On α-nonexpansive mappings in Banach spaces

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On α-nonexpansive mappings in Banach spaces

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ABSTRACT. In 2011 Aoyama and Kohsaka introduced the α-nonexpansive mappings. Here we present a further study about them and their relationships with other classes of generalized nonexpansive mappings.

1. INTRODUCTION

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Basic definitions

$(X, \|\cdot\|)$ Banach space. $T : C \subset X \to X$

Some types of mappings

- (Bruck'73). **λ-firmly nonexpansive** if there exist $\lambda \in [0,1]$ s.t.

$$\|Tx - Ty\| \leq \|(1-\lambda)(x-y) + \lambda(Tx - Ty)\|.$$

- (Koshaka-Takahashi'08). **Non-spreading** if

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2.$$

- (Takahashi'10). **Hybrid** if

$$3\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 + \|x - y\|^2.$$

Basic definitions

- (Takahashi-Yao'11). **TJ-1** if

$$2\,\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 .$$

- (Takahashi-Yao'11). **TJ-2** if

$$3\,\|Tx - Ty\|^2 \leq 2\,\|Tx - y\|^2 + \|Ty - x\|^2 .$$

α-nonexpansive mappings

α-nonexpansive mappings

(Aoyama-Kohsaka'11)

Let $\alpha < 1$. A mapping $T : C \to X$ is **α-nonexpansive** if

$$\|Tx - Ty\|^2 \leq \alpha \|Tx - y\|^2 + \alpha \|Ty - x\|^2 + (1 - 2\alpha) \|x - y\|^2$$

for all $x, y \in C$

Straightforward consequences of the definition

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- Every constant mapping is α-nonexpansive provided that $0 \leq \alpha \leq \frac{2}{3}$.

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- Every constant mapping is α-nonexpansive provided that $0 \leq \alpha \leq \frac{2}{3}$.

  $T : B_X \to B_X,\ Tx = O_X$ not is α-nonexpansive for any $\alpha > \frac{2}{3}$.

Straightforward consequences of the definition

Proposition.

Let $T : C \to C$ be a mapping. If there exists $x \in C$ such that $T^2x = x$ then either

- x is a fixed point for T,

or

- For any $\alpha \in (0, 1)$, T is not α-nonexpansive.

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Proposition.

Let $T : C \to C$ be a mapping. Assume that there exists a point $x \in C$ such that $\|x - Tx\| = \|Tx - T^2x\| = \|T^2x - x\|$. Then either

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- For any $\alpha \in (0, 1)$, T is not α-nonexpansive.

Some further properties of α-nonexpansive mappings

For $\alpha < 1$ let $\mathcal{N}_\alpha(C) := \{\, T : C \to X \mid T \text{ is } \alpha\text{-nonexpansive} \,\}$.

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An interpolation-type property

If $T \in \mathcal{N}_{\alpha_1}(C) \cap \mathcal{N}_{\alpha_2}(C)$ with $\alpha_1 < \alpha_2$, then $T \in \mathcal{N}_\alpha(C)\ \forall\, \alpha \in [\alpha_1, \alpha_2]$.

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Question: Is the identity mapping the only element in this set?

Answer: NO. For example $T : [0,1] \times [0,1] \to \mathbb{R}^2$, $T(x_1, x_2) = (x_1, 0)$ belongs to $\bigcap_{\alpha \in [0,1)} \mathcal{N}_\alpha([0,1])$.

In general, we have the following obvious result.

Proposition.

For each $i = 1, 2$, let C_i be a nonempty subset of a normed space $(X_i, \|\cdot\|_i)$ and $\alpha \in [0, 1)$. Assume that, for each $i = 1, 2$, $T_i : C_i \to C_i$ is an α-nonexpansive mapping with respect to the norm $\|\cdot\|_i$. Then, the mapping $T : C_1 \times C_2 \to C_1 \times C_2$, defined by

$$T(x_1, x_2) := \big(T_1(x_1), T_2(x_2)\big),$$

is α-nonexpansive with respect to the product norm

$$\|(x_1, x_2)\| := \left[\|x_1\|_1^2 + \|x_2\|_2^2\right]^{\frac{1}{2}}.$$

Relationships with other classes of mappings

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- Continuous mappings
- λ-firmly nonexpansive mappings
- Contractive mappings
- Generalized nonexpansive mappings
- Mean nonexpansive mappings
- (L)-type mappings

Relationship with continuous or λ-firmly nonexpansive mappings

Continuous mappings VS α-nonexpansive mappings

It is obvious that every 0-nonexpansive mapping is continuous. However, for $\alpha > 0$ there exists no relationship between α-nonexpansiveness and continuity.

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λ-firmly nonexpansive mappings VS α-nonexpansive mappings

Let C be a nonempty subset of a normed space X, and $\lambda \in [0, 1)$. If $T : C \to X$ is λ-firmly nonexpansive mapping, then $T \in \mathcal{N}_\alpha$ for all $\alpha \in [0, \hat{\lambda}]$, where

$$\hat{\lambda} := \begin{cases} \lambda & \text{if} \quad 0 \leq \lambda \leq \frac{1}{2}, \\ \\ \frac{\lambda}{1+\lambda} & \text{if} \quad \frac{1}{2} < \lambda < 1. \end{cases}$$

Contractive mappings VS α-nonexpansive mappings

Proposition.

If $T : C \to X$ is k-contractive for $k \in (\frac{1}{3}, 1)$, then T is α-nonexpansive for every $\alpha \in \left[0, \frac{1-k}{1+k}\right]$.

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The range of values of α might not be sharp, even in Hilbert spaces.

Example.

Let H be a Hilbert space. $T : B_H \to B_H$, $Tx = \frac{1}{2}x$ is α-nonexpansive for every $\alpha \in \left[0, \frac{3}{4}\right]$.

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If $T : C \to X$ is k-contractive, with $k \in \left[0, \frac{1}{3}\right]$, then T is α-nonexpansive for every $\alpha \in \left[0, \frac{1}{2}\right]$.

Generalized nonexpansive mappings VS α-nonexpansive mappings

$T : C \to X$ is **generalized nonexpansive** if there exist $a, b, c \in \mathbb{R}$ with $a + 2b + 2c \leq 1$ such that for $x, y \in C$

$$\|Tx - Ty\| \leq a\|x - y\| + b\big(\|x - Tx\| + \|y - Ty\|\big) + c\big(\|x - Ty\| + \|y - Tx\|\big).$$

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Example.

$T : [0, \frac{2}{3}] \to [0, \frac{2}{3}]$, $Tx = x^2$, is α-nonexpansive, for all $\frac{5}{6} \leq \alpha < 1$. However, T fails to be generalized nonexpansive.

Mean nonexpansive mappings

In 2007 Goebel and Japón Pineda introduced a new class of mappings called a-mean nonexpansive mappings, which is wider than the class of the nonexpansive mappings. $T : C \to C$ is a-mean nonexpansive if

$$\sum_{i=1}^{n} a_i \left\| T^i x - T^i y \right\| \leq \|x - y\|, \quad \text{for all} x, y \in C,$$

where the multi-index $a = (a_1, a_2, \ldots, a_n)$ satisfies $a_i \geq 0$ and $\sum_{i=1}^{n} a_i = 1$.

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Definition

Let $a \in (0, 1]$. $T : C \to C$ is ***a*-mean nonexpansive** if

$$a \|Tx - Ty\| + (1 - a) \left\| T^2 x - T^2 y \right\| \leq \|x - y\| \qquad \text{for all } x, y \in C.$$

Remark.

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None of the classes a-mean nonexpansive mappings and α-nonexpansive mappings is included in the other one.

Example (Goebel-Japon'07)

Let C be the unit ball in the space $\mathbb{R}^4$ endowed with the ℓ_1-norm. Let $\tau : \mathbb{R} \to [-\frac{1}{3}, \frac{1}{3}]$ be the function that truncates the argument on the levels $-\frac{1}{3}$ and $\frac{1}{3}$, that is, $\tau(t) = \max\left\{-\frac{1}{3}, \min\left\{\frac{1}{3}, t\right\}\right\}$. Define $T : C \to C$ by

$$T(x_1, x_2, x_3, x_4) = \left(\tau(\tfrac{2}{3}x_4), \tau(2x_1), 0, \tau(\tfrac{6}{5}x_3)\right).$$

- T is a-mean nonexpansive, for all $a \in (0, 1]$.
- For any $\alpha \in (0, 1)$, T is not α-nonexpansive.

Condition (L) VS α-nonexpansive mappings

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Definition. (Dhompongsa-Nanan,011)

A mapping $T : C \to X$ satisfies **condition** (A) on C whenever there exists an a.f.p.s. for T in each nonempty, closed, convex and T-invariant subset D of C, that is, if $\inf\{\|x - Tx\| : x \in D\} = 0$ for every such subset D.

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Definition. (Llorens-Moreno'11)

A mapping $T : C \to C$, where C is a nonempty closed bounded subset of X, satisfies **condition** (L) (or it is an (L)**-type mapping**) on C if

(C_1) T satisfies condition (A) on C.

(C_2) For any a.f.p.s. (x_n) of T in C and each $x \in C$

$$\limsup_{n\to\infty} \|x_n - Tx\| \leq \limsup_{n\to\infty} \|x_n - x\| .$$

Condition (L) VS α-nonexpansive mappings

Theorem.

Let $T : C \to C$ be an α-nonexpansive mapping for some $0 \leq \alpha < 1$, where C is a nonempty closed bounded subset of X. If T satisfies condition (A) on C, then it satisfies condition (L).

If either the set C or the Banach space X have suitable properties, we can obtain fixed point results for α-nonexpansive mappings.

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Corollary 1.

Let C be a nonempty compact convex subset of a Banach space X. If $T : C \to C$:

(i) T is α-nonexpansive for some $0 \leq \alpha < 1$, and

(ii) T satisfies condition (A) on C.

Then, T has a fixed point.

Corollary 2.

Let C be a nonempty weakly compact convex subset of a Banach space X with normal structure. If $T : C \to C$:

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It is unclear whether every α-nonexpansive self-mapping defined on a closed, convex and bounded subset C of a Banach space satisfies condition (A) on C. In other words, we do not know whether assumption (ii) of the above corollaries is essential for the fixed point result.

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- If $\alpha = 0$, property (A) is fulfilled.
- What happens for $0 < \alpha < 1$?

Do α-nonexpansive mappings have an a.f.p.s.?

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Theorem.

Let C be a convex closed bounded subset of a Banach space X. If $T : C \to C$ is $\frac{1}{2}$-nonexpansive, then for every $x \in C$ the sequence $(T^n x)$ is an a.f.p.s. for T.

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Define the constants $c_{n,k}$, with $1 \leq k \leq n$, by

$$\begin{cases} c_{1,1} = c_{2,1} = c_{2,2} = 1, \\ c_{n,1} := 1, c_{n,n} := c_{n,n-1} \quad \text{for } n \geq 3, \\ c_{n,k} := c_{n-1,1} + c_{n-1,2} + \cdots + c_{n-1,k} \quad \text{for } k = 2, \ldots, n-1. \end{cases}$$

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Then, for all $x \in C$ and $n \geq 1$,

$$\|T^{n+1}x - T^n x\|^2 \leq \frac{c_{n,1}}{2^n}\|T^{n+1}x - x\|^2 + \frac{c_{n,2}}{2^{n+1}}\|T^n x - x\|^2 + \cdots + \frac{c_{n,n}}{2^{2n-1}}\|T^2 x - x\|^2.$$

A final remark

Corollary.

Let C be a nonempty weakly compact convex subset of a Banach space X with normal structure. If $T : C \to C$ is nonspreading (i.e, $\frac{1}{2}$-nonexpansive), then T has a fixed point.

A final remark

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Remark.

According to Theorem 4.1 in Kohsaka-Takahashi (2008), if C is a nonempty closed convex and bounded subset of a smooth strictly convex Banach space X, and $T : C \to C$ is a nonspreading mapping, then, T has a fixed point. The above result does not require the assumptions on smoothness and strict convexity for the set C in presence of normal structure.

But, what happens for $0 < \alpha < 1$ with $\alpha \neq \frac{1}{2}$??!

Thank you for your attention!