A further fixed point theorem in orthogonally convex Banach spaces

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1. Classes of mappings
   - Condition (L)
   - 1-set contractive mappings
   - Relation between the two classes
2. Fixed point result
   - Maurey lemma for type (L)+1-set contractive mappings
   - Fixed point theorem in OC Banach spaces

C will be a nonempty, closed, convex and bounded set of a Banach space $(X, \|\cdot\|)$.

An almost fixed point sequence for $T : C \to C$ (a.f.p.s. from now on) is a sequence (x_n) on C such that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Definition (S. Dhompongsa, N. Nanan (2011))

Given a mapping $T : C \to X$ we will say that T satisfies condition (A) on C whenever there exists an a.f.p.s. for T in each nonempty, closed, convex and T-invariant subset D of C, that is, if $.\inf\{\|x - Tx\| : x \in D\} = 0.$

Each nonexpansive mapping satisfies condition (A) on any nonempty, closed, bounded and convex subset of its domain.

Definition

Let $T : C \to C$ be a mapping such that

- *T satisfies condition (A) on C.*
- *For each a.f.p.s. (x_n) para T en C y cada $x \in C$*

$$\limsup_{n\to\infty} \|x_n - Tx\| \leq \limsup_{n\to\infty} \|x_n - x\|.$$

We say that T satisfies condition (L) or is an (L)-type mapping.

This condition does not imply the existence of fixed points in itself. However, there are some results that guarantee the existence of such points under geometric conditions over the space in which the mappings are defined. We recall some of them.

Theorem

Let C be a nonempty compact convex subset of a Banach space X and $T : C \to C$ a mapping satisfying condition (L). Then, T has a fixed point.

Theorem

Let X be a Banach space with normal structure. Let K be a nonempty, weakly compact and convex subset of X. Let $T : K \to K$ be a mapping satisfying condition (L). Then, T has a fixed point.

Lemma

Let K be a nonempty, weakly compact and convex subset of a Banach space X. Let $T : K \to K$ be a mapping satisfying condition (L). Let C be a minimal subset for T on K. Hence, there exists $\lambda > 0$ such that for any a.f.p.s. for T on C (x_n) and any $x \in C$,

$$\limsup_{n \to \infty} \|x_n - x\| = \lambda.$$

A **measure of non-compactness** is a mapping $\mu : \mathcal{P}_b(X) \to [0, \infty)$ such that relatively compact subsets get the measure 0, and other sets get measures that are bigger according to "how far" they are removed from compactness.

A mapping T from a subset M of X into X is said to be k-**set contractive** with respect to a measure of non-compactness μ for a certain $k > 0$ if it is continuous and for any bounded, non-relatively compact subset A of M it holds that

$$\mu(T(A)) \leq k\mu(A).$$

Under the same conditions, a mapping is called **condensing** (with respect to μ) whenever

$$\mu(T(A)) < \mu(A).$$

The main result for this kind of mappings is the so called Darbo's theorem.

Theorem (1955,Darbo)

Let $(X, \|\cdot\|)$ be a Banach space and μ be a measure of non-compactness. Let C represent a nonempty, closed, bounded and convex subset of X. If T is a k-set contractive mapping with respect to μ from C to C with $k < 1$, then T has a fixed point in C.

In 1967 Sadovskii generalices the above result for condensing mappings. However, for the weaker case of $k = 1$, that is, for 1-set contractive mappings, several classic examples show that the existence of fixed point does not remain true.

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1-set contractive mappings satisfy condition (A).

In particular, whenever $T : C \to C$ is a 1-set contractive mapping, the mapping $T_{\varepsilon,x_0} : C \to C$, defined by

$$T_{\varepsilon,x_0}x := (1-\varepsilon)Tx + \varepsilon x_0$$

for any $x_0 \in C$ and $\varepsilon \in (0,1)$ has a fixed point in any closed bounded convex and T_{ε,x_0}-invariant subset of C. This property will play a major role in the proof of an important lemma below.

In the Banach space ℓ_2, let B represent its unit ball, and $T : B \to B$ be the mapping given by

$$T(x) = T(x_1, x_2, x_3, \ldots) = (\sqrt{1 - \|x\|_2^2}, x_1, x_2, x_3, \ldots).$$

This is a well known example of a fixed point free mapping (see page 17 in [1], and Example 12.2. in [3]). In [2], Example 7, it is shown that such mapping is a 1-set contractive mapping for Kuratowski's measure of noncompactness α.

Example

Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space. Consider for any constant $c > 0$ the set $B_c = \{x \in X : \|x\| \leq c\}$. Let us define the following mapping $S : B_2 \to B_2$.

$$S(x) = \begin{cases} -x & \text{si } \|x\| \leq \frac{1}{8} \\ (2 - 8\|x\|)(-x) + (8\|x\| - 1)(1 - \|x\|)x & \text{si } \frac{1}{8} \leq \|x\| \leq \frac{1}{4} \\ (1 - \|x\|)x & \text{si } \frac{1}{4} \leq \|x\| \leq 1 \\ (1 - \frac{1}{\|x\|^2})x & \text{si } 1 \leq \|x\| \leq 2. \end{cases}$$

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Lemma

Let C be a closed bounded and convex subset of a Banach space. Let $T : C \to C$ be a 1-set contractive with respect to a measure of non-compactness μ that in turn satisfies condition (L). Hence, if (x_n) and (y_n) are a.f.p.s. for T in C, there exists another a.f.p.s. for T in C, say (z_n), such that

$$\lim_{n\to+\infty} \|x_n - z_n\| = \lim_{n\to+\infty} \|y_n - z_n\| = \frac{1}{2} \lim_{n\to+\infty} \|x_n - y_n\|$$

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Definition (Jiménez-Melado, Llorens-Fuster, 1988)

Let $(X, \|\cdot\|))$ be a Banach space. Consider for a pair of points x, $y \in X$ the subset defined, for any $\lambda > 0$, by

$$M_\lambda(x, y) = \{z \in X : \max\{\|z - x\|, \|z - y\|\} \leq \frac{1}{2}(1 + \lambda)\|x - y\|\}.$$

If A is a bounded subset of X, let $|A| = \sup\{\|x\| : x \in A\}$, and for a bounded sequence (x_n) in X and $\lambda > 0$, let

$$D[(x_n)] = \limsup_{n\to\infty} \limsup_{m\to\infty} \|x_n - x_m\|$$

$$A_\lambda[(x_n)] = \limsup_{n\to\infty} \limsup_{m\to\infty} |M_\lambda(x_n, x_m)|.$$

A Banach space is said to be **orthogonally convex** (OC) if for each sequence (x_n) in X which converges weakly to 0 and for which $D[(x_n)] > 0$, there exists $\lambda > 0$ such that $A_\lambda[(x_n)] < D[(x_n)]$.

Theorem

Let X be an orthogonally convex Banach space. Let K be a weakly compact closed convex subset of X and let $T : K \to K$ be a 1-set contractive mapping with respect to a measure of noncompactness μ that in turn satisfies condition (L).
Then T has a fixed point.

Proof

Let us suppose that T is fixed point free. There is no loss of generality in assuming that K is minimal w.r.t. the conditions stated. Since T has no fixed points, K is not a singleton and we can assume that $\operatorname{diam}(K) = 1$ and that $0 \in K$.

Since T satisfies condition (L), there is an a.f.p.s (x_n) for T in K, that is,

$$x_n - Tx_n \to 0_X.$$

We can assume again that $x_n \rightharpoonup 0_X$. By the lemma above, for any $x \in K$,

$$\lim_{n\to+\infty} \|x - x_n\| = \lambda \leq \operatorname{diam}(K) = 1.$$

Hence, $D[(x_n)] = \limsup_{n\to\infty}(\limsup_{m\to\infty} \|x_n - x_m\|) \leq \lambda$.

Since X is orthogonally convex, there exists some $\gamma > 0$ such that

$$A_\gamma[(x_n)] < D[(x_n)] \leq \lambda$$

and we can choose $b < \lambda$ such that

$$A_\gamma[(x_n)] = \limsup_{n\to\infty}(\limsup_{m\to\infty}|M_\gamma(x_n, x_m)|) < b$$

and hence there exists some positive integer n_0 such that for any $n \geq n_0$,

$$\limsup_{m\to\infty}|M_\gamma(x_n, x_m)| < b.$$

Fixing $n \geq n_0$ we can obtain another positive integer m_n such that

$$|M_\gamma(x_n, x_{m_n})| < b.$$

Moreover, such m_n can be chosen so that $m_n < m_{n+1}$ for all $n \geq n_0$. We can assume, by passing to subsequences if necessary, that $(\|x_n - x_{m_n}\|)_n$ is convergent to, say, L.

By the lemma above applied to the sequences (x_n) and (x_{m_n}), there is an a.f.p.s. (z_n) for T in K such that

$$\lim_{n\to+\infty}\|x_n - z_n\| = \lim_{n\to+\infty}\|x_{m_n} - z_n\| = \frac{1}{2}L.$$

Hence, there is some $n_1 \geq n_0$ such that for any $n \geq n_1$

$$\max\{\|x_n - z_n\|, \|x_{m_n} - z_n\|\} \leq \frac{1+\lambda}{2}\|x_n - x_{m_n}\|,$$

which implies that for $n \geq n_1$, $z_n \in M_\gamma(x_n, x_{m_n})$.

Consequently, for $n \geq n_1$,

$$\|z_n\| \leq |M_\gamma(x_n, x_{m_n})| < b < \lambda.$$

On the other hand, since $0_X \in K$, by Lemma 1,

$$\|z_n\| \to \lambda,$$

which is a contradiction.

A prominent role in the results proved in this paper is played by the property that the convex combination of a 1-set contractive mapping with a constant in its domain has necessarily a fixed point. If this property were satisfied by (L) type mappings itself, it would suffice to ask for this condition instead of it in combination with 1-set contractiveness. Unfortunately, (L) type mappings do not satisfy this property in general, as seen with this example.

Example

Let $T : [0,1] \to [0,1]$ given by

$$Tx = \begin{cases} 1 - x & x \in [0, \frac{1}{3}) \\ \frac{x+1}{3} & x \in [\frac{1}{3}, 1]. \end{cases}$$

T is a mapping that satisfies condition (L), but the mapping $T_{1/2}$,

$$T_{1/2}x = \begin{cases} \frac{1-x}{2} & x \in [0, \frac{1}{3}) \\ \frac{x+1}{6} & x \in [\frac{1}{3}, 1]. \end{cases}$$

does not have a fixed point.

- D.R. Smart, Fixed point theorems, Cambridge Tracts in Mathematics, No. 66. Cambridge University Press, London-New York, 1974
- J. Ayerbe Toledano, T. Domínguez Benavides, G. ,López Acedo, Measures of Noncompactness in Metric Fixed Point Theory, Birkhäuser, 1997.
- J. Appell, *Measures of noncompactness, condensing operators and fixed points: an application-oriented survey*, Fixed Point Theory, Volume 6, No. 2, 2005, 157-229 http://www.math.ubbcluj.ro/nodeacj/sfptcj.htm