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Fixed point theory for 1-set contractive and pseudocontractive mappings

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ABSTRACT

The purpose of this paper is to study the existence and uniqueness of fixed point for a class of nonlinear mappings defined on a real Banach space, which, among others, contains the class of separate contractive mappings, as well as to see that an important class of 1-set contractions and of pseudocontractions falls into this type of nonlinear mappings. As a particular case, we give an iterative method to approach the fixed point of a nonexpansive mapping. Later on, we establish some fixed point results of Krasnoselskii type for the sum of two nonlinear mappings where one of them is either a 1-set contraction or a pseudocontraction and the another one is completely continuous, which extend or complete previous results. In the last section, we apply such results to study the existence of solutions to a nonlinear integral equation.

1. Introduction

From a mathematical point of view, many problems arising form diverse areas of natural science involve the existence of solutions of nonlinear equations with either the form

$$Au = u, \quad u \in M, \text{ or } Au + Bu = u, \quad u \in M, \tag{1}$$

where M is a closed and convex subset of a Banach space X, and $A, B : M \to X$ are nonlinear mappings. For instance, in [3,6,9,10,14,16–18,25,26,35,36] several of such results are applied to boundary value problems and for determining solutions of nonlinear integral equations. Fixed point Theory plays an important role in order to solve Eqs. (1). This Theory has two main branches: On the one hand we may consider the results that are obtained by using topological properties and on the other hand those results which may be deduced from metric assumptions.

Regarding the topological branch, the main two theorems are Brouwer's Theorem and its infinite dimensional version, Schauder's fixed point theorem (see for example [10]). In both theorems compactness plays an essential role. In 1955, Darbo [11] extended Schauder's theorem to the setting of noncompact operators, introducing the notions of k-set-contraction with $0 \leqslant k < 1$.

Concerning the metric branch, the most important metric fixed point result is the Banach contraction principle (see Chapter 1 in [23] where Kirk gives an overview of the sharpening of this result). Since 1965 considerable effort has been done to study the fixed point theory for nonexpansive mappings (for instance see [19,23]).

Although historically the two branches of the fixed point theory have had a separated development, in 1958, Krasnoselskii [24] establishes that the sum of two operators $A + B$ has a fixed point in a nonempty closed convex subset C of a Banach space

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$(X, \|.\|)$,whenever: (i) $Ax + By \in C$ for all $x, y \in C$, (ii) C is bounded, (iii) A is completely continuous on C and (iv) B is a k-contraction on X with $0 \leqslant k < 1$.

This result combines both Banach contraction principle and Schauder's fixed point theorem and thus it is a blend of the two branches. Nevertheless, it is not hard to see that Krasnoselskii's theorem is a particular case of Darbo's theorem. Namely, it appears that $A + B$ is a k-set contraction with respect to the Kuratowski measure of noncompactness. In 1967, Sadovskii [34] gave a more general fixed point result than Darbo's theorem using the concept of condensing map, see [15,20,32,35] for a sharpening of such results. In this framework, it is well known that for the limit case i.e., mappings which are 1-set contraction for some measure of noncompactness, it is not possible to obtain a similar fixed point result like in the above cases. Indeed, it was proved in [4] that given an infinite dimensional Banach space $(X, \|\cdot\|)$ and for an arbitrary measure of noncompactness Φ on X there exists a fixed point free $\Phi - 1$-set contraction self-mapping of the unit ball of X (also see [29]). Therefore, to develop a theory of fixed point for 1-set contractions similar to the same for nonexpansive mappings does not work. Nevertheless, there exists a degree theory for semiclosed 1-set contraction mappings (for instance see [15,20]).

In this paper, we study the existence and uniqueness of fixed point for a class of nonlinear mappings, which, among others, contains the class of separate contractive mappings [27]. Moreover, we show, without invoking degree theory, that an important class of semiclosed 1-set contractions as well as an important class of pseudocontractive mappings, has a unique fixed point (for instance, this allows us to generalize Proposition 3.4 in [17]). Later we use the results obtained to establish new fixed point results of Krasnoselskii type for the sum of two nonlinear mappings where one of them is either a 1-set contraction, or a pseudocontraction and the other one is completely continuous, which extend or complete previous results (for instance, we give generalizations of Theorem 2.1 in [26], Theorem 2.1 in [27] as well as a generalization of Theorem 3.7 in [17]). Finally, we apply such results to study the existence of solutions to a nonlinear integral equation of the form

$$u(t) = g(t, u(t)) + \int_0^t f(\tau, u(\tau))d\tau, \quad u \in C(0, T; X),$$

where X is Banach space and $g : [0, T] \times X \to X$ is not necessary a Lipschitzian map with respect to the second variable. Thus, the case where $g(t, \cdot)$ is a separate contraction is a particular case of our framework.

2. Preliminaries

Throughout this paper we assume that $(X, \|.\|)$ is a real Banach space. As usual, we will denote by $B_R(x_0)$ and $S_R(x_0)$, the closed ball, and the sphere, with radius R and center $x_0 \in X$, respectively.

Definition 2.1. Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{B}(X)$ the family of bounded subsets of X. By a measure of noncompactness on X, we mean a function $\Phi : \mathcal{B}(X) \to \mathbb{R}^+$ satisfying:

(1) $\Phi(\Omega) = 0$ if and only if Ω is relatively compact in X.
(2) $\Phi(\overline{\Omega}) = \Phi(\Omega)$.
(3) $\Phi(\text{conv}(\Omega)) = \Phi(\Omega)$, for all bounded subsets $\Omega \in \mathcal{B}(X)$, where conv denotes the convex hull of Ω.
(4) for any subsets $\Omega_1, \Omega_2 \in \mathcal{B}(E)$ we have

$$\Omega_1 \subseteq \Omega_2 \Rightarrow \Phi(\Omega_1) \leqslant \Phi(\Omega_2),$$

(5) $\Phi(\Omega_1 \cup \Omega_2) = \max\{\Phi(\Omega_1), \Phi(\Omega_2)\}$, $\Omega_1, \Omega_2 \in \mathcal{B}(X)$.
(6) $\Phi(\lambda\Omega) = |\lambda|\Phi(\Omega)$ for all $\lambda \in \mathbb{R}$ and $\Omega \in \mathcal{B}(X)$.
(7) $\Phi(\Omega_1 + \Omega_2) \leqslant \Phi(\Omega_1) + \Phi(\Omega_2)$.

The most important examples of measures of noncompactness are the *Kuratowski measure of noncompactness* (or set measure of noncompactness)

$$\alpha(\Omega) = \inf\{r > 0 : \Omega \text{ may be covered by finitely many sets of diameter } \leqslant r\}$$

and the *Hausdorff measure of noncompactness* (or ball measure of noncompactness)

$$\beta(\Omega) = \inf\{r > 0 : \text{ there exists a finite r-net for } \Omega \text{ in } X\}.$$

A detailed account of theory and applications of measures of noncompactness may be found in the monographs [2,5] (see also [3]).

Let $\mathcal{W}(X)$ be the subset of $\mathcal{B}(X)$ consisting of all weakly compact subsets of X. Recall that the notion of the measure of weakly non-compactness was introduced by De Blasi [12] and it is the map $w : \mathcal{B}(X) \to [0, \infty[$ defined by

$$\omega(M) := \inf\{r > 0 : \text{ there exists } W \in \mathcal{W}(X) \text{ with } M \subseteq W + B_r\}$$

for every $M \in \mathcal{B}(X)$. It is well known that w fulfills the conditions of Definition 2.1 replacing conditions (1) and (2) of such definition by

(1) $\omega(M_1) = 0$ if and only if, $\overline{M_1}^w \in \mathcal{W}(X)$ ($\overline{M_1}^w$ means the weak closure of M_1),
(2) $\omega(\overline{M_1}^w) = \omega(M_1)$,

respectively.

Definition 2.2. Let Φ be a measure of non-compactness on X and let D be a nonempty subset of X. A mapping $T : D \to X$ is said to be a $\Phi - k$-set contraction (w–k-set contraction resp.), $k \in (0, 1]$, if T is continuous and if, for all bounded subsets C of D, $\Phi(T(C)) \leqslant k\Phi(C)$ ($w(T(C)) \leqslant kw(C)$ resp.). T is said to be Φ-condensing (w-condensing resp.) if T is continuous and $\Phi(T(A)) < \Phi(A)$ ($w(T(A)) < w(A)$ resp.) for every bounded subset A of D with $\Phi(A) > 0$ ($w(A) > 0$ resp.).

- A mapping $T : D(T) \subseteq X \to X$ is said to be nonexpansive if the inequality $\|T(x) - T(y)\| \leqslant \|x - y\|$ holds for every $x, y \in D(T)$. Recall that a Banach space X is said to have the fixed point property for nonexpansive mappings (FPP for short) if for each nonempty bounded closed and convex subset C of X, every nonexpansive self-mapping T has a fixed point (see [19,23]).
- The mapping T is said to be pseudocontractive if for every $x, y \in D(T)$ and for all $r > 0$, the inequality

$$\|x - y\| \leqslant \|(1 + r)(x - y) + r(Ty - Tx)\|$$

holds. Pseudocontractive mappings are easily seen to be more general than nonexpansive ones. The interest in these mappings also stems from the fact that they are firmly connected to the well known class of accretive mappings. Specifically, T is pseudocontractive if and only if $I - T$ is accretive, where I is the identity mapping.

Recall that a mapping $A : D(A) \to X$ is said to be *accretive* if the inequality $\|x - y + \lambda(Ax - Ay)\| \geqslant \|x - y\|$ holds for all $\lambda \geqslant 0$, $x, y \in D(A)$. If, in addition, $\Re(I + \lambda A)$ (i.e., the range of the operator $I + \lambda A$) is for one, hence for all, $\lambda > 0$, precisely X, then A is called m-accretive. If $\overline{D(A)} \subseteq \bigcap_{\lambda>0} \Re(I + \lambda A)$, then A is said to have the range condition. Accretive operators were introduced by Browder [7] and Kato [22] independently.

We say that the mapping $T : D(T) \to X$ is weakly inward on $D(T)$ if

$$\lim_{\lambda\to 0^+} d((1 - \lambda)x + \lambda T(x), D(T)) = 0$$

for all $x \in D(T)$. Such condition is always weaker than the assumption of T mapping the boundary of $D(T)$ into $D(T)$. Recall that if $A : D(A) \to X$ is a continuous accretive mapping, $D(A)$ is convex and closed and $I - A$ is weakly inward on $D(A)$, then A has the range condition, (see [30]).

The following theorems will be the key in the proof of some of our results. The first one was proved by Sadovskii [34] in 1967. In 1955 Darbo [11] proved the same result for $\Phi - k$-set contractions, $k < 1$. Such mappings are obviously Φ-condensing. The second one is a sharpening of the first one and it is due to Petryshyn [32].

Theorem 2.1 (*Darbo–Sadovskii*). *Suppose M is a nonempty bounded closed and convex subset of a Banach space X and suppose $T : M \to M$ is Φ-condensing. Then T has a fixed point.*

Theorem 2.2 (*Petryshyn*). *Let C be a closed, convex subset of a Banach space X such that $0 \in C$. Consider $T : C \to C$ a Φ-condensing mapping. If there exists $r > 0$ such that $Tx \neq \lambda x$ for any $\lambda > 1$ whenever $x \in C$, $\|x\| = r$, then T has a fixed point in C.*

Motivated by a nonlinear equation arising in transport theory, Latrach et al. [25] established generalizations of Darbo fixed point theorem for the weak topology. Their analysis uses the concept of the De Blasi measure of weak compactness but they neither assume the weak continuity nor the sequentially weak continuity of the mapping. Here we shall use the following sharpening of such result, which can be found in [16].

Theorem 2.3. *Let C be a nonempty closed and convex subset of a Banach space X and suppose $T : C \to C$ is an ω-condensing mapping satisfying $(\mathcal{A}_1)$. If there exists $x_0 \in C$ and $R > 0$ such that $Tx - x_0 \neq \lambda(x - x_0)$ for every $\lambda > 1$ and for every $x \in C \cap S_R(x_0)$, then T has a fixed point.*

Recall that a mapping $T : D(T) \subseteq X \to X$ is said to have condition $(\mathcal{A}_1)$ if for each $(x_n)_{n\in\mathbb{N}} \subseteq D(T)$ which is weakly convergent in X, then $(Tx_n)_{n\in\mathbb{N}}$ has a strongly convergent subsequence in X. This condition was already considered in the papers [17,18,25,36].

Definition 2.3. A mapping $A : D(A) \subseteq X \to X$ is said to be ϕ-expansive if there exists a function $\phi : [0, \infty[\to [0, \infty[$ such that for every $x, y \in D(A)$, the inequality $\|Ax - Ay\| \geqslant \phi(\|x - y\|)$ holds with ϕ satisfying

- $\phi(0) = 0$.
- $\phi(r) > 0$ for $r > 0$.
- Either it is continuous or it is nondecreasing.

Theorem 2.4 (*Remark 3.8 in [17]*). *Let $A : D(A) \to X$ be an m-accretive operator ϕ-expansive. Then A is surjective.*

Remark 2.1. In the sequel, when we use the symbol ϕ to represent a function, such function will be under the conditions of Definition (2.3). This type of mappings was already considered in the papers [17,18].

3. Existence of zeroes for ϕ-expansive mappings

Proposition 3.1. *Let X be a Banach space and C a closed subset of X. Let $T : C \to X$ be an injective and continuous mapping such that $T^{-1} : \Re(T) \to C$ is uniformly continuous. Then $\Re(T)$ is a closed subset of X.*

Proof. Let (x_n) be a sequence of elements of $\Re(T)$ such that x_n converges to x_0. We have to prove that $x_0 \in \Re(T)$. Indeed, since (x_n) is a Cauchy sequence and T^{-1} is uniformly continuous it is easy to see that $(T^{-1}(x_n))$ is also a Cauchy sequence. Hence we may assume that $(T^{-1}(x_n))$ converges to $y_0 \in C$ because C is a closed subset of X.

Finally, by using that T is a continuous mapping we conclude that (x_n) converges to Ty_0 which means that $x_0 = Ty_0 \in \Re(T)$. $\square$

Lemma 3.1. *Let C be a nonempty bounded closed subset of a Banach space X and let $T : C \to X$ be a ϕ-expansive mapping. Then T is injective and $T^{-1} : \Re(T) \to C$ is uniformly continuous.*

Proof. if $x, y \in C, x \neq y$, then

$$\|T(x) - T(y)\| \geqslant \phi(\|x - y)\|) > 0$$

and thus T is injective. To prove that T^{-1} is uniformly continuous we argue as follows:

Suppose that there exists $\epsilon_0 > 0$ such that for each $n \in \mathbb{N}$ we can find $x_n, y_n \in \Re(T)$ satisfying both $\|x_n - y_n\| < \frac{1}{n}$ and $\|T^{-1}x_n - T^{-1}y_n\| > \epsilon_0$.

Since T is ϕ-expansive it is clear that

$$\phi(\|T^{-1}x_n - T^{-1}y_n\|) \leqslant \|x_n - y_n\| < \frac{1}{n}. \tag{2}$$

Now assume that ϕ is a nondecreasing function. In this case, we have that $0 < \phi(\epsilon_0) \leqslant \phi(k)$ for every $k \geqslant \epsilon_0$. Hence, we get a contradiction because on one hand, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n} < \phi(\epsilon_0)$ for all $n \geqslant n_0$ and on the other hand, by (2), we have that $\phi(\epsilon_0) \leqslant \phi(\|T^{-1}x_n - T^{-1}y_n\|) \leqslant \|x_n - y_n\| < \frac{1}{n}$.

Otherwise, ϕ will be a continuous function. In this case, since C is a bounded subset we know that the sequence $(T^{-1}x_n - T^{-1}y_n)$ is bounded and therefore we may, without loss of generality, assume that

$$\lim_{n\to\infty} \|T^{-1}x_n - T^{-1}y_n\| = r \geqslant \epsilon_0.$$

Consequently, $\lim_{n\to\infty} \phi(\|T^{-1}x_n - T^{-1}y_n\|) = \phi(r) > 0$ which is a contradiction since (2) yields $\lim_{n\to\infty} \phi(\|T^{-1}x_n - T^{-1}y_n\|) = 0$. $\square$

The next result is an easy consequence of Proposition 3.1 and Lemma 3.1.

Corollary 3.1. *Let C be a closed bounded subset of a Banach space X and let $T : C \to X$ be a continuous ϕ-expansive mapping. If there exists (x_n) in $\Re(T)$ such that (x_n) converges to zero, then $0 \in \Re(T)$.*

Lemma 3.2. *Let C be a closed subset of a Banach space X and let $T : C \to X$ be a continuous mapping. If there exists an almost fixed point sequence (x_n) of T in C and if the inverse mapping $(I - T)^{-1} : \Re(I - T) \to C$ exists and it is uniformly continuous, then T has a unique fixed point $x_0 \in C$. Furthermore, $x_n \to x_0$ as $n \to \infty$.*

Proof. Since $I - T : C \to X$ is continuous injective and $(I - T)^{-1} : \Re(I - T) \to C$ is uniformly continuous by Proposition 3.1, we obtain that $\Re(I - T)$ is closed. Hence, because $x_n - T(x_n) \to 0$ as $n \to \infty$, we obtain that $0 \in \Re(I - T)$. Consequently, there exists an $x_0 \in C$ such that $(I - T)(x_0) = 0$, that is, $T(x_0) = x_0$. This fixed point is unique because $I - T$ is injective. Finally, since $(I - T)^{-1}$ is continuous and $(I - T)(x_n) \to (I - T)(x_0)$ we obtain that $x_n \to x_0$. $\square$

The next example shows the importance of the assumption $(I - T)^{-1}$ be uniformly continuous in Lemma 3.2.

Example 3.1. Consider the function $T : [1, \infty[\to [1, \infty[$ defined by $Tx = x + \frac{1}{x}$. Clearly T is a fixed point free mapping and however $(I - T)^{-1} : [-1, 0) \to [1, \infty)$ is given by $(I - T)^{-1}(x) = -\frac{1}{x}$ which is continuous but not uniformly continuous. Finally, T admits almost fixed point sequences.

Corollary 3.2. *Let C be a closed bounded subset of a Banach space X and let $T : C \to C$ be a continuous mapping such that $I - T : C \to X$ is ϕ-expansive. If there exists an almost fixed point (a.f.p. in short) sequence (x_n) of T in C, then T has a unique fixed point $x_0 \in C$. Furthermore, $x_n \to x_0$ as $n \to \infty$.*

Proof. Since $I - T : C \to X$ is ϕ-expansive and continuous, then, by Lemma 3.1, we know that $(I - T)^{-1} : \Re(I - T) \to C$ is uniformly continuous. Thus, applying Lemma 3.2 we achieve the conclusion. $\square$

Remark 3.1. In the above corollary we are assuming that $I - T$ is a ϕ-expansive mapping. If ϕ is nondecreasing then it is easy to see that $(I - T)^{-1} : \Re(T) \to C$ is uniformly continuous although C becomes unbounded (see proof of Lemma 3.1). Nevertheless, if ϕ is continuous the conclusion of Corollary 3.2 remains true if we remove the boundedness of C but we add the existence of a bounded almost fixed point sequence for T.

Corollary 3.3. *Let Ω be a bounded closed convex subset of a Banach space X. Assume that $F : \Omega \to X$ is weakly inward on Ω, continuous pseudo-contractive mapping such that $(I - F)^{-1} : \Re(I - F) \to \Omega$ is uniformly continuous (in particular, if $I - F$ is ϕ-expansive). Then F has a unique fixed point in Ω.*

Proof. Since F is a continuous pseudo-contractive-mapping weakly inward on Ω, it is well known that $A := I - F : \Omega \to X$ is an accretive operator with the range condition.

Consequently, the resolvent $J_1 := (I + A)^{-1} : \Omega \to \Omega$ is a single valued and nonexpansive mapping. Moreover, since Ω is bounded closed and convex there exists a sequence (w_n) in Ω such that $w_n - J_1(w_n) \to 0$.

If we let $x_n = J_1(w_n)$, then $x_n + x_n - F(x_n) = w_n$ and therefore

$$x_n - F(x_n) = w_n - J_1(w_n),$$

which implies that (x_n) is a bounded a.f.p. sequence in Ω for F. Now, invoking Lemma 3.2 we obtain the result. $\square$

At this point, let us recall that $T : M \subseteq X \to X$ is a separate contraction mapping (see [27]) if there exist two functions $\varrho, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying.

1. $\psi(0) = 0, \psi$ is nondecreasing,
2. $\|Tx - Ty\| \leqslant \varrho(\|x - y\|)$,
3. $0 < \psi(r) \leqslant r - \varrho(r)$ for $r > 0$.

In the original definition of separate contraction mapping appears that ψ is strictly increasing, which is a particular case of the above definition. Anyway, if T is a separate contraction, then T is a nonexpansive mapping and $I - T$ is ϕ-expansive with $\phi = \psi$. Consequently

- If C is a bounded closed convex subset of a Banach space and $T : C \to X$ is a weakly inward separate contraction on C, then T has a unique fixed point in C due to Corollary 3.3.
- If C is a closed subset of X and $T : C \to C$ is a separate contraction, we recapture Theorem 2.1 in [27] since on one hand, ϕ is nondecreasing and on the other hand, if $x_0 \in C$ then it is not hard to see that $(T^n x_0)$ is an a.f.p. sequence. Therefore we may apply Corollary 3.2 and Remark 3.1.
- In [33] Sadiq Basha introduces the concept of weak contraction of the first kind. Clearly this type of mappings falls into the class of separate contractions and therefore Corollary 3.3 in [33] is an easy consequence of the above comment.

On the other hand, a mapping $T : M \subseteq X \to X$ is said to be expansive (see [37]) if there exists a constant $h > 1$ such that $\|Tx - Ty\| \geqslant h\|x - y\|$, for all $x, y \in M$. In this case, clearly $(I - T)$ is ϕ-expansive with $\phi(t) = (h - 1)t$. Moreover, there exists $T^{-1} : \Re(T) \to M$ and it is a $\frac{1}{h}$-contraction. This implies that $I - T^{-1} : \Re(T) \to X$ is ϕ-expansive with $\phi(r) = (1 - \frac{1}{h})r$. Thus, if $M \subseteq \Re(T)$ we infer that T^{-1} has a unique fixed point.

Next result represents a completion of Theorem 2.3 in the above section and Theorem 2.3 in [14] in the sense that it works with $w - 1$-set contractions.

Theorem 3.1. *Let C be a closed and convex subset of a Banach space X and let $T : C \to C$ be a mapping such that:*

(i) *T satisfies $(\mathcal{A}_1)$.*
(ii) *T is an $\omega - 1$-set contraction.*
(iii) *There exist $R > 0$ and $x_0 \in C$ such that for all $x \in C \cap S_R(x_0)$ and for all $\lambda > 1$ we have that $T(x) - x_0 \neq \lambda(x - x_0)$. Then there exists an almost fixed point sequence (x_n) of T. Furthermore, if:*
(iv) *$(I - T) : C \to X$ is ϕ-expansive, then T has a unique fixed point $x \in C$ and $x_n \to x$.*

Proof. We define $B_R^C(x_0) = \{x \in C : \|x - x_0\| \leqslant R\}$ and let $\rho : C \to B_R^C(x_0)$ be the mapping given by

$$\rho(x) = \begin{cases} x, & \text{if } \|x - x_0\| \leqslant R, \\ \frac{R}{\|x - x_0\|} x + \left(1 - \frac{R}{\|x - x_0\|}\right) x_0, & \text{if } \|x - x_0\| > R. \end{cases}$$

$B_R^C(x_0)$ is a nonempty bounded closed and convex subset of C and the mapping ρ is a continuous retraction of C on $B_R^C(x_0)$.

For each integer $n \geqslant 2$ we define the mapping $T_n : C \to C$ given by

$$T_n(x) = \frac{1}{n}x_0 + \left(1 - \frac{1}{n}\right)T(x).$$

It is clear that T_n is an $w - (1 - \frac{1}{n})$-set contraction satisfying condition $(\mathcal{A}_1)$, for any $n \geqslant 2$.

Now we define the mappings $T_{n,\rho} : B_R^C(x_0) \to B_R^C(x_0)$ as $T_{n,\rho}(x) = \rho(T_n(x))$. The mapping $T_{n,\rho}$ is continuous with condition $(\mathcal{A}_1)$. Furthermore, $T_{n,\rho}$ is an $w - (1 - \frac{1}{n})$- set contraction because T_n is an $w - (1 - \frac{1}{n})$- set contraction and ρ is an $w - 1$-set contraction.

By Theorem 2.3 we have that $T_{n,\rho}$ has a fixed point, say $T_{n,\rho}(x_n) = x_n$.

We will verify that $T_n(x_n) = x_n$. In order to prove this we will prove that $\|T_n(x_n) - x_0\| \leqslant R$. Assume for a contradiction that $\|T_n(x_n) - x_0\| > R$. Hence

$$x_n = \rho(T_n(x_n)) = \frac{R}{\|T_n(x_n) - x_0\|}T_n(x_n) + \left(1 - \frac{R}{\|T_n(x_n) - x_0\|}\right)x_0$$

and thus

$$\frac{R}{\|T_n(x_n) - x_0\|}(T_n(x_n) - x_0) = x_n - x_0.$$

Consequently $x_n \in C \bigcap S_R(x_0)$. We also have that

$$T(x_n) - x_0 = \frac{n}{n-1}(T_n(x_n) - x_0) = \lambda_n(x_n - x_0),$$

where $\lambda_n = \frac{n}{n-1}\frac{\|T_n(x_n)-x_0\|}{R} > 1$, which contradicts condition **(ii)**. Hence we have that $\|T_n(x_n) - x_0\| \leqslant R$ and therefore

$$x_n = \rho(T_n(x_n)) = T_n(x_n).$$

Next we shall prove that (x_n) is an almost fixed point sequence for T. First we note that

$$\|x_0 - T(x_n)\| \leqslant \|x_0 - T_n(x_n)\| + \|T_n(x_n) - T(x_n)\| \leqslant R + \frac{1}{n}\|x_0 - T(x_n)\|$$

and thus $\|x_0 - T(x_n)\| \leqslant \frac{n}{n-1}R$. Therefore by the following inequality

$$\|x_n - T(x_n)\| = \frac{1}{n}\|x_0 - T(x_n)\| \leqslant \frac{R}{n-1},$$

we obtain that (x_n) is a bounded almost fixed point sequence for T. Finally, if T satisfies condition (**iii**) then T has a unique fixed point $x \in C$ and $x_n \to x$ invoking Remark 3.1 □

Remark 3.2. In Theorem 3.1 we will obtain the same conclusion if we replace assumption (iii) by $(I - T)^{-1} : \Re(I - T) \to C$ exists and it is uniformly continuous due to Lemma 3.2.

Next result can be considered as a completion of Theorem 2.2. Moreover, it is a generalization of Proposition 3.4 in [17]. It is worth noting that its proof is similar to the proof of Theorem 3.1, but using Theorem 2.1 instead of Theorem 2.3.

Theorem 3.2. *Let C be a closed and convex subset of a Banach space X and let $T : C \to C$ be a mapping such that:*

(i) *T is a $\Phi - 1$-set contraction.*
(ii) *There exist $R > 0$ and $x_0 \in C$ such that for all $x \in C \cap S_R(x_0)$ and for all $\lambda > 1$ we have that $T(x) - x_0 \neq \lambda(x - x_0)$.*
Then there exists an almost fixed point sequence (x_n) of T. Furthermore, if:
(iii) *$I - T : C \to \Re(I - T)$ is ϕ-expansive then T has a unique fixed point $x \in C$ and $x_n \to x$.*

Although the condition (iii) in Theorem 3.2 implies that $I - T$ is semiclosed (see for instance [15,20]), in this theorem we obtain the uniqueness of the fixed point as well as the convergence to it for every almost fixed point sequence.

As we said in the introduction given an infinite dimensional Banach space X and for an arbitrary measure of noncompactness on X there exists a 1-set contraction self-mapping fixed point free on the unit ball of X. Next example illustrates the importance of assuming ϕ-expansiveness or uniform continuity of $(I - T)^{-1}$ in Theorems 3.1 and 3.2.

Example 3.2. Consider the mapping $T : B_{l_1} \to B_{l_1}$ given by

$$T(x) = \left(1 - \sum_{i=1}^{\infty}|x_i|, x_1, x_2, x_3, \ldots\right)$$

for each $x = (x_i) \in B_{l_1}$. T is an $\alpha - 1$-set contraction (and an $\omega - 1$-set contraction) and satisfies $(\mathcal{A}_1), I - T : B_{l_1} \to l_1$ is injective and however T does not have fixed points.

Indeed, notice that $T = S + R$ where $S : B_{l_1} \to l_1$ is given by $S(x) = \left(1 - \sum_{i=1}^{\infty}|x_i|\right)e_1$ and $R : B_{l_1} \to B_{l_1}$ is defined as $R(x) = \sum_{i=1}^{\infty} x_i e_{i+1}$. Clearly S is compact and R is a nonexpansive linear mapping, in particular R is a 1-set contraction with respect to α and w. Thus, T is also a 1-set contraction with respect to both measures of noncompactness. Clearly T is continuous, in fact, for every $x = (x_i)$ and $y = (y_i)$ in B_{l_1}

$$\|T(x) - T(y)\|_1 = \left|\sum_{i=1}^{\infty}(|y_i| - |x_i|)\right| + \sum_{i=1}^{\infty}|x_i - y_i| \leqslant 2\|x - y\|_1.$$

Since l_1 is a Schur-space and T is continuous we have that T enjoys condition $(\mathcal{A}_1)$.

Now we will prove that $I - T$ is injective. Let $x = (x_i)$ and $y = (y_i)$ in B_{l_1} such that $(I - T)(x) = (I - T)(y)$. We have that

$$\left(x_1 - y_1 + \sum_{i=1}^{\infty}(|x_i| - |y_i|), x_2 - y_2 - (x_1 - y_1), x_3 - y_3 - (x_2 - y_2), \ldots\right) = (0, 0, 0, \ldots),$$

then, for every positive integer i, we have $x_{i+1} - y_{i+1} = x_i - y_i$ and hence $x_i - y_i = 0$, that is, $x = y$. Finally, if $T(x) = x$ for some $x = (x_i) \in B_{l_1}$ we have

$$\left(1 - \sum_{i=1}^{\infty}|x_i|, x_1, x_2, x_3, \ldots\right) = (x_1, x_2, x_3, \ldots)$$

and then $1 - \sum_{i=1}^{\infty}|x_i| = x_1$ and $x_i = x_{i+1}$ for every i. Hence $x_i = 0$ for every i and consequently

$$1 = 1 - \sum_{i=1}^{\infty}|x_i| = x_1 = 0,$$

which is not possible.

Next we are going to present an example where it is not possible to apply Theorems 2.1, 2.2 or 2.3 and nevertheless such example fulfills the assumptions of Theorem 3.2.

Example 3.3. Let X be a infinite dimensional Banach space and let $T : X \to X$ be the mapping defined as

$$T(x) = \begin{cases} -x, & \text{if } \|x\| \leqslant 1, \\ -\frac{x}{\|x\|}, & \text{if } \|x\| > 1. \end{cases}$$

The mapping T is a $\Phi - 1$-set contraction (and an $\omega - 1$-set contraction). We also have that $I - T$ is ϕ-expansive. To see T is a Φ−1-set contraction we will verify that for every subset K of X we have $T(K) = conv(-K \cup \{0\})$. Indeed, let $x \in K$. If $\|x\| \leqslant 1$ then $T(x) = -x \in -K$. If $\|x\| > 1$, then

$$T(x) = \frac{1}{\|x\|}(-x) + \left(1 - \frac{1}{\|x\|}\right)0 \in conv(-K \cup \{0\}).$$

By this we obtain that $\Phi(T(K)) \leqslant \Phi(conv(-K \cup \{0\})) = \Phi(K)$.

Now we shall prove that $I - T$ is ϕ-expansive. Let $S = I - T$. Clearly

$$S(x) = \begin{cases} 2x, & \text{if } \|x\| \leqslant 1, \\ \left(1 + \frac{1}{\|x\|}\right)x, & \text{if } \|x\| > 1. \end{cases}$$

Let $x, y \in X$. Case (1): $x, y \in B_X$. In this case $\|S(x) - S(y)\| = 2\|x - y\|$. Case (2): $x \in B_X, y \in X \setminus B_X$. Since $\|x\| \leqslant 1$ if and only if $-(\|y\| - \|x\|) \leqslant -(\|y\| - 1)$, we obtain

$$\|S(x) - S(y)\| = \left\|2x - y - \frac{y}{\|y\|}\right\| = \left\|2(x - y) + \frac{(\|y\| - 1)y}{\|y\|}\right\| \geqslant 2\|x - y\| - (\|y\| - 1) \geqslant 2\|x - y\| - (\|y\| - \|x\|) \geqslant \|x - y\|.$$

Case (3): $x, y \in X \setminus B_X$. As in case (2) we have $\|S(x) - S(y)\| \geqslant \|x - y\|$.

On the other hand, T is not condensing because $T(B_X) = B_X$ and T is nonexpansive if and only if X is a Hilbert space (see [13]). Therefore, when X is not a Hilbert space Proposition 3.4 in [17] does not apply.

3.1. Nonexpansive mappings

It turns out that property (FPP) closely depends upon geometric properties of the Banach spaces under consideration. Even when C is a weakly compact convex subset of X, a nonexpansive self-mapping of C needs not have fixed points (see Chapter 2 in [23] where Sims collects together examples of fixed point free nonexpansive mappings in a variety of Banach spaces, also see [19]). In this section we will give an iterative method for approaching the fixed point of a nonexpansive mapping T such that $I - T$ becomes ϕ-expansive. In order to show this we need the following lemma found in [38].

Lemma 3.3. *Assume (a_n) is a sequence in $[0,\infty)$ such that*

$$a_{n+1} \leqslant (1-\gamma_n)a_n + \delta_n, \quad n \geqslant 0,$$

where (γ_n) is a sequence in $(0,1)$ and (δ_n) is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty}\gamma_n = \infty$,
(2) $\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n} \leqslant 0$ or $\sum_{n=1}^{\infty}|\delta_n| < \infty$. *Then* $\lim_{n\to\infty}a_n = 0$.

Theorem 3.3. *Let C be a closed and convex subset of a Banach space X and let $T : C \to C$ be a nonexpansive mapping such that:*

(i) *$I - T : C \to \Re(I-T)$ is ϕ-expansive,*
(ii) *There exist $R > 0$ and $x_0 \in C$ such that for all $x \in C\bigcap S_R(x_0)$ and for all $\lambda > 1$ we have that $T(x) - x_0 \neq \lambda(x - x_0)$.*
Assume (α_n) is a sequence in $(0,1)$satisfying:

(i) $\sum_{n=1}^{\infty}\alpha_n = \infty$,
(ii) $\lim_{n\to\infty}\alpha_n = 0$,
(iii) $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$.
(For example $\alpha_n = \frac{1}{n}$). *Let $x_1 \in C$ and define $x_{n+1} = \alpha_n x_1 + (1-\alpha_n)T(x_n)$ for each positive integer n. Then (x_n) converges to the unique fixed point of T.*

Proof. It is well known that if T is nonexpansive, then it is 1-set contractive for the Kuratowskii measure of noncompactness and thus, T satisfies the assumptions of Theorem 3.2 which implies that there exists a unique fixed point $p \in C$ for T.

We claim that (x_n) is a bounded sequence in C.

$$\|x_n - p\| = \|\alpha_n(x_1 - p) + (1-\alpha_n)(Tx_{n-1} - p)\| \leqslant \alpha_n\|x_1 - p\| + (1-\alpha_n)\|x_{n-1} - p\| \leqslant \max\{\|x_1 - p\|, \|x_{n-1} - p\|\}.$$

By induction we infer that $\|x_n - p\| \leqslant \|x_1 - p\|$ for every $n \in \mathbb{N}$. This means that (x_n) is bounded as we claimed.

On the other hand, by definition of (x_n) we obtain that (Tx_n) is also a bounded sequence. Let M be an upper bounded of the sequence $(\|x_1 - Tx_n\|)$. Then

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\alpha_n x_1 + (1-\alpha_n)T(x_n) - (\alpha_{n-1}x_1 + (1-\alpha_{n-1})T(x_{n-1}))\| \\ &= \|(\alpha_n - \alpha_{n-1})x_1 + (1-\alpha_n)(T(x_n) - T(x_{n-1})) + (\alpha_{n-1} - \alpha_n)T(x_{n-1})\| \\ &= \|(\alpha_n - \alpha_{n-1})(x_1 - T(x_{n-1})) + (1-\alpha_n)(T(x_n) - T(x_{n-1}))\| \leqslant M\|\alpha_n - \alpha_{n-1}| + (1-\alpha_n)\|x_n - x_{n-1}\|.\end{aligned}$$

By Lemma 3.3 we have that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Finally, since $x_{n+1} - T(x_n) = \alpha_n(x_1 - T(x_n)) \to 0$ as $n \to \infty$ and by the following inequality

$$\|x_n - T(x_n)\| \leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - T(x_n)\|$$

we obtain that (x_n) is a bounded almost fixed point sequence. Finally, according to Theorem 3.2 the sequence (x_n) converges to the unique fixed point of T. □

Remark 3.3. By Theorem 2.4 it is well known that if $T : X \to X$ is a nonexpansive mapping such that $I - T : X \to X$ is ϕ-expansive, then T has a unique fixed point. Thus, the same technique as in the proof of Theorem 3.3 shows that the sequence defined in such result converges to the unique fixed point of T.

Finally, if C is a nonempty closed bounded and convex subset of a Banach space X and if $T : C \to C$ be nonexpansive. For every $\alpha \in (0,1)$ we define the mapping $T_\alpha : C \to C$ as $T_\alpha(x) = \alpha x + (1-\alpha)T(x)$. It is known (see [21]) that T_α is asymptotically regular, that is, for every $x \in C$ we have that $T_\alpha^{n+1}(x) - T_\alpha^n(x) \to 0$ as $n \to \infty$ and so $\left(T_\alpha^n(x)\right)_n$ is an almost fixed point sequence for T_α. It is not difficult to see that $\left(T_\alpha^n(x)\right)_n$ is also an almost fixed point sequence for T. According to Lemma 3.2, if $(I-T)^{-1}$ exists and it is uniformly continuous then $\left(T_\alpha^n(x)\right)_n$ converges to the unique fixed point of T.

Llorens-Fuster and Moreno-Gálvez in [28] defined a class of generalized nonexpansive mappings as follows.

Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ satisfies condition (L) on C provided that it fulfills the following two conditions.

1. If a set $D \subseteq C$ is nonempty, closed, convex and T-invariant, then there exists an almost fixed point sequence (a.f.p.s. in short) for T in D,
2. For any a.f.p.s. (x_n) of T in C and each $x \in C$

$$\limsup_{n\to\infty}\|x_n - Tx\| \leqslant \limsup_{n\to\infty}\|x_n - x\|.$$

If C is a bounded closed convex subset of a Banach space X and $T : C \to C$ is a continuous mapping with condition (L) such that $I - T$ is ϕ-expansive, then after Lemma 3.2, T has a unique fixed point in C. In this sense, since T can be noncontinuous it is worth noting the following result, which can be found in [31].

Theorem 3.4. *Let X be a Banach space and C a bounded, closed and convex subset of X, if $T : C \to C$ is a mapping with condition (L) and $I - T$ is ϕ-expansive, then T has a unique fixed point in C.*

Now we give an example where Theorem 3.4 does not work and however such example fulfills the assumption of Theorem 3.1.

Example 3.4. Consider $K = \left\{x \in l_1 : \|x\|_1 \leqslant 2, \sum_{i=2}^{\infty}|x_i| \leqslant 1\right\}$ and the mapping $T : K \to K$ given by

$$T(x) = \left(1 - \sum_{i=2}^{\infty}|x_i|, -x_2, -x_3, -x_4, \ldots\right)$$

for each $x = (x_i) \in K$. It is clear that K is a closed bounded and convex subset of l_1. Proceeding similarly as in Example 3.2 we can prove that T is an $\alpha - 1$-set contraction (and an $\omega - 1$-set contraction) and satisfies $(\mathcal{A}_1)$. Moreover, $I - T$ is ϕ-expansive and in particular $(I - T)^{-1} : \Re(I - T) \to K$ exists and it is uniformly continuous. Indeed, let $x = (x_i)$ and $y = (y_i)$ in K. We have that

$$(I - T)(x) - (I - T)(y) = \left(x_1 - y_1 + \sum_{i=2}^{\infty}(|x_i| - |y_i|)\right)e_1 + 2\sum_{i=2}^{\infty}(x_i - y_i)e_i$$

and then

$$\begin{aligned}\|(I - T)(x) - (I - T)(y)\|_1 &= \left|x_1 - y_1 + \sum_{i=2}^{\infty}(|x_i| - |y_i|)\right| + 2\sum_{i=2}^{\infty}|x_i - y_i| \geqslant |x_1 - y_1| - \left|\sum_{i=2}^{\infty}(|x_i| - |y_i|)\right| + 2\sum_{i=2}^{\infty}|x_i - y_i| \\ &= \|x - y\|_1 - \left|\sum_{i=2}^{\infty}(|x_i| - |y_i|)\right| + \sum_{i=2}^{\infty}|x_i - y_i| \geqslant \|x - y\|_1 - \sum_{i=2}^{\infty}|x_i - y_i| + \sum_{i=2}^{\infty}|x_i - y_i| = \|x - y\|_1.\end{aligned}$$

Hence $I - T$ is ϕ-expansive. By Theorem 3.1, T has a unique fixed point. In fact, it is easy to see that $x = e_1$ is the unique fixed point of T. On the other hand, T is neither condensing, nor satisfies property (L).

To see that T is not Φ-condensing, define $C = \{e_i\}_{i \geqslant 2}$, where (e_i) are the elements of the standard Schauder basis in l_1. We know that C is not relatively compact and since $T(C) = -C$ we have $\Phi(T(C)) = \Phi(C)$.

Finally we will see T does not enjoy property (L). Let (x_n) be the constant sequence $x_n = e_1$ and let $x = \frac{2}{3}e_1 + \frac{1}{2}e_2 \in K$. We obtain $x_n - x = \frac{1}{3}e_1 - \frac{1}{2}e_2$ and so $\|x_n - x\|_1 = \frac{5}{6}$. On the other hand $x_n - T(x) = \frac{1}{2}e_1 + \frac{1}{2}e_2$ and then $\|x_n - T(x)\|_1 = 1$. Thus

$$\liminf_{n\to\infty} \|x_n - T(x)\|_1 > \liminf_{n\to\infty} \|x_n - x\|_1.$$

Since (x_n) is an almost fixed point sequence for T we get that T does not satisfy property (L). Consequently, Theorem 3.4 does not ensure the existence of a fixed point for T.

4. Fixed point results for the sum of two operators

Krasnoselskii's fixed point Theorem [24] was one of the first results for solving equations of the form (1). Nevertheless, in several applications, the verification of conditions (i), (ii), (iii) and (iv) defined in the Introduction section is, in general, either quite hard to be done or even some of them fails. As a tentative approach to overcome those difficulties, many interesting articles have appeared relaxing some of the assumptions (i), (ii), (iii) or (iv). For instance, in [9], Burton and Kirk used an alternative Leray–Schauder type to avoid the boundedness of C. In [8], Burton replaced assumption (i) by: (If $u = Bu + Ay$ with $y \in C$, then $u \in C$). Condition (iv) is also quite restrictive, so some authors have replaced it by a more general condition (many times B is assumed to be a separate contraction [27]). In this section, we present three results in this sense. In the first one we replace the assumption (iii) (i.e B is either a contraction or a separate contraction) by a more general condition: B is either a 1-set contraction or a continuous pseudocontractive mapping and $I - B$ is ϕ-expansive. In fact, it is also a generalization of Theorem 3.7 in [17]. The second one uses the conditions given in [8,9]. The last one extends Theorem 2.1 in [26].

Theorem 4.1. *Let C be a closed bounded and convex subset of a Banach space X and let $A, B : C \to X$ be continuous mappings such that*

(**i**) *A is compact.*
(**ii**) *B is either a $\Phi - 1$-set contraction or a pseudo-contractive mapping.*
(**iii**) *$(I - B)^{-1} : \Re(I - B) \to C$ exists and it is uniformly continuous.*
(**iv**) *$A(C) + B(C) \subset C$. Then there exists $x \in C$ such that $A(x) + B(x) = x$.*

Proof. Fix an arbitrary $y \in C$ and define $S_y : C \to C$ as $S_y(x) = A(y) + B(x)$. Clearly S_y is well defined and it is continuous.

Next note that $(I - S_y)^{-1} : \mathfrak{R}(I - S_y) \to X$ exists and it is uniformly continuous. Indeed, clearly $(I - S_y)^{-1}$ is injective and uniformly continuous because $(I - B)^{-1}$ satisfies such conditions.

If B is a $\Phi - 1$-set contraction, then for every bounded subset $K \subset C$ we have that

$$\Phi\big(S_y(K)\big) = \Phi(B(K)) \leqslant \Phi(K),$$

that is, S_y is also a $\Phi - 1$-set contraction.

If B is a pseudocontractive, then $\|x - z\| \leqslant \|(1 + r)(x - z) - r(B(x) - B(z))\|$ for every $r > 0$. Consequently

$$\|x - z\| \leqslant \|(1 + r)(x - z) - r(B(x) + Ay - (B(z) + A(y))\| = \|(1 + r)(x - z) + r(S_y(x) - S_y(z))\|.$$

That is, S_y is also pseudocontractive.

Now we shall prove that $A(C) \subset \mathfrak{R}(I - B)$.

Indeed, let $y \in C$.

If B is either a $\Phi - 1$-set contraction or a pseudocontractive mapping, by either step 1 and Remark 3.2, or Corollary 3.3, respectively, we obtain that there exists $x_0 \in C$ such that $S_y(x_0) = x_0$, that is, $A(y) + B(x_0) = x_0$. Then $(I - B)(x_0) = A(y)$ and thus $A(y) \in \mathfrak{R}(I - B)$.

From this we have that $(I - B)^{-1} \circ A : C \to C$ is well defined. Clearly $\Big((I - B)^{-1} \circ A\Big)(C)$ is relatively compact because A is compact and $(I - B)^{-1}$ is continuous. By Schauder's theorem we get that there exists an $x \in C$ such that $((I - B)^{-1} \circ A)(x) = x$ and consequently $A(x) = (I - B)(x)$. Hence $A(x) + B(x) = x$. $\square$

Remark 4.1. Notice that in ([1], Theorem 2.19, Corollary 2.25) the authors gave similar results to Theorem 4.1. However, the condition on the operator B in the above result is, in fact, more general than the non expansiveness which is assumed in ([1],Corollary 2.25).

Theorem 4.2. *Let C be a closed and convex subset of a Banach space X with $0 \in C$ and let $A : C \to X$ and $B : X \to X$ be a continuous mappings such that*

(**i**) *A is compact.*
(**ii**) *B is Φ-1-set contractive with $B(X)$ bounded,*
(**iii**) *$(I - B)^{-1} : \mathfrak{R}(I - B) \to X$ exists and it is uniformly continuous.*
(**iv**) *If $x = A(y) + B(x)$ for some $y \in C$ then $x \in C$. Then, either*

1. *The equation $x = B(x) + A(x)$ has a solution, or*
2. *The set $\{x \in C : x = \lambda B(\frac{x}{\lambda}) + \lambda A(x), \lambda \in (0, 1)\}$ is unbounded.*

Proof. First, let us see that $A(C) \subseteq \mathfrak{R}(I - B)$. Indeed, given $y \in C$ we define $S_y : X \to X$ as $S_y x = Bx + Ay$.

By using the same arguments as in the above theorem we obtain that such mapping is Φ-1-set contractive and that $(I - S_y)^{-1}$ is uniformly continuous.

On the other hand, since $B(X)$ is a bounded subset of X, there exists $M > 0$ such that $\|B(x)\| \leqslant M$ for all $x \in X$. Thus, if we call $R = M + \|Ay\|$, we have that

$$S_y x = Bx + Ay \neq \lambda x \quad \text{for all } x \in S_R \quad \text{and for all } \lambda > 1.$$

The above facts yield that S_y is under the conditions given in Theorem 3.2, and then there exists a unique fixed point, $x \in X$, for S_y. This means that $x = Bx + Ay$, hence by assumption (iv) we derive that $x \in C$. Which allows us to define $T := (I - B)^{-1} \circ A : C \to C$.

On the other hand, since $(I - B)^{-1}$ is a continuous mapping and by assumption (i) A is compact then T is also a compact mapping.

Finally, in order to obtain the result we apply Theorem 3.2. Indeed, if $x \in C$ and $x = \lambda B(\frac{x}{\lambda}) + \lambda A(x)$ for some $\lambda \in (0, 1)$, then taking $\mu = \frac{1}{\lambda} > 1$, we have $T(x) - 0 = \mu(x - 0)$ and vice versa. Consequently if (2) fails, we may find $R > 0$ such that $T(x) - 0 \neq \mu(x - 0)$ for all $x \in C \cap S_R(0)$ and for all $\mu > 1$. $\square$

Remark 4.2. The conclusion of Theorems 4.1 and 4.2 also holds if we replace assumption (iii) by $I - B$ is a ϕ-expansive mapping. If, moreover, we replace in Theorem 4.2 condition (ii) by B is a continuous pseudocontractive mappings we obtain the same conclusion without assuming that $B(X)$ is bounded. This is a consequence of Theorem 2.4 and thus we may consider Theorem 4.2 as a generalization of Theorem 2.3 in [27].

Next theorem represents an extension of Theorem 2.1 in [26]. Notice that such result was used in order to get a solution for a nonlinear transport equation (see [36]).

Theorem 4.3. *Let C be a bounded closed and convex subset of a Banach space X and let $A : C \to X$ and $B : C \to X$ be continuous mappings such that*

(**i**) *A has the condition $(\mathcal{A}_1)$,*
(**ii**) *B is pseudocontractive,*
(**iii**) *$I - B$ ϕ-expansive,*
(**iv**) *There exists $\gamma \in [0, 1)$ such that $w(A(S) + B(S)) \leqslant \gamma w(S)$ for all $S \subseteq C$.*
(**v**) *$A(C) + B(C) \subseteq C$.*
Then, the equation $x = B(x) + A(x)$ has a solution.

Proof. Let $y \in C$ be fixed but arbitrary and define $S_y : C \to C$ as $S_y(x) = A(y) + B(x)$. Clearly S_y is well defined and it is a continuous pseudocontractive mapping with $I - S_y$ ϕ-expansive. Therefore, by Corollary 3.3, we obtain that there exists a unique $x_0 \in C$ such that $S_y(x_0) = x_0$. This fact allows us to show that $T := (I - B)^{-1} \circ A : C \to C$ is well defined.

On the other hand, we know that given a subset K of C the following inclusion

$$(I - B)^{-1} \circ A(K) \subseteq A(K) + B((I - B)^{-1} \circ A(K))$$

holds.

Consequently,

$$T(K) \subseteq A(K \cup T(K)) + B(K \cup T(K)),$$

by hypothesis (iv), we have

$$w(T(K)) \leqslant w(A(K \cup T(K)) + B((K \cup T(K)) \leqslant \gamma w(K \cup T(K)) = \gamma \max\{w(K), w(T(K))\} = \gamma w(K).$$

This means that $T := (I - B)^{-1} \circ A : C \to C$ is an w-condensing mapping satisfying condition $(\mathcal{A}_1)$. Hence we may apply Theorem 2.3 to achieve the conclusion. $\square$

4.1. Application

Let $(X, \|\cdot\|)$ be a Banach space and let $T > 0$. We will now study the existence of solutions for the integral equation

$$\psi(t) = g(t, \psi(t)) + \int_0^t f(s, \psi(s))ds, \tag{3}$$

on $C(0, T; X)$, the space of X-valued continuous functions on the interval $[0, T]$, where the functions f and g satisfy the following conditions:

(E1) The function $g : [0, T] \times X \to X$ is uniformly continuous on the bounded subsets of $[0, T] \times X, g(t, \cdot)$ is a pseudocontractive mapping and let $M_r := \max\{\|g(t, x)\| : \|x\| \leqslant r \text{ and } t \in [0, T]\}$,
(E2) $I - g(t, \cdot) : X \to X$ is ϕ-expansive,
(E3) The function $f : [0, T] \times X \to X$ is a compact Carathéodory function and there exist two functions $m, s \in L^1(0, T; \mathbb{R}^+)$ and an increasing function $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|f(t, x)\| \leqslant m(t)\Omega(\|x\|) + s(t)$.
(E4) $\lim_{r\to\infty} \frac{\|m\|_1 \Omega(r) + M_r}{r} < 1$.

Theorem 4.4. *Eq. (3) has a solution in $C(0, T; X)$ whenever the conditions (E1)–(E4) are satisfied.*

Proof. For this purpose we define

$$A : C(0, T; X) \to C(0, T; X),$$
$$\psi \mapsto A(\psi)(t) = \int_0^t f(\tau, \psi(\tau))d\tau$$

and

$$B : C(0, T; X) \to C(0, T; X),$$
$$\psi \mapsto B(\psi)(t) = g(t, \psi(t)).$$

Our task consists into see that $A + B$ has a fixed point.

We infer that both operators are well defined by assumptions (E1) and (E3).

The continuity of A follows from Lebesgue's convergence theorem. Assumption (E3) allows us to apply the Ascoli–Arzela theorem in order to show that the operator A is also compact.

On the other hand, since g is uniformly continuous on the bounded subsets of $[0, T] \times X$ and $g(t, \cdot)$ is pseudocontrative, we easily obtain that B is a continuous pseudocontractive mapping on $C(0, T; X)$.

Next, let us see that $I-B$ is a ϕ-expansive operator on $C(0,T;X)$. Indeed, since $I-g(t,\cdot)$ is ϕ-expansive on X, we have

$$\|u(t)-g(t,u(t))-(v(t)-g(t,v(t)))\| \geqslant \phi(\|u(t)-v(t)\|) \tag{4}$$

for all $u,v\in C(0,T;X)$ and for all $t\in[0,T]$.

If ϕ is nondecreasing, inequality (4) implies

$$\|u-Bu-(v-Bv)\|_\infty \geqslant \sup\{\phi(\|u(t)-v(t)\|): t\in[0,T]\} = \phi(\|u-v\|_\infty).$$

Otherwise, ϕ is continuous. In this case, there exists $t_0\in[0,T]$ such that $\phi(\|u(t_0)-v(t_0)\|)=\max\{\phi(\|u(t)-v(t)\|): t\in[0,T]\}$. By inequality (4), it is clear that

$$\|u-Bu-(v-Bv)\|_\infty \geqslant \|u(t_0)-g(t_0,u(t_0))-(v(t_0)-g(t_0,v(t_0)))\| \geqslant \phi(\|u(t_0)-v(t_0)\|) \geqslant \phi(\|u-v\|_\infty).$$

Finally, let us show that there exists $r_0>0$ such that $A(B_{r_0}(0))+B(B_{r_0}(0))\subseteq B_{r_0}(0)$.

Otherwise, for every $r>0$ we can find $u_r, v_r\in B_r(0)$ with $\|Au_r+Bv_r\|_\infty>r$. This means that $\frac{1}{r}\|Au_r+Bv_r\|_\infty>1$.

Then we may assume that $\liminf_{r\to\infty}\frac{1}{r}\|Au_r+Bv_r\|_\infty\geqslant 1$. However,

$$\|Au_r(t)+Bv_r(t)\| \leqslant M_r+\int_0^t\|f(\tau,u_r(\tau))\|d\tau \leqslant M_r+\int_0^T(m_1(\tau)\Omega(r)+s(\tau))d\tau \leqslant M_r+\|m\|_1\Omega(r)+\|s\|_1.$$

Consequently,

$$\liminf_{r\to\infty}\frac{1}{r}\|Au_r+Bv_r\|_\infty \leqslant \liminf_{r\to\infty}\frac{\|m\|_1\Omega(r)+M_r}{r}<1.$$

Thus, by Theorem 4.1 we obtain the result. $\square$

Example 4.1. Consider the following nonlinear integral equation:

$$u(t)=\sin(u(t))+\int_0^t\frac{1}{s+1}\sqrt[3]{|u(s)|^2+1}ds, \quad u\in C[0,T]. \tag{5}$$

Let us show that Eq. (5) has a solution in the Banach space $C[0,T]$. In order to show this, we will check that the conditions of Theorem 4.4 are satisfied.

Define $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ by $f(t,x)=\frac{1}{t+1}\sqrt[3]{|x|^2+1}$. This function is clearly continuous and $|f(t,x)|\leqslant m(t)\Omega(|x|)$, where $m(t)=\frac{1}{t+1}$ and $\Omega(r)=\sqrt[3]{r^2+1}$.

Consider the mapping $g:\mathbb{R}\to\mathbb{R}, g(r)=\sin(r)$. It is clear that g is nonexpansive. We shall prove that $I-g$ is ϕ-expansive, where $\phi:\mathbb{R}^+\to\mathbb{R}^+$ is given by.

$$\phi(r)=\begin{cases} r-2\sin\left(\frac{r}{2}\right), & \text{if } 0\leqslant r\leqslant 2\pi,\\ 2\pi, & \text{if } r>2\pi.\end{cases}$$

The mapping ϕ is nondecreasing, continuous and it satisfies that $\phi(0)=0$ and $\phi(r)>0$ for all $r>0$.

Let $x,y\in\mathbb{R}$ and suppose $y\leqslant x$.

Case (i): $|x-y|\leqslant 2\pi$. Since $0\leqslant\frac{x-y}{2}\leqslant\pi$ we have that $\sin\left(\frac{x-y}{2}\right)\geqslant 0$ and thus

$$\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)\leqslant\sin\left(\frac{x-y}{2}\right). \tag{6}$$

Therefore, by (6) and by the trigonometric identity

$$\sin(x)-\sin(y)=2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right),$$

we obtain

$$\sin(x)-\sin(y)\leqslant 2\sin\left(\frac{x-y}{2}\right). \tag{7}$$

Given that $I-g$ is increasing we get by (7) that

$$|(I-g)(x)-(I-g)(y)|=x-\sin(x)-y+\sin(y)\geqslant x-y-2\sin\left(\frac{x-y}{2}\right)=\phi(|x-y|).$$

Case (ii): $|x-y|\geqslant 2\pi$. It is easy to see that the mapping $r\mapsto r-2\left|\sin\left(\frac{r}{2}\right)\right|$ is increasing in $[2\pi,\infty)$. Keeping in mind this we obtain that

$$\begin{aligned}|(I-g)(x)-(I-g)(y)| &\geqslant |x-y|-|\sin(x)-\sin(y)|=x-y-2\left|\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)\right|\geqslant x-y-2\left|\sin\left(\frac{x-y}{2}\right)\right|\geqslant 2\pi\\ &=\phi(|x-y|).\end{aligned}$$

By using the above functions, Eq. (5) can be rewritten as follows

$$u(t) = g(u(t)) + \int_0^t f(s, u(s))ds,$$

where f and g are under the conditions of Theorem 4.4 and thus we can conclude that this equation has a solution.

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