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FIXED POINT THEORY FOR MULTIVALUED GENERALIZED NONEXPANSIVE MAPPINGS

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A very general class of multivalued generalized nonexpansive mappings is defined. We also give some fixed point results for these mappings, and finally we compare and separate this class from the other multivalued generalized nonexpansive mappings introduced in the recent literature.

1. INTRODUCTION

Fixed point theory for multivalued mappings has many useful applications in various fields, in particular game theory and mathematical economics. Thus, it is natural to extend the known fixed point results for single-valued mappings to the setting of multivalued mappings. Some famous results of existence of fixed points for single-valued mappings (e.g. Banach's Contraction Principle, Schauder Fixed Point Theorem) have already been extended to the multivalued case. Nevertheless, the fixed point theory of multivalued nonexpansive mappings is much more complicated and difficult than the corresponding theory of single-valued nonexpansive mappings and many problems remain unsolved in it.

Although nonexpansive mappings are perhaps one of the most important topics in the so called metric fixed point theory, one can find in the literature considerable amount of research about more general classes of mappings than the nonexpansive ones. For instance, Tomonari Suzuki [**25**] defined in 2008 a class of generalized nonexpansive mappings, which he called (C)-type mappings, whose setvalued version was defined and studied in [**1, 2, 22, 26**]. In 2011, in [**11**], some fixed point results for two classes of single-valued mappings enlarging the family

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of (C)-type mappings were presented. Again these new classes were generalized to the setvalued case in [**5, 14, 19**] and [**4**].

Finally, in [**20**] fixed point results for a class of (single-valued) generalized nonexpansive mappings were studied. This class properly contains Suzuki's (C)-type mappings as well as several of its generalizations given in [**11**]. The aim of these notes is to extend the class of mappings introduced in [**20**] to the multivalued case and to give some fixed point results for multivalued mappings in this setting. In the last section we discuss some relationships between the proposed class of multivalued mappings and several others which have been recently introduced in [**9, 10, 14**] as well as in [**5, 14, 19**].

2. NOTATIONS AND PRELIMINARIES

We assume throughout this paper that $(X, \|\cdot\|)$ is a Banach space and C is a nonempty closed convex bounded subset of X. Let D be a nonempty subset of X. We use the following symbols:

$\mathcal{P}(D) = \{Y \subset D \mid Y \text{ is nonempty}\}$ $\quad$ $\mathcal{P}_b(D) := \{Y \in \mathcal{P}(D) \mid Y \text{ is bounded}\}$
$\mathcal{P}_{c\ell}(D) := \{Y \in \mathcal{P}(D) \mid Y \text{ is closed}\}$ $\quad$ $\mathcal{P}_{b,c\ell}(D) := \mathcal{P}_b(D) \cap \mathcal{P}_{c\ell}(D)$
$\mathcal{P}_{cp}(D) := \{Y \in \mathcal{P}(D) \mid Y \text{ is compact}\}$ $\quad$ $\mathcal{P}_{cv}(D) := \{Y \in \mathcal{P}(D) \mid Y \text{ is convex}\}$
$\mathcal{P}_{c\ell,cv}(D) := \mathcal{P}_{c\ell}(D) \cap \mathcal{P}_{cv}(D)$

For a multivalued mapping $T : D \to \mathcal{P}(D)$, the T-image of a set $Y \in \mathcal{P}(D)$ is defined as $T(Y) := \bigcup_{x \in Y} T(x)$.

On $\mathcal{P}_{b,c\ell}(X)$ one defines the Hausdorff distance H

$$H(A, B) := \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\},$$

where $d(a, B) := \inf\{\|a - b\| : b \in B\}$ is the standard distance from the point a to the subset B. For more on the Hausdorff metric see [**13**].

Finally we recall some geometric properties of normed spaces that appear in this paper.

1. A normed space $(X, \|\cdot\|)$ is said to satisfy the *Opial condition* if for any sequence (x_n) in X such that $x_n \rightharpoonup x_0$ it follows that $\forall y \in X$, $y \neq x_0$,

   $$\liminf_{n\to\infty} \|x_n - x_0\| < \liminf_{n\to\infty} \|x_n - y\|.$$

   It can be readily established, on the extraction of appropriate subsequences, that the lower limits can be replaced with upper limits in the above definition.

2. A Banach space $(X, \|\cdot\|)$ is said to have *normal structure* if for each bounded, convex, subset C of X with $\operatorname{diam}(C) > 0$ there exists a *nondiametral* point $p \in C$, that is, a point $p \in C$ such that

   $$\sup\{\|p - x\| : x \in C\} < \operatorname{diam}(C).$$

This property was introduced in 1948 by Brodskii and Milman. Since 1965, it has been widely studied due to its relevance in fixed point theory for nonexpansive mappings. For more information see, for instance, [**12**].

3. (L)-TYPE MAPPINGS

A class of (single-valued) non-expansive generalized mappings has been recently considered in [**20**], under the name of (L)-type mappings. Such class properly contains several other classes of mappings which in turn are more general than the class of nonexpansive mappings. Recall that if $T : C \to X$ is a mapping, a sequence (x_n) in C is called an almost fixed point sequence (a.f.p.s for short) for T in C whenever $x_n - T(x_n) \to 0_X$.

Definition 1. *A mapping $T : C \to X$ satisfies condition (L), (or it is an (L)-type mapping), on C provided that it fulfills the following two conditions*

1. *If a set $D \subset C$ is nonempty closed convex and T-invariant, (i.e. $T(x) \in D$ for any $x \in D$), then there exists an a.f.p.s. for T in D.*
2. *For any a.f.p.s. (x_n) of T in C and each $x \in C$*

$$\limsup_{n\to\infty} \|x_n - T(x)\| \le \limsup_{n\to\infty} \|x_n - x\|.$$

In order to extend this concept to multivalued mappings we need to precise the meaning of a.f.p.s. in this setting.

Definition 2. *Given a mapping $T : C \to \mathcal{P}_{b,c\ell}(X)$, a sequence (x_n) in C is called an a.f.p.s. for T provided that*

$$d(x_n, T(x_n)) \to 0.$$

If

$$H(\{x_n\}, T(x_n)) \to 0$$

we say that (x_n) is a strong a.f.p.s. for T.

Definition 3 ([**10**], Remark 3.15)**.** *Given a mapping $T : C \to \mathcal{P}(X)$ we say that T satisfies condition (A) on C whenever there exists an a.f.p.s. for T in each nonempty, closed, convex and T-invariant subset D of C.*

Here T-invariant means $T(x) \subset D$ for any $x \in D$.

Next we introduce two classes of nonlinear mappings, which are a direct way to extend the (L)-type mappings defined in [**20**] to the multivalued case.

Definition 4. *A mapping $T : C \to \mathcal{P}_{c\ell}(C)$ satisfies condition (L), (or it is an (L)-type mapping), on C provided that it fulfills Condition (A) on C and*

(B) *For any a.f.p.s. (x_n) of T in C and each $x \in C$*

$$\limsup_{n\to\infty} d(x_n, T(x)) \le \limsup_{n\to\infty} \|x_n - x\|.$$

If $x_0 \in C$ is a fixed point for the mapping $T : C \to \mathcal{P}_{c\ell}(C)$, and this mapping satisfies Condition (L) on C, taking $x_n = x_0$ for every positive integer n, we obtain an a.f.p.s. for T, and from Condition (B), one has for all $x \in C$,

$$d(x_0, Tx) = \limsup_{n\to\infty} d(x_n, Tx) \leq \limsup_{n\to\infty} \|x_n - x\| = \|x_0 - x\|,$$

in other words, T is a *quasi-nonexpansive* mapping in this case.

Definition 5. *A mapping $T : C \to \mathcal{P}_{c\ell}(C)$ satisfies strong condition (L), (or it is an (SL)-type mapping), on C provided that it fulfills Condition (A) on C and*

(B_s) *For any a.f.p.s. (x_n) of T in C and each $x \in C$*

$$\limsup_{n\to\infty} H(\{x_n\}, T(x)) \leq \limsup_{n\to\infty} \|x_n - x\|.$$

It is obvious that mappings that satisfy condition (SL) also satisfy condition (L). Of course, in the single-valued case, both classes (L) and (SL) coincide and recover the original definition given in [**20**].

Remark 1. According to [**8**], a mapping $T : C \to \mathcal{P}_{c\ell}(C)$ satisfies condition (A) on C whenever one of the following statements holds.

1. T is (Hausdorff) nonexpansive on C, that is $H(Tx, Ty) \leq \|x - y\|$ for $x, y \in C$.
2. T is 1-set contractive on C. Recall that $T : C \to \mathcal{P}_{c\ell}(C)$ is 1-set contractive if T is Hausdorff continuous and $\alpha(T(M)) \leq \alpha(M)$ for every $M \subset C$, where α stands for the Kuratowski measure of noncompactness on $\mathcal{P}_b(X)$. (See [**8**], Lemma 4).
3. T satisfies condition (C_λ) on C. Following [**2, 3**], for $\lambda \in (0, 1)$ a mapping $T : C \to \mathcal{P}_{c\ell}(C)$ is said to satisfy condition (C_λ) in C provided that for every $x, y \in C$, $\lambda d(x, Tx) \leq \|x - y\| \Rightarrow H(Tx, Ty) \leq \|x - y\|$. According to [**3**, Lemma 2.6] or [**14**, Lemma 2.8], if C is weakly compact and $D \subset C$ is nonempty weakly compact convex and T-invariant, we can assure the existence of an a.f.p.s. for T in D, that is, T satisfies condition (A) on C.

First, we show that every multivalued nonexpansive mapping is an (L)-type mapping. In the last section we study other relationships between the class of (L)-type mappings and some classes of generalized nonexpansive setvalued mappings.

Lemma 1 ([**13**], Theorem 1.15)**.** *Let $A, B \subset X$, and let $x \in X$. Then*

$$d(x, A) \leq d(x, B) + H(B, A).$$

Proposition 1. *If $T : C \to \mathcal{P}_{c\ell}(C)$ is nonexpansive then it satisfies condition (L).*

Proof. Let $T : C \to \mathcal{P}_{c\ell}(C)$ be a nonexpansive mapping. It is well known that if D is a closed convex T-invariant subset of C, then T has a.f.p. sequences in D. Moreover, since T is nonexpansive on D, for every a.f.p.s. (x_n) for T and every $x \in C$,

$$\limsup_{n\to\infty} d(x_n, Tx) \leq \limsup_{n\to\infty} (d(x_n, Tx_n) + H(Tx_n, Tx)) \leq \limsup_{n\to\infty} \|x_n - x\|.$$

Then, T satisfies condition (L). □

In the following example we show that a nonexpansive mapping need not satisfy condition (SL).

EXAMPLE 1. [**16**]. In the Banach space $(\mathbb{R}^2, \|\cdot\|_2)$ consider the mapping $T : [0,1]\times[0,1] \to \mathcal{P}_{c\ell}([0,1]\times[0,1])$ defined by

$$T(x,y) = \overline{conv}\{(0,0),(x,0),(0,y)\}.$$

The sequence (p_n) defined by $p_n = \left(1, \dfrac{1}{n}\right)$ is an a.f.p.s. for T on $[0,1]\times[0,1]$, since

$$d(p_n, T(p_n)) = d\left(\left(1,\frac{1}{n}\right), \overline{conv}\left\{(0,0),(1,0),\left(0,\frac{1}{n}\right)\right\}\right) \le \frac{1}{n},$$

which clearly tends to zero.

It is not a strong a.f.p.s., since

$$\begin{aligned} H(\{p_n\}, T(p_n)) &= H\left(\left\{\left(1,\frac{1}{n}\right)\right\}, \overline{conv}\left\{(0,0),(1,0),\left(0,\frac{1}{n}\right)\right\}\right) \\ &= \sup\left\{\left\|\left(1,\frac{1}{n}\right) - x\right\|_2 : x \in \overline{conv}\left\{(0,0),(1,0),\left(0,\frac{1}{n}\right)\right\}\right\} \\ &= d\left(\left(1,\frac{1}{n}\right),(0,0)\right) = \sqrt{1+\frac{1}{n^2}} \end{aligned}$$

which does not converge to 0.

Since T is nonexpansive on C (see [**16**]) then, from Proposition 1, T satisfies condition (L) on C.

However, this mapping does not satisfy condition (SL) on C. Since $(0,0) \in Tx$ for any $x \in [0,1]\times[0,1]$,

$$\limsup_{n\to\infty} H(\{p_n\}, T(1,0)) \ge \limsup_{n\to\infty} \|p_n - (0,0)\|_2 = 1 > \limsup_{n\to\infty} \|p_n - (1,0)\|_2 = 0.$$

REMARK 2. Let $T : C \to \mathcal{P}_{c\ell}(C)$ be a mapping which satisfies condition (SL) on C. Let x_0 be a fixed point of T. Then taking $x_n = x_0$ for every positive integer n we have an a.f.p.s. (x_n) for T on C. From condition (B_s) we obtain

$$H(\{x_0\}, Tx_0) = \limsup_{n\to\infty} H(\{x_n\}, Tx_0) \le \limsup_{n\to\infty} \|x_n - x_0\| = 0.$$

Thus, $Tx_0 = \{x_0\}$, that is, x_0 is a *stationary point* (or *endpoint*) for T.

4. FIXED POINT RESULTS

Theorem 1. *Let C be a nonempty compact convex subset of a Banach space X and $T : C \to \mathcal{P}_{c\ell}(C)$ a mapping satisfying condition (L). Then, T has a fixed point.*

Proof. Since C is nonempty closed bounded convex and T-invariant, there exists an a.f.p.s. for T, say (x_n), in C. Since C is compact, there exists a subsequence (x_{n_j}) of (x_n) such that (x_{n_j}) converges to some $z \in C$. By condition (B) of Definition 4,

$$\limsup_{j\to\infty} d(x_{n_j}, Tz) \le \limsup_{j\to\infty} \|x_{n_j} - z\| = 0,$$

and hence, $d(z, Tz) = 0$, that is $z \in Tz$.

Remark 3. Notice that in the above proof, Condition (A) of Definition 4 can be replaced by the weaker assumption

(A′) There exists an a.f.p.s. for T on C.

Theorem 2. *Let C be a nonempty closed bounded and convex subset of a Banach space $(X, \|\cdot\|)$ which satisfies the Opial condition. Let $T : C \to \mathcal{P}_{cp}(C)$ be a mapping satisfying condition (L). Then, if (x_n) is an a.f.p.s. for T such that it converges weakly to $x \in C$, we have that x is a fixed point for T.*

Proof. Let (x_n) be an a.f.p.s. for T on C which converges weakly to $x \in C$. Since Tx is a compact set, there exists a point $w_n \in Tx$ such that

$$\|x_n - w_n\| = d(x_n, Tx).$$

Again from the compactness of the set Tx we may assume that $w_n \to w \in Tx$. Suppose that $w \neq x$. Then, since T satisfies condition (L) and X satisfies the Opial condition,

$$\begin{aligned}\limsup_{n\to\infty} \|x_n - w\| &\le \limsup_{n\to\infty} \|x_n - w_n\| + \limsup_{n\to\infty} \|w_n - w\| \\ &= \limsup_{n\to\infty} \|x_n - w_n\| = \limsup_{n\to\infty} d(x_n, Tx) \\ &\le \limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - w\|.\end{aligned}$$

This is a contradiction which leads to $x = w \in Tx$, which means that x is a fixed point for T.

Corollary 1. *Let C be a nonempty weakly compact and convex subset of a Banach space $(X, \|\cdot\|)$ which satisfies the Opial condition. Let $T : C \to \mathcal{P}_{cp}(C)$ be a mapping satisfying condition (L). Then, T has a fixed point in C.*

Since every nonexpansive mapping is an (L)-type mapping, the above result is a generalization of Theorem 3.2. in Lami Dozo [**18**]. On the other hand, one can note that in the above proof, Condition (A) of Definition 4 can be replaced by the weaker assumption (A′).

Remark 4. Let C be a nonempty closed bounded and convex subset of a Banach space $(X, \|\cdot\|)$. Let $T : C \to \mathcal{P}_{c\ell}(C)$ be a mapping satisfying condition (SL) on C. Then, if (x_n) is an a.f.p.s. for T, the level sets defined as

$$D_r := \{x \in C : \limsup_{n\to\infty} \|x_n - x\| \le r\}$$

are T-invariant whenever they are nonempty. Indeed, take $x \in D_r$, and take $y \in Tx$. It follows that

$$\begin{aligned}\limsup_{n\to\infty} \|x_n - y\| &\le \limsup_{n\to\infty}\ \sup\{\|x_n - w\| : w \in Tx\} \\ &= \limsup_{n\to\infty} H(\{x_n\}, Tx) \le \limsup_{n\to\infty} \|x_n - x\| \le r,\end{aligned}$$

and then $Tx \subseteq D_r$.

Let C be a nonempty closed and convex subset of a Banach space X, let (x_n) be a bounded sequence on X. Recall that the *asymptotic radius of* (x_n) *at* $x \in X$ is the number

$$r(x,(x_n)) = \limsup_n \|x - x_n\|.$$

In the same way, the *asymptotic radius of* (x_n) *in* C is the number

$$r(C,(x_n)) = \inf\{\limsup_n \|x_n - x\| : x \in C\} = \inf\{r(x,(x_n)) : x \in C\},$$

and the *asymptotic center* of (x_n) in C is defined as the (possibly empty) set

$$A(C,(x_n)) = \{x \in C : \limsup_n \|x_n - x\| = r(C,(x_n))\}.$$

Since $A(C,(x_n)) = \{x \in C : \limsup_n \|x_n - x\| \leq r(C,(x_n))\}$, a consequence of the above remark is the following.

Proposition 2. *Let C be a nonempty closed bounded and convex subset of a Banach space $(X, \|\cdot\|)$. Let $T : C \to \mathcal{P}_{c\ell}(C)$ be a mapping satisfying condition (SL). If (x_n) is an a.f.p.s. for T on C, then $A(C,(x_n))$ is T-invariant whenever it is nonempty.*

Corollary 2. *Let C be a nonempty closed bounded and convex subset of a Banach space $(X, \|\cdot\|)$. Let $T : C \to \mathcal{P}_{cp}(C)$ be a mapping satisfying condition (SL) on C. Then, if there exists an a.f.p.s. (x_n) for T on C such that $A(C,(x_n))$ is nonempty and compact, T has a fixed point in C.*

Proof. By our assumption $A(C,(x_n))$ is nonempty and compact. Since the mapping T satisfies condition (SL), the asymptotic center is T-invariant. From Theorem 4.1, given that the mapping T satisfies condition (L), T has a fixed point. $\square$

It is well known, (see [**12**] for instance), that $A(C,(x_n)) \neq \emptyset$ whenever C is weakly compact, and that if C is convex, then $A(C,(x_n))$ is convex. On the other hand, we do not know a complete characterization of those spaces in which asymptotic centers of bounded sequences are compact. Nevertheless, there are some partial answers. For example, k-uniformly convex Banach spaces satisfy that condition [**15**]. However, an example given by Kuczumov and Prus [**17**] shows that in nearly uniformly convex spaces, the asymptotic center of a bounded sequence with respect to a closed bounded convex subset is not necessarily compact.

Nevertheless, the level sets, as well as the asymptotic centers of bounded a.f.p. sequences, need not be invariant under mappings satisfying condition (L), as seen in the following example.

Example 2. Let $(X, \|\cdot\|)$ be the Banach space $(\mathbb{R}^2, \|.\|_\infty)$ where $\|.\|_\infty$ is the sup norm. If

$$C := \{(x_1, x_2) :\in \mathbb{R}^2 : |x_1| + |x_2| \leq 1\},$$

let $T : C \to \mathcal{P}_{cp,cv}(C)$ be the mapping given by

$$T((x_1, x_2)) := \{x_1\} \times [|x_1| - 1, 1 - |x_1|].$$

One can see that $T(x)$ is the largest vertical segment included in C which contains the point x. Hence the set of fixed points of T is C.

Let H_∞ be the Hausdorff metric associated to the sup metric $d_\infty(x,y) = \|x-y\|_\infty$. First let us point out that, if $A = \{x\} \times [a,b] \subset \mathbb{R}^2$, then it is straightforward to check that, given $\varepsilon > 0$,

$$A_\varepsilon := \{y \in \mathbb{R}^2 : \operatorname{dist}_\infty(y, A) < \varepsilon\} = (x-\varepsilon, x+\varepsilon) \times (a-\varepsilon, b+\varepsilon).$$

We claim that

$$H_\infty(T((x_1,x_2)), T((y_1,y_2))) \leq |x_1 - y_1|.$$

Indeed, let $\varepsilon' > \varepsilon := |x_1 - y_1|$.

As $|y_1| - |x_1| \leq |x_1 - y_1| < \varepsilon'$, then

$$\left.\begin{array}{l} |y_1| - \varepsilon' - 1 < |x_1| - 1 \\ 1 - |x_1| < 1 + \varepsilon' - |y_1| \end{array}\right\} \Rightarrow [|x_1| - 1, 1 - |x_1|] \subset (|y_1| - 1 - \varepsilon', 1 - |y_1| + \varepsilon').$$

This inclusion, together with the fact that $x_1 \in (y_1 - \varepsilon', y_1 + \varepsilon')$ yields

$$\begin{aligned} T((x_1,x_2)) &:= \{x_1\} \times [|x_1| - 1, 1 - |x_1|] \\ &\subset (y_1 - \varepsilon', y_1 + \varepsilon') \times (|y_1| - 1 - \varepsilon', 1 - |y_1| + \varepsilon'). \end{aligned}$$

Thus, for all $\varepsilon' > \varepsilon$,

$$T((x_1,x_2)) \subset (T((y_1,y_2)))_{\varepsilon'}$$

and, by the same argument,

$$T((y_1,y_2)) \subset (T((x_1,x_2)))_{\varepsilon'}.$$

Therefore, for all $\varepsilon' > \varepsilon$

$$H_\infty(T((x_1,x_2)), T((y_1,y_2))) \leq \varepsilon',$$

which proves the claim.

As a direct consequence we have that T is $\|.\|_\infty$-nonexpansive, that is

$$H_\infty(T((x_1,x_2)), T((y_1,y_2))) \leq |x_1 - y_1| \leq \|(x_1,x_2) - (y_1,y_2)\|_\infty.$$

From Proposition 1 the mapping T satisfies condition (L) with respect to the norm $\|\cdot\|_\infty$. However, it does not satisfy condition (SL) with respect to any equivalent norm $\|\cdot\|$ on $\mathbb{R}^2$. Indeed, taking $x_n \equiv x \notin \{(1,0), (-1,0)\}$, of course (x_n) is an a.f.p.s. for T, while

$$\limsup_n H(\{x_n\}, Tx) = H(\{x\}, Tx) > 0 = \limsup \|x_n - x\|.$$

Bearing in mind that any point is a fixed point for the mapping T, and hence any arbitrary sequence on C is an almost fixed point sequence for T, we can consider the level subsets with respect to the a.f.p.s. sequence $(x_n) \equiv (0_{\mathbb{R}^2})$ for any $0 \leq r < 1/2$

$$D_r = \{x \in C : \limsup_{n\to\infty} \|x_n - x\|_\infty \leq r\}.$$

Taking $x = (x_1, x_2) \in D_r$, we have that $(x_1, 1 - |x_1|) \in Tx \backslash D_r$ and then $Tx \nsubseteq D_r$, that is the level set D_r is not T-invariant. In particular, for $r = 0$ we obtain that

$$A(C, (x_n)) = D_0 = \{(0, 0)\}$$

is not T-invariant.

Let C be a nonempty weakly compact and convex subset of a Banach space X. Let $T : C \to \mathcal{P}_{c\ell}(C)$ be a mapping. Since K is weakly compact, a standard application of Zörn's Lemma yields a subset K of C which is minimal, that is, with no closed convex T-invariant nontrivial subsets of K.

The next proposition can be regarded as a multivalued weaker version of the classical Goebel-Karlovitz well known Lemma for single-valued nonexpansive mappings.

Proposition 3. *Let C be a nonempty weakly compact and convex subset of a Banach space X. Let $T : C \to \mathcal{P}_{c\ell}(C)$ be a mapping satisfying condition (SL). Let K be a minimal subset of C for T. Then, there exists $k \in \mathbb{R}$ such that for any a.f.p.s. (x_n) for T in K and any $x \in K$,*

$$\limsup_{n\to\infty} \|x_n - x\| = k.$$

Proof. Since T satisfies condition (SL), we can consider an almost fixed point sequence (x_n) for T on K, and the function

$$\begin{array}{cccl} \phi_{(x_n)}: & K & \to & \mathbb{R} \\ & x & \mapsto & \phi_{(x_n)}(x) = \limsup\limits_{n\to\infty} \|x_n - x\|. \end{array}$$

Suppose that such function is not constant. Hence, there are two points z_1, z_2 in the minimal set K, such that $\phi_{(x_n)}(z_1) = \alpha_1$, $\phi_{(x_n)}(z_2) = \alpha_2$ with $\alpha_1 < \alpha_2$. Consider the set,

$$K' = \left\{ z \in K : \phi_{(x_n)}(z) \le \frac{\alpha_1 + \alpha_2}{2} \right\}.$$

This subset K' of K is nonempty (since $z_1 \in K'$) closed bounded convex and T-invariant, by Remark 4. Moreover, $z_2 \notin K'$, since $\phi_{(x_n)}(z_2) = \alpha_2 > \dfrac{\alpha_1 + \alpha_2}{2}$, which contradicts the minimality of K. Consequently, the function $\phi_{(x_n)}$ is constant on the minimal set K.

Let us see now that, in addition, such constant does not depend on the almost fixed point sequence which we previously chose. Take two almost fixed point sequences (x_n) and (y_n) for T on K. We call $k_1 = \limsup\limits_{n\to\infty} \|x_n - x\|$ and $k_2 = \limsup\limits_{n\to\infty} \|y_n - x\|$ for any $x \in K$. Since K is weakly compact, after passing to subsequences if necessary we may assume that $x_n \rightharpoonup x_0 \in K$, $y_n \rightharpoonup y_0 \in K$. Since K is convex then $\dfrac{x_0 + y_0}{2} \in K$. Then, by the weak lower semicontinuity of the norm, we have

$$\left\|\frac{x_0+y_0}{2}-x_n\right\| \le \frac{1}{2}\|x_0-x_n\|+\frac{1}{2}\|y_0-x_n\| \le \frac{1}{2}\|x_0-x_n\|+\frac{1}{2}\liminf_{m\to+\infty}\|y_m-x_n\|$$
$$\le \frac{1}{2}\|x_0-x_n\|+\frac{1}{2}\limsup_{m\to\infty}\|y_m-x_n\| = \frac{1}{2}\|x_0-x_n\|+\frac{1}{2}k_2.$$

Taking upper limits on n we obtain

$$k_1=\limsup_{n\to\infty}\left\|\frac{x_o+y_0}{2}-x_n\right\| \le \frac{1}{2}\limsup_{n\to\infty}\|x_0-x_n\|+\frac{1}{2}k_2=\frac{1}{2}k_1+\frac{1}{2}k_2.$$

Considering now the analogous reasoning starting from $\left\|\frac{x_0+y_0}{2}-y_n\right\|$ and taking limits on n we also get

$$k_2 \le \frac{1}{2}k_1+\frac{1}{2}k_2,$$

and then $k_1=k_2$.

Remark 5. Notice that the constant k in the above result does not depend on the almost fixed point sequence (x_n) nor the point $x\in K$. We do not know if for this constant k, it happens that $k=\operatorname{diam}(K)$, as in the single-valued nonexpansive case.

Theorem 3. *Let X be a Banach space with normal structure. Let C be a nonempty weakly compact and convex subset of X. Let $T : C \to \mathcal{P}_{c\ell}(C)$ be a mapping satisfying condition (SL) on C. Then, T has a stationary point.*

Proof. Since C is weakly compact, it contains a closed convex T-invariant minimal subset, say K. If $K=\{x_0\}$, then x_0 is a stationary point of T. Otherwise, since T satisfies condition (SL) there exists an a.f.p.s. (x_n) for T in K. This sequence is either eventually constant, that is $x_n=x_0\in K$ for $n\ge n_0$, and hence x_0 is a fixed point of T (and hence a stationary point according Remark 2), or it is non constant. In this case, since X has normal structure, from Corollary 1 of [**6**], the real function $g : K\to[0,\infty)$ given by

$$g(x) := \limsup_{n}\|x-x_n\|$$

is not constant in $\operatorname{conv}\{x_n : n=1,2,\dots\}\subset K$, which contradicts the above proposition.

5. RELATED CLASSES OF MAPPINGS

Closely patterned on the nonexpansive type generalized mappings defined in [**11**] (which in turn extended the class of such mappings defined by T. Suzuki in [**25**]), in [**14, 1**] the following classes of multivalued mappings were introduced.

Definition 6 ([**14**]). *Given $\lambda\in(0,1)$, a mapping $T : C\to\mathcal{P}_{c\ell,b}(X)$ is said to satisfy condition (C_λ) on C if for any $x,y\in C$ such that $\lambda d(x,Tx)\le\|x-y\|$ then*

$$H(Tx,Ty)\le\|x-y\|.$$

The particular case $\lambda = \dfrac{1}{2}$ was also studied in [**1, 2**]. Of course, every nonexpansive mapping satisfies condition (C_λ) for each $\lambda \in (0,1)$.

Definition 7 ([**4**])**.** *For $\mu \geq 1$, a mapping $T : C \to \mathcal{P}_{b,c\ell}(X)$ is said to satisfy condition (E_μ) if, for any $x, y \in C$,*

$$d(x, Ty) \leq \mu d(x, Tx) + \|x - y\|.$$

We say that T satisfies condition (E) on C whenever T satisfies (E_μ) for some $\mu \geq 1$.

Every nonexpansive mapping satisfies condition (E_1). Moreover, if $x_0 \in C$ is a fixed point of the mapping $T : C \to \mathcal{P}_{b,c\ell}(X)$, and this mapping satisfies Condition (E_μ) on C, then for all $x \in C$,

$$d(x_0, Tx) \leq \|x_0 - x\|.$$

In other words, T is a *quasi-nonexpansive* mapping.

Also inspired on [**25, 11**], but in a different way, in [**5, 22**] the following class of mappings is studied.

Definition 8 ([**22**], see also [**5**])**.** *Let X be a metric space and C a nonempty subset of X. A mapping $T : C \to \mathcal{P}(X)$ is said to satisfy condition (C) if, for each $x, y \in C$ and $u_x \in T(x)$ such that $\dfrac{1}{2}d(x, u_x) \leq d(x, y)$, there exists $u_y \in T(y)$ such that*

$$d(u_x, u_y) \leq d(x, y).$$

Definition 9 ([**5**])**.** *Let X be a Banach space and $C \in \mathcal{P}(X)$. A mapping $T : C \to \mathcal{P}(X)$ is said to satisfy condition (E_μ) for some $\mu \geq 1$ if for each $x, y \in C$ and $u_x \in T(x)$ there exists $u_y \in T(y)$ such that*

$$\|x - u_y\| \leq \mu\|x - u_x\| + \|x - y\|.$$

Again mappings satisfying condition (E_μ) are quasi-nonexpansive provided that they have fixed points. Of course these concepts coincide with the respective standard ones in the single-valued case.

On the other hand, apart from the evident confusion of names, the relationship between the classes of mappings obtained by these two ways of generalization is unclear. In order to clarify the notation we refer to the two last classes as (C') and (E'_μ), respectively.

Of course, if T takes compact values it is easy to see that condition $(C_{1/2})$ implies condition (C') and that condition (E_μ) implies condition (E'_μ).

Remark 6. The names of the above classes of mappings are far from being unified in the literature. For instance, the family of mappings which is called $C_{1/2}$ in [**4**], is also referred is also referred to as a family that satisfies condition (D) in [**26**], condition (C) in [**1**] and condition (E) in [**2**].

On the other hand, the name 'Condition (E)' is used in [**26**] for the class of mappings satisfying the so called condition I in [**21, 23**]. This condition I refers to mappings in a very different setting.

First we study some direct relationships between these classes of mappings.

Proposition 4. *Let $T : C \to \mathcal{P}_{c\ell}(C)$ be a mapping satisfying condition $(C_{1/2})$. Then, T satisfies condition (E_3).*

Proof. Let $x \in C$. Since for any $z \in Tx$,

$$d(x, Tx) \leq \|x - z\|, \tag{1}$$

then $\frac{1}{2}d(x, Tx) \leq \frac{1}{2}\|x - z\| \leq \|x - z\|$. This implies, by condition $(C_{1/2})$ that for any $x \in C$ and any $z \in Tx$

$$H(Tx, Tz) \leq \|x - z\|. \tag{2}$$

Consider now $x, y \in C$, $z \in Tx$. Then we claim that either $\frac{1}{2}d(x, Tx) \leq \|x - y\|$ or $\frac{1}{2}H(Tx, Tz) \leq \|y - z\|$. On the contrary, bearing in mind (1) and (2),

$$\begin{aligned} \|x - z\| \leq \|x - y\| + \|y - z\| &< \frac{1}{2}d(x, Tx) + \frac{1}{2}H(Tx, Tz) \\ &\leq \frac{1}{2}\|x - z\| + \frac{1}{2}\|x - z\| = \|x - z\|, \end{aligned}$$

which is a contradiction.

By our claim, in the first case, since $\frac{1}{2}d(x, Tx) \leq \|x - y\|$, we have by condition $(C_{1/2})$ that $H(Tx, Ty) \leq \|x - y\|$ and hence

$$d(x, Ty) \leq d(x, Tx) + H(Tx, Ty) \leq d(x, Tx) + \|x - y\| \leq 3d(x, Tx) + \|x - y\|.$$

In the second case, $\frac{1}{2}H(Tx, Tz) \leq \|y - z\|$ and then

$$\begin{aligned} \frac{1}{2}d(z, Tz) &\leq \frac{1}{2}\sup_{w \in Tx} d(w, Tz) \\ &\leq \frac{1}{2}\max\{\sup_{w \in Tx} d(w, Tz), \sup_{w \in Tz} d(w, Tx)\} = \frac{1}{2}H(Tx, Tz) \leq \|y - z\| \end{aligned}$$

and applying condition $(C_{1/2})$

$$H(Ty, Tz) \leq \|z - y\|. \tag{3}$$

Then, bearing in mind (1), (2) and (3),

$$\begin{aligned} d(x, Ty) &\leq d(x, Tx) + H(Tx, Ty) \\ &\leq d(x, Tx) + H(Tx, Tz) + H(Tz, Ty) \leq \|x - z\| + \|x - z\| + \|z - y\| \\ &\leq 2\|x - z\| + \|z - x\| + \|x - y\| = 3\|x - z\| + \|x - y\|. \end{aligned}$$

Since this is accomplished for any $z \in Tx$, by taking the infimum

$$d(x, Ty) \leq 3d(x, Tx) + \|x - y\|,$$

T satisfies condition (E_3). □

In [**5**, Lemma 3.2] a similar result for conditions (C') and (E'_3) is given.

Proposition 5. *Let $T : C \to \mathcal{P}_{b,c\ell}(X)$ be a mapping satisfying conditions (E) and (A) on C. Then T satisfies condition (L) on C.*

Proof. Let (x_n) be any a.f.p.s. for T in C. Let $x \in C$. Then,

$$\begin{aligned}\limsup_{n\to\infty} d(x_n, Tx) &\leq \limsup_{n\to\infty}(\mu d(x_n, Tx_n) + \|x_n - x\|)\\ &\leq \mu \limsup_{n\to\infty} d(x_n, Tx_n) + \limsup_{n\to\infty} \|x_n - x\| = \limsup_{n\to\infty} \|x_n - x\|\end{aligned}$$

Proposition 6. *Let $T : C \to \mathcal{P}_{c\ell}(C)$ be a mapping which fails to satisfy condition (E) on C. Then T contains an a.f.p.s. on C, that is T satisfies condition (A') in Remark* 3.

Proof. Since T fails to satisfy condition (E) on C, for every positive integer n there exist $x_n, y_n \in C$ such that

$$d(x_n, Ty_n) > n\, d(x_n, Tx_n) + \|x_n - y_n\|.$$

Then,

$$0 \leq n\, d(x_n, Tx_n) < d(x_n, Ty_n) - \|x_n - y_n\| \leq \operatorname{diam}(C).$$

Therefore, $\limsup_n d(x_n, Tx_n) \leq \lim_n \dfrac{\operatorname{diam}(C)}{n} = 0$, that is, (x_n) is an a.f.p.s. for T on C.

Corollary 3. *Let $T : C \to \mathcal{P}_{c\ell}(C)$ be a mapping satisfying on C both condition (C_λ), for some $\lambda \in (0,1)$, and condition (E) on C. Then T satisfies condition (L) on C.*

Proof. By Remark 1, since T satisfies condition (C_λ), then the mapping T also satisfies condition (A) on C. Since T also satisfies condition (E) on C, the result now follows from Proposition 5.

Corollary 4. *Let $T : C \to \mathcal{P}_{c\ell}(C)$ be a mapping satisfying condition $(C_{1/2})$, then it satisfies condition (L).*

Proof. By Proposition 4, T satisfies condition (E_3), and then, by the corollary above, T satisfies condition (L).

Proposition 7. *Let $C \in \mathcal{P}_{c\ell}(X)$, $T : C \to \mathcal{P}_{c\ell}(X)$ a mapping which satisfies conditions (A) and (E'_μ), for some $\mu \geq 1$, on C. Then T satisfies condition (L).*

Proof. Consider an almost fixed point sequence (x_n) for T on C. For each positive integer n there exists $u_n \in Tx_n$ such that

$$d(x_n, Tx_n) + \frac{1}{n} > \|x_n - u_n\|.$$

Since T satisfies condition (E'_μ), for x_n, u_n and $y \in C$, there exists $u_{y_n} \in Ty$ such that

$$d(x_n, Ty) \leq \|x_n - u_{y_n}\| \leq \mu\|x_n - u_n\| + \|x_n - y\| < \mu\left(d(x_n, Tx_n) + \frac{1}{n}\right) + \|x_n - y\|,$$

and consequently

$$\begin{aligned}\limsup_{n\to\infty} d(x_n, Ty) &\leq \mu\left(\limsup_{n\to\infty}\left(d(x_n, Tx_n) + \frac{1}{n}\right)\right) + \limsup_{n\to\infty}\|x_n - y\| \\ &= \limsup_{n\to\infty}\|x_n - y\|,\end{aligned}$$

that is, T satisfies condition (L). □

The following examples are given to compare and even separate (L)-type multivalued mappings from the above classes of mappings.

Example 3. Let $(X, \|\cdot\|)$ be a Banach space and let B_X its closed unit ball. Let $T : B_X \to \mathcal{P}_{cl,cv}(B_X)$ be defined as

$$T(x) = \begin{cases} B_X & x = 0_X \\ B\left[-\frac{x}{\|x\|}, 1 - \|x\|\right] \cap B_X & x \neq 0_X, \end{cases}$$

(where it is assumed that $B[y, 0] = \{y\}$ if $y \in X$).

We show that for this mapping T it holds that

1. It satisfies condition E_μ on B_X for every $\mu \geq 1$.
2. If $\dim(X) < \infty$ then it satisfies condition E'_μ on B_X for every $\mu \geq 1$.
3. It fails to satisfy condition (C_λ) on B_X for every $\lambda \in (0, 1)$.
4. It satisfies condition (L) on B_X.
5. It fails to satisfy condition (SL) on B_X.
6. It fails to satisfy condition (C') on B_X.

For all $x \in B_X$ one has that $d(x, T(x)) = 2\|x\|$ and $-x \in T(x)$. Since for every $x, y \in B_X$,

$$\begin{aligned}d(x, T(y)) &\leq d(x, -y) = \|x + y\| \leq \|x\| + \|y\| \\ &\leq \|x\| + \|y - x\| + \|x\| = d(x, T(x)) + \|x - y\|,\end{aligned}$$

the mapping T satisfies condition (E_1) on B_X. If $\dim(X) < \infty$ then T is compact-valued and hence T also satisfies condition (E'_1).

On the other hand, for every $\lambda \in (0,1)$, condition (C_λ) for T reads

$$\lambda 2\|x\| \leq \|x-y\| \Rightarrow H(T(x),T(y)) \leq \|x-y\|.$$

If we take $x \in B_X$ such that $\|x\| = \frac{1}{2}$ and $y = -x$, it is obvious that $\lambda = \lambda 2\|x\| \leq \|x-y\| = 1$ while

$$H(T(x),T(-x)) = \frac{3}{2}.$$

Hence, T does not satisfy condition (C_λ) on B_X. In particular T fails to be nonexpansive on B_X.

Finally, the unique T-invariant closed convex subset of B_X is just B_X. Since $d(x,T(x)) = 2\|x\|$, a sequence (x_n) in B_X is an a.f.p.s. for T if and only if $x_n \to 0_X$. Then, for every $x \in B_X$,

$$\limsup_{n\to\infty} d(x_n,T(x)) = d(0_X,T(x)) = \|x\| = \limsup_{n\to\infty} \|x_n - x\|.$$

This means that T satisfies condition (L) on B_X.

Moreover, if $\|x\| < 1$,

$$\limsup_{n\to\infty} H(x_n,T(x)) = H(0_X,T(x)) = 1 > \limsup_{n\to\infty} \|x_n - x\| = \|x\|,$$

that is, T is not a (SL)-type mapping.

Finally, take $x \in B_X$ with $\|x\| = \frac{1}{2}$, and choose $y = u_x = -2x \in T(x)$. One has that $T(y) = \{-y\} = \{2x\}$ and the unique element in $T(y)$ is just $u_y = 2x$ for which

$$d(u_x,u_y) = d(-2x,2x) = 2 > \frac{3}{2} = d(x,-2x) = d(x,y).$$

Hence T fails to satisfy condition (C') on B_X.

The following example shows that the converse of Proposition 5 is not generally true.

Example 4. Let $f : [-1,1] \to [-1,1]$ be the mapping given by

$$f(x) = \begin{cases} \frac{x}{1+|x|} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Let $T : [-1,1] \to \mathcal{P}_{c\ell,cv}([-1,1])$ be defined by

$$T(x) = \left[\frac{1}{2}f(x) \wedge f(x), \frac{1}{2}f(x) \vee f(x)\right],$$

where $a \wedge b$, $a \vee b$ denote respectively the minimum and the maximum of the real numbers a, b.

It is easy to check that 0 is the only fixed point of T.

For this mapping T we will see that:

1. It fails to satisfy condition E_μ on $[-1,1]$ for every $\mu \geq 1$.
2. It fails to satisfy condition $C_{1/2}$ on $[-1,1]$.
3. It satisfies condition (L) on $[-1,1]$.
4. It satisfies condition (SL) on $[-1,1]$.

If we take for each positive integer $x_n := \dfrac{1}{2\pi n + \pi/2}$ and $y_n := \dfrac{1}{2\pi n}$, then we have $T(x_n) = \left[\dfrac{1}{2}\dfrac{x_n}{1+|x_n|}, \dfrac{x_n}{1+|x_n|}\right]$. Since $\dfrac{x_n}{1+|x_n|} < x_n$, we can choose $u_{x_n} = \dfrac{x_n}{1+|x_n|}$ and

$$|x_n - u_{x_n}| = d(x_n, T(x_n)) = x_n - \frac{x_n}{1+|x_n|}.$$

Moreover, $T(y_n) = \{0\}$, and for every $u_{y_n} \in T(y_n)$ we have.

$$\begin{aligned}\frac{|x_n - u_{y_n}| - |x_n - y_n|}{|x_n - u_{x_n}|} &= \frac{x_n - (y_n - x_n)}{x_n - \dfrac{x_n}{1+|x_n|}} \\ &= \frac{x_n - (y_n - x_n)}{\dfrac{x_n^2}{1+x_n}} = \frac{(1+x_n)\left(2 - \dfrac{y_n}{x_n}\right)}{x_n} \to +\infty\end{aligned}$$

Consequently, the mapping T does not satisfy condition (E'_μ) on $[-1,1]$ for any $\mu \geq 1$. Since T is compact-valued then T fails condition (E_μ) on $[-1,1]$ for any $\mu \geq 1$. Hence T fails conditions (C') and $(C_{1/2})$.

On the other hand, let $D \subset [-1,1]$ be a closed convex T-invariant set. We claim that $0 \in D$. If $D = \{0\}$ our claim is obvious. Starting from $x_0 \in D$, choose $x_{n+1} = f(x_n) \in T(x_n)$. In this way we have built a sequence in D. Our claim is also obvious if there exists a positive integer n_0 such that $x_{n_0} = 0$. Otherwise for all positive integers n, $x_n \neq 0$, and

$$x_{n+1} = \frac{x_n}{1+|x_n|} \sin\left(\frac{1}{x_n}\right).$$

Hence

$$|x_{n+1}| = \frac{|x_n|}{1+|x_n|}\left|\sin\left(\frac{1}{x_n}\right)\right| = \frac{|x_n|}{1+|x_n|}\left|\sin\left(\frac{1}{|x_n|}\right)\right|.$$

Since $|f(x)| \leq |x|$ for every $x \in [-1,1]$, there exists $\lim\limits_{n\to\infty} |x_n| =: a$. If $a = 0$ then $\lim x_n = 0 \in D$ and again our claim holds. Finally, if $a \neq 0$ then, from the above equality we obtain that

$$a = \frac{a}{1+a}\left|\sin\left(\frac{1}{|a|}\right)\right|,$$

which implies that

$$1 + a = \left|\sin\left(\frac{1}{|a|}\right)\right|,$$

a contradiction for $a > 0$.

Thus, $0 \in D$ for every closed T-invariant subset of $[-1, 1]$. Since $T(0) = \{0\}$, the sequence $(x_n) \equiv (0)$ is an a.f.p.s. in D for T, and the mapping T satisfies condition (A).

We claim that (x_n) is an a.f.p.s. for T if and only if $x_n \to 0$. It is obvious that (x_n) is an a.f.p.s. if $x_n \to 0$. If (x_n) is an a.f.p.s. and $x_n \not\to 0$ we may suppose that $x_n \to x \neq 0$. Since

$$d(x_n, T(x_n)) = \min\left\{d(x_n, f(x_n)), d\Big(x_n, \frac{1}{2}f(x_n)\Big)\right\},$$

and f is continuous, then

$$\min\left\{d(x_n, f(x_n)), d\Big(x_n, \frac{1}{2}f(x_n)\Big)\right\} \to \min\left\{d(x, f(x)), d\Big(x, \frac{1}{2}f(x)\Big)\right\}.$$

But the only fixed point for f or for $\frac{1}{2}f$ is just $x = 0$. Then

$$\min\left\{d(x, f(x)), d\Big(x, \frac{1}{2}f(x)\Big)\right\} > 0,$$

and we have a contradiction because $d(x_n, T(x_n)) \to 0$.

Let (x_n) be an a.f.p.s. for T, and $x \in [-1, 1]$. If $x_n \in T(x)$ then $d(x_n, T(x)) = 0$. If $x_n \notin T(x)$, $d(x_n, T(x)) = \min\left\{d(x_n, f(x)), d\Big(x_n, \frac{1}{2}f(x)\Big)\right\}$. In any case

$$\limsup d(x_n, T(x)) = \min\left\{d(0, f(x)), d\Big(0, \frac{1}{2}f(x)\Big)\right\} \leq |x| = \limsup |x_n - x|,$$

which yields that T is an (L)-type mapping on $[-1, 1]$.

Finally, if (x_n) is an a.f.p.s. for T, and $x \in [-1, 1]$, $H(\{x_n\}, T(x)) = \max\left\{d(x_n, f(x)), d\Big(x_n, \frac{1}{2}f(x)\Big)\right\}$, and

$$\limsup H(\{x_n\}, T(x)) = \max\left\{d(0, f(x)), d\Big(0, \frac{1}{2}f(x)\Big)\right\} \leq |x| = \limsup |x_n - x|,$$

which yields that T is an (SL)-type mapping on $[-1, 1]$.

5.1. Remarks on other classes of multivalued mappings

In 2005, N. Shahzad and A. Lone introduced in [**24**] a class of multivalued mappings which is seemingly near the class of (SL) mappings.

Definition 10. *Let C be a nonempty weakly compact convex subset of X. The continuous map $T : C \to \mathcal{P}_{cp}(X) \cap \mathcal{P}_{cv}(X)$ is called subsequentially limit-contractive if for every a.f.p.s. (x_n) in C we have*

$$\limsup H(T(x_n), T(x)) \leq \limsup \|x_n - x\|$$

for all $x \in A(C, (x_n))$, the asymptotic center of (x_n) in C.

Of course, for the consistence of this definition, it is required that the domain C of such a mapping needs to be a nonempty weakly compact convex subset of X. Otherwise the asymptotic center of (x_n) in C could be empty. Even more, for the consistence of this definition, the existence of one a.f.p.s. in C should be required for T. Notice that our condition (SL) does not require weak compactness of the domain C nor continuity of the mapping T.

Very recently, S. Dhompongsa and N. Nanan [**10**] introduced two further classes of setvalued mappings. The first one, which is inspired on a class of mappings defined in [**7**], (which in turn properly contains the class of mappings defined in [**25**]) is:

Definition 11. *Let C be a nonempty subset of a Banach space X. A mapping $T : C \to \mathcal{P}_{b,c\ell}(X)$ is said to satisfy condition (*) if*

1. *T has an a.f.p.s. in C,*
2. *T has an a.f.p.s. in $A(C, (x_{n'}))$ for some subsequence $(x_{n'})$ of any given a.f.p.s. (x_n) for T in C.*

The second one, which is not too far from the single-valued case in [**20**], is the following.

Definition 12. *Let $\mathcal{U}$ be a free ultrafilter defined on $\mathbb{N}$. Let C be a bounded closed and convex subset of a Banach space X. A mapping $T : C \to \mathcal{P}_{b,c\ell}(X)$ is said to satisfy condition (**) if it fulfills the following conditions*

$$T \text{ has an a.f.p.s. in } C; \tag{4}$$

If (x_n) is an a.p.f.s. for T in C and $x \in C$, then

$$\lim_{n \to \mathcal{U}} H(Tx_n, Tx) \leq \lim_{n \to \mathcal{U}} \|x_n - x\|. \tag{5}$$

Notice that assumption (4) is just condition (A$'$) and hence it is weaker than condition (A) in our definitions. Nevertheless, if there exists a T-invariant subset, say D, of C, such that condition (4) is not satisfied in D, then T could satisfy Definition 12 in C. This means that, among other things, we cannot use this type of condition in order to obtain results such as Theorem 4.3.

Next, we see that the use of ultrafilters in (5) is not necessary, in the following sense: condition (5) is equivalent to

$$\limsup_n H(Tx_n, Tx) \leq \limsup_n \|x_n - x\|. \tag{6}$$

Indeed, if there exists (x_n) an a.f.p.s. for T in C such that

$$\limsup_n H(Tx_n, Tx) > \limsup_n \|x_n - x\|,$$

then, there exists a subsequence (x_{n_k}) of (x_n) such that

$$\lim_k H(Tx_{n_k}, Tx) = \limsup_n H(Tx_n, Tx) > \limsup_n \|x_n - x\|,$$

then

$$\lim_k H(Tx_{n_k}, Tx) > \limsup_n \|x_n - x\| \geq \limsup_k \|x_{n_k} - x\|.$$

Since there exists a subsequence $(x_{n_{k_s}})$ of (x_{n_k}) such that

$$\limsup_k \|x_{n_k} - x\| = \lim_s \|x_{n_{k_s}} - x\|,$$

then

$$\lim_s H(Tx_{n_{k_s}}, Tx) > \lim_s \|x_{n_{k_s}} - x\|.$$

In other words, if (6) does not hold for some a.f.p.s. (x_n) in C and some $x \in C$, then there exists a subsequence, say (x_{n_k}), such that

$$\lim_k H(Tx_{n_k}, Tx) > \lim_k \|x_{n_k} - x\|.$$

Moreover, if (x_n) is an a.f.p.s. for T in C such that it admits a subsequence, say (x_{n_k}), such that

$$\lim_k H(Tx_{n_k}, Tx) > \lim_k \|x_{n_k} - x\|,$$

then T does not satisfy (5) for every free ultrafilter $\mathcal{U}$ defined on $\mathbb{N}$. Indeed, take $y_k = x_{n_k}$ for $k = 1, 2, \ldots$. Since (y_k) is again an a.f.p.s. for T, then, for every free ultrafilter $\mathcal{U}$ defined on $\mathbb{N}$,

$$\lim_{k\to\mathcal{U}} H(Ty_k, Tx) = \lim_k H(Tx_{n_k}, Tx) > \lim_k \|x_{n_k} - x\| = \lim_{k\to\mathcal{U}} \|y_k - x\|,$$

and T does not satisfy condition (5), as claimed.

Conversely, if it does not have Condition (5), that is, if there exists a free ultrafilter $\mathcal{U}$ defined on $\mathbb{N}$ such that

$$a := \lim_{n\to\mathcal{U}} H(Tx_n, Tx) > \lim_{n\to\mathcal{U}} \|x_n - x\| =: b,$$

for some a.f.p.s. (x_n) for T in C and some $x \in C$, then for every $\varepsilon > 0$, the set

$$\{n \in \mathbb{N} : |H(T(x_n), Tx) - a| < \varepsilon\} \cap \{n \in \mathbb{N} : |\|x_n - x\| - b| < \varepsilon\} \in \mathcal{U}.$$

Moreover, $\mathcal{U}$ cannot contain finite sets. Then there exists the natural number

$$\min(\{n \in \mathbb{N} : |H(T(x_n), Tx) - a| < \varepsilon\} \cap \{n \in \mathbb{N} : |\|x_n - x\| - b| < \varepsilon\}).$$

Hence we may obtain a subsequence (x_{n_k}) of (x_n) such that

$$a = \lim_k H(Tx_{n_k}, Tx) > \lim_k \|x_{n_k} - x\| = b.$$

That is, equality (5) does not hold if and only if there exists an a.f.p.s. (x_n) such that

$$\lim_k H(Tx_{n_k}, Tx) > \lim_k \|x_{n_k} - x\|,$$

for some subsequence (x_{n_k}). In particular, taking $y_k = x_{n_k}$ for $k = 1, 2, \ldots$, since (y_k) is again an a.f.p.s. for T, then,

$$\limsup_k H(Ty_k, Tx) = \lim_k H(Tx_{n_k}, Tx) > \lim_k \|x_{n_k} - x\| = \limsup_k \|y_k - x\|,$$

which means that it fails to satifsy (6).

Remark 7. Notice that in our Definition 3.5 it is required that for all a.f.p.s. (x_n) for T in C and any $x \in C$, then

$$\limsup_n H(\{x_n\}, Tx) \leq \limsup_n \|x_n - x\|. \tag{7}$$

while in Definition 5.7 it it is required that for all a.f.p.s. (x_n) for T in C and any $x \in C$, then

$$\limsup_n H(Tx_n, Tx) \leq \limsup_n \|x_n - x\|. \tag{8}$$

First, notice that every nonexpansive mapping satisfies condition (8). Hence the nonexpansive mapping considered in Example 1 satisfies condition (8), but not condition (7). Hence condition (8) does not imply condition (7).

On the other hand, we do not know whether the converse statement holds, that is, if condition (7) implies condition (8).

Moreover, if a mapping $T : C \to \mathcal{P}_{c\ell}(C)$ satisfies condition (8), then it is easy to check that it satisfies the following condition.

For all a.f.p.s. (x_n) for T in C and any $x \in C$, then

$$\limsup_n d(x_n, Tx) \leq \limsup_n \|x_n - x\|. \tag{9}$$

The converse statement does not hold, because the mapping considered in Example 3 satisfies condition (9) but it fails condition (8). Indeed, for the a.f.p.s. $(x_n) \equiv 0_X$, if $x \in C$ one has

$$\limsup_{n\to\infty} H(T(x_n), T(x)) = H(B_X, T(x)) = 1 + \|x\| > \|x\| = \limsup_{n\to\infty} \|x_n - x\|.$$

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