteq \bigcap_{r>0} \mathcal{R}(I + rA)$). Suppose that every weakly compact convex subset of X has the fixed point property for nonexpansive mappings. Let C be a closed convex subset of X such that $C \subseteq \mathcal{R}(I + rA)$ and C is J_r^A-invariant for some $r > 0$. If $0 \in \mathcal{R}(A)$, then for each $x \in C$ the strong $\lim_{r\to\infty} J_r^A x$ exists and belongs to $A^{-1}0$.

As we can notice both above results works for general m-accretive operators but however the framework of the Banach spaces where they must be defined is not too large. In this paper, we yield special classes of accretive operators and we study the strong convergence of their resolvents without any restriction on the Banach spaces where they are defined.

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2. Preliminaries

Throughout this paper we assume that X is a real Banach space and denote by $X^\star$ the dual space of X. As it is usual we will denote by $B[x,r]$ and $S[x,r]$ the closed ball and the sphere of the Banach space X with radius r and center $x \in X$, respectively. We define the normalized duality mapping by

$$J(x) := \{j \in X^\star : \langle x, j\rangle = \|x\|^2, \|j\| = \|x\|\}.$$

Let $\langle y, x\rangle_+ := \max\{\langle y, j\rangle : j \in J(x)\}$.

Given an operator $A : D(A) \to 2^X$, we define

$$J_\lambda^A : = (I + \lambda A)^{-1} : \mathcal{R}(I + \lambda A) \to D(A),$$

$$A_\lambda : = \frac{I - J_\lambda^A}{\lambda},$$

$$|Ax| : = \inf\{\|u\| : \ u \in Ax\}.$$

The operators J_λ^A (J_λ for short) and A_λ are the resolvent and the Yosida approximant of A, respectively.

We now recall some important facts regarding accretive operators which will be used in this paper (see, for instance, [1]):

Proposition 2.1.

(i) *$A \subset X \times X$ is accretive if and only if $\langle u - v, x - y\rangle_+ \geq 0$ for all $(x, u), (y, v) \in A$.*

(ii) *$A \subset X \times X$ is accretive if and only if for each $\lambda > 0$, the resolvent J_λ^A is a single-valued nonexpansive mapping.*

(iii) *For all $x \in \mathcal{R}(I + \lambda A)$ with $\lambda > 0$, $A_\lambda x \in AJ_\lambda^A x$.*

3. Two different classes of accretive operators

In order to proceed, we shall first give the following definitions.

Definition 3.1. (see [2]) Let $\phi : [0, \infty[\to [0, \infty[$ be a continuous function such that $\phi(0) = 0$ and $\phi(r) > 0$ for $r > 0$. Let X be a Banach space. An operator $A : D(A) \to 2^X$ is said to be ϕ-strongly accretive if for every $(x, u), (y, v) \in A$, then

$$\phi(\|x - y\|)\|x - y\| \leq \langle u - v, x - y\rangle_+.$$

Definition 3.2. (see [3]) Let X be a Banach space. An accretive operator $A : D(A) \to 2^X$ is said to be ϕ-expansive if for every $(x, u), (y, v) \in A$, then

$$\phi(\|x - y\|) \leq \|u - v\|.$$

Definition 3.3. An accretive operator $A : D(A) \to 2^X$ is said to be ϕ-accretive at zero whenever there exists $z \in X$ such that the inequality

$$\langle u, x - z\rangle_+ \geq \phi(\|x - z\|) \text{ for all } (x, u) \in A. \tag{3.1}$$

holds.

Next, we shall study the relationship between the above classes of accretive operators.

Proposition 3.4. *Let X be a Banach space and $A \subseteq X \times X$ a ϕ-strongly accretive operator. Then A is ϕ-expansive.*

Proof. Let $(x, u), (y, v) \in A$. Since A is ϕ-strongly accretive, we have

$$\phi(\|x - y\|)\|x - y\| \leq \langle u - v, x - y\rangle_+.$$

On the other hand, it is well known that $\langle u - v, x - y\rangle_+ \leq \|u - v\|\|x - y\|$, therefore

$$\phi(\|x - y\|)\|x - y\| \leq \|u - v\|\|x - y\|,$$

which implies that

$$\phi(\|x - y\|) \leq \|u - v\|.$$

Next example shows that there exist accretive operators which are ϕ-expansive and they cannot become ϕ-accretive at zero neither ϕ-strongly accretive for any ϕ.

Example 3.5. Let X be the real Hilbert space $\mathbb{R}^2$ with the usual Euclidean inner product. Define $A : \mathbb{R}^2 \to \mathbb{R}^2$ by $A(x, y) = (y, -x)$. Then A is accretive and ϕ-expansive with $\phi(r) = r$. However, A is not ϕ-accretive at zero for any possible function ϕ as described above.

Example 3.6. Consider the Banach space $(l_2, \|.\|_2)$ and let $(e_n = (\delta_{i,n}))$ be the usual Schauder basis of such space.

Define the following operator:

$$T(\sum_{k=1}^{\infty} x_k e_k) = \sum_{i=1}^{\infty} z_i e_i$$

where

$$z_i = \begin{cases} x_{2k} & i = 2k - 1 \\ -x_{2k-1} & i = 2k \end{cases}$$

Let us see that T is accretive:

$$\langle T(x) - T(y), x - y\rangle =$$

$$= \langle \sum_{k=1}^{\infty} (x_{2k} - y_{2k})e_{2k-1} + \sum_{k=1}^{\infty} (y_{2k-1} - x_{2k-1})e_{2k}, \sum_{k=1}^{\infty} (x_k - y_k)e_k\rangle =$$

$$= \sum_{k=1}^{\infty} (x_{2k} - y_{2k})(x_{2k-1} - y_{2k-1}) + \sum_{k=1}^{\infty} (y_{2k-1} - x_{2k-1})(x_{2k} - y_{2k}) = 0.$$

This argument allows us to see that T is an accretive operator which fails to be both ϕ-strongly accretive and ϕ-accretive at zero for any ϕ. Nevertheless, if we take $\phi(t) = t$ we may derive

$$\|T(x) - T(y)\| = \|\sum_{k=1}^{\infty}(x_{2k} - y_{2k})e_{2k-1} + \sum_{k=1}^{\infty}(y_{2k-1} - x_{2k-1})e_{2k}\| = \\ = \|x - y\| = \phi(\|x - y\|).$$

In the following example we will prove that there exist operators which are ϕ-accretive at zero but not ϕ-expansive for any ϕ.

Example 3.7 (see [4]). Let X be a Banach space. Consider the following operator on X :

$$\begin{array}{rccl} T: & X & \longrightarrow & 2^X \\ & x & \mapsto & T(x) = \begin{cases} \frac{x}{\|x\|}, & x \neq 0 \\ B_X, & x = 0. \end{cases} \end{array}$$

Where B_X denotes the unit ball of X. It is easy to see that this operator is m-accretive on X, ϕ-accretive at zero for $\phi(r) = r$ but it fails to be ψ-expansive for any ψ.

Proposition 3.8. *Let Ω be a bounded open domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Consider β a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ with $0 \in \beta(0)$ such that $\beta(r)r \geq |r|^\alpha$ (this means that for each $s \in \beta(r)$ we have that $sr \geq |r|^\alpha$) for some $\alpha > 2$. Then, for each $1 \leq p < \infty$ the operator $B_p : D(B_p) \subseteq L^p(\Omega) \to 2^{L^p(\Omega)}$, where*

$$D(B_p) := \{u \in L^p(\Omega) \ : \ \exists v \in L^p(\Omega) \ : \ v(x) \in \beta(u(x)) \ a.e.\}$$

and defined by

$$B_p(u) := \{v \in L^p(\Omega) \ : \ v(x) \in \beta(u(x)) \ a.e.\},$$

is m-accretive and ϕ-accretive at zero on $L^p(\Omega)$.

Proof. It is well known that B_p is an m-accretive operator on $L^p(\Omega)$ (for instance see [1]).

Thus, we will only prove that B_p is ϕ-accretive at zero on $L^p(\Omega)$.

Case $p = 1$. To see this, since $0 \in B_1 0$, if we consider $u \in D(B_1)$ and $v \in B_1(u)$ we have to study $\langle v, u\rangle_+$.

The normalized duality map on $L^1(\Omega)$ is give by

$$J(u) = \|u\|_1\{j : \ j \in L^\infty(\Omega), \ |j| \leq 1, \ ju = |u| \ a.e.\}.$$

Hence, there exists $j \in J(u)$ such that

$$\langle v, u\rangle_+ = \langle v, j\rangle.$$

Consequently, Hölder's inequality yields $K > 0$ such that

$$\begin{aligned}\langle v,u\rangle_+ = \langle v,j\rangle &= \|u\|_1 \int_\Omega v(t)j(t) = \\ &= \|u\|_1 (\int_{\{t\in\Omega\,:\,u(t)\neq 0\}} v(t)\frac{u(t)}{|u(t)|} + \int_{\{t\in\Omega\,:\,u(t)=0\}} |v(t)|dt) \geq \\ &\geq \|u\|_1 \int_\Omega |u(t)|^{\alpha-1} \geq K\|u\|_1^\alpha.\end{aligned}$$

Therefore, if we define $\phi(t) = Kt^\alpha$, we obtain that B_1 is ϕ-accretive at zero on $L^1(\Omega)$.

Case $1 < p < \infty$. Since (see for instance [1]) the normalized duality map on $L^p(\Omega)$ is given by

$$J(u) = \|u\|_p^{2-p}|u|^{p-2}u.$$

For each $(u,v) \in B_p$ we have

$$\langle v,u\rangle_+ = \|u\|_p^{2-p}\int_\Omega v(t)u(t)|u(t)|^{p-2}dt \geq \|u\|_p^{2-p}\int_\Omega |u(t)|^{p+\alpha-2}dt$$

Then by Hölder's inequality there exists $K > 0$ such that $\langle v,u\rangle_+ \geq K\|u\|_p^\alpha$, and thus , it is sufficient to take the function $\phi(t) = Kt^\alpha$. □

Remark 3.9. In [4] we can find several examples of operators which are ϕ-accretive at zero.

4. The behavior of the resolvents

Theorem 4.1. *(see* [5]*, Theorem 6) Let X be a Banach space, let $A : D(A) \subseteq X \to 2^X$ be ϕ-accretive operator at zero and let $z \in X$ satisfy condition (3.1). Then*

(a) *If $z \in \mathcal{R}(I + A)$, then $0 \in \mathcal{R}(A)$.*
(b) *If X is reflexive with the fixed point property for spheres and $\overline{co}(D(A)) \subseteq \mathcal{R}(I + A)$, then $0 \in \mathcal{R}(A)$.*

Proof. Since the operator A is accretive, then it is well known that the resolvent:

$$g := (I + A)^{-1} : R(I + A) \to D(A)$$

is a single-valued and nonexpansive mapping.

We claim that for every $y \in \mathcal{R}(I + A)$, g satisfies $\|g(y) - z\| \leq \|y - z\|$.

Consider $y \in R(I + A)$, then there exists $x \in D(A)$ such that $x = g(y)$. Hence, $y \in x + Ax$, which means that there exists $u \in Ax$ satisfying that $y = x + u$. Since A is ϕ-accretive at zero (i.e., $0 \leq \phi(\|x - z\|) \leq \langle u, x - z\rangle_+$) we obtain

$$\|g(y) - z\| = \|x - z\| \leq \|x - z + u\| = \|y - z\|.$$

(a) If $z \in R(I + A)$, then $g(z) = z$ and hence $z \in z + Az$, which implies that $0 \in Az$.

(b) We may consider the set

$$C := \overline{\text{co}}(D(A)) \cap B[z, \text{dist}(z, \overline{\text{co}}(D(A))]$$

Since X is a reflexive Banach space, then C is a nonempty, weakly compact, convex subset and it is not difficult to see that it is also g-invariant. Then the set

$$K = P_C(z) := \{x \in C : \|x - z\| = \text{dist}(z, C) =: d\}$$

is nonempty. Moreover $K = C \cap B[z, d] = C \cap S[z, d]$ is closed and convex. For every $x \in K$, $gx \in C$ and then $d \leq \|gx - z\| \leq \|x - z\| = d$. Hence $gx \in K$ which implies that K is g invariant. Since X has the fixed point property for spheres and K lies in a sphere, then g has a fixed point on K which is a zero of A. □

Theorem 4.2. *Let X be a Banach space, let $A : D(A) \subseteq X \to 2^X$ be a ϕ-expansive operator such that $\overline{co}(D(A)) \subseteq \bigcap_{\lambda>0} \mathcal{R}(I + \lambda A)$. If*

$$\lim_{\|x\| \to \infty} |Ax| = \infty, \tag{4.1}$$

then $0 \in \mathcal{R}(A)$.

Proof. Consider $x_0 \in D(A)$, it is well known that the function $\lambda \to \|A_\lambda x_0\|$ is decreasing. Consequently there exists $k > 0$ such that

$$\lim_{\lambda \to \infty} \|A_\lambda x_0\| = k.$$

Since, on the other hand, it is clear that $A_\lambda x_0 \in AJ_\lambda x_0$, by (4.1) we deduce that the set $\{J_\lambda x_0 \ : \ \lambda \geq 0\}$ is bounded.

Now, by using both that $\lim_{\lambda \to 0^+} J_\lambda x_0 = x_0$ and that the function $\lambda \to J_\lambda x_0$ is continuous, we may deduce that there exists a bounded neighborhood $\mathcal{U}$ of x_0 such that

$$t(x - x_0) \notin A(x) \quad \text{for } x \in \partial\mathcal{U} \cap D(A) \text{ and } t < 0.$$

Therefore we may apply Theorem 13 of [3] and thus we obtain the result. □

Corollary 4.3. *Let X be a Banach space, let $A : D(A) \subseteq X \to 2^X$ be a ϕ-expansive operator such that $\overline{co}(D(A)) \subseteq \bigcap_{\lambda>0} \mathcal{R}(I + \lambda A)$. If*

$$\lim_{r \to \infty} \phi(r) = \infty, \tag{4.2}$$

then $0 \in \mathcal{R}(A)$.

Theorem 4.4. *Let X be a Banach space, let $A : D(A) \subseteq X \to 2^X$ be an m-accretive and either ϕ-expansive or ϕ-accretive at zero. Then there exists a unique $z \in D(A)$ such that $0 \in A(z)$. In addition,*

(i) $\lim_{\lambda \to \infty} J_\lambda x = z$ *for each* $x \in X$.
(ii) $\lim_{n \to \infty} J_\lambda^n x = z$ *for each* $\lambda > 0$ *and* $x \in X$.

Proof. If we assume that A is ϕ-expansive, then by ([3], Theorem 8) we know that A is surjective and therefore $0 \in \mathcal{R}(A)$. The uniqueness is an easy consequence of the definition of such type of operators.

When A is ϕ-accretive at zero, we can use the above Theorem to show that A has a zero. The uniqueness is a consequence of condition (3.1).

To see part (i), let $x \in X$. Since A is an m-accretive operator it is well known that for all $\lambda > 0$, $X = \mathcal{R}(I + \lambda A)$. Then $x_\lambda = J_\lambda x \in D(A)$. This means,

$$x \in x_\lambda + \lambda A(x_\lambda) \quad \text{for all } \lambda > 0.$$

Since $\|x_\lambda - z\| \leq \|x - z\|$, the set $\{x_\lambda \,:\, \lambda > 0\}$ is bounded, and consequently $\lambda^{-1}(x - x_\lambda) \to 0$ as $\lambda \to \infty$.

Now, if we assume that A is ϕ-expansive, since $A_\lambda x \in AJ_\lambda x$, we have

$$\phi(\|x_\lambda - z\|) \leq \|A_\lambda x - 0\| \leq \|\lambda^{-1}(x - x_\lambda)\|.$$

Therefore $J_\lambda x \to z$, as $\lambda \to \infty$.

When A is ϕ-accretive at zero, it is clear that

$$\phi(\|x_\lambda - z\|) \leq \langle A_\lambda x, x_\lambda - z\rangle_+ \leq \|\frac{x_\lambda - x}{\lambda}\|\|x_\lambda - z\|.$$

Since J_λ^A is nonexpansive, we obtain

$$\lambda\phi(\|x_\lambda - z\|) \leq 2\|x - z\|^2 \quad \forall \lambda > 0,$$

which means that $\phi(\|x_\lambda - z\|) \to 0$ as $\lambda \to \infty$, and hence we may conclude that $x_\lambda \to z$.

To see (ii), let $x \in X$ and let $R > 0$ be such that $\|x - z\| < R$. Then for a fixed $\lambda > 0$, the mapping J_λ maps $\overline{co}(D(A)) \cap B[z, R]$ into itself. Since J_λ is firmly nonexpansive with a fixed point z, then it is asymptotically regular (see Corollary 1 of [7]). In addition, for every $\lambda > 0$ and every $x \in X$ we know that $A_\lambda J_\lambda^n x \in AJ_\lambda^{n+1} x$.

If A is ϕ-expansive, we have

$$\lambda\phi(\|J_\lambda^{n+1} x - z\|) \leq \|J_\lambda^n x - J_\lambda^{n+1} x\| \to 0 \quad \text{as} \quad n \to \infty.$$

If A is ϕ-accretive at zero, then

$$\lambda\phi(\|J_\lambda^{n+1} x - z\|) \leq \|J_\lambda^n x - J_\lambda^{n+1} x\|\|x - z\| \leq R\|J_\lambda^n x - J_\lambda^{n+1} x\| \to 0 \text{ as } n \to \infty.$$

which completes the proof. □

Remark 4.5. As a consequence of Proposition 3.4 and ([3], Theorem 8) it is worthy to observe that every m-accretive and ϕ-strongly accretive operator is in fact ψ-accretive at zero for $\psi(t) = t\phi(t)$.

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