On never nonexpansive maps

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In occasion of Enrique's 70'th birthday

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Joint work with Enrique

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The Problem

$$\left\{ \begin{array}{lcl} X & \equiv & \textit{Reflexive} \\ K & \equiv & \textit{Bounded closed convex} \ \subset \ X \\ T: \ K \to K & \equiv & \textit{Fixed point free} \end{array} \right\}$$

$$??? \ \Downarrow \ ???$$

T is never nonexpansive

By the adverb never we mean for no renorming of X.

The classical maps

Kakutani's map (1943)

Kakutani's (generalized) map is

$$T_K(x) = \varepsilon(1 - \|x\|) \cdot e_1 + Rx, \qquad x \in B_{\ell_2}, \quad 0 < \varepsilon \leqslant 1$$

Known properties of T_K
- *Fixed point free*
- $\sqrt{1+\varepsilon^2}$ – *Lipschitz*
- *Not uniformly Lipschitz in* $B^+_{\ell_2}$ [**Enrique** (2001)]

Recent properties
- *Loose map*
- *Fixable map*
- *Not uniformly Lipschitz in any* 0 – *bcc set in which it is a self – map*

Nirenberg's map (????)

Nirenberg's map is

$$T_N(x) = \sqrt{1- \| x \|^2} \cdot e_1 + Rx, \qquad x \in B_{\ell_2}.$$

$$\text{Known properties of } T_N \begin{cases} \textit{Fixed point free} \\ \textit{Not Lipschitz} \end{cases}$$

Recent properties

$$\left\{ \begin{array}{l} \textit{Loose map} \\ \textit{Fixable map} \\ \textit{Determined map} \\ \textit{Not uniformly Lipschitz in any } 0-\textit{bcc set in which it is a self}-\textit{map} \end{array} \right\}$$

Goebel-Kirk-Thele's map (1973)

Goebel-Kirk-Thele's map is

$$T_{GKT}(x) = \frac{(1-\| x \|)\cdot e_1 + Rx}{\| (1-\| x \|)\cdot e_1 + Rx \|}, \qquad x \in B_{\ell_2}.$$

$$UL(T,K) := \inf\{M > 0 : \ T \text{ is } M-\text{unif.Lipschitz in } K \text{ for some renorming of } X\}$$

Known properties of T_{GKT} $\begin{cases} \text{Fixed point free} \\ UL(T_{GKT}, B^+_{\ell_2}) \geqslant \sqrt{2} \end{cases}$ [**Enrique** (2001)]

Recent properties

$$\begin{cases} \textit{Loose map} \\ \textit{Fixable map} \\ \textit{Determined map} \\ UL(T_{GKT}, K) \geqslant \sqrt{\frac{71+17\sqrt{17}}{54}} \approx 1.616424928, \ \textit{for} \begin{cases} K \equiv 0-bcc \\ T_{GKT}(K) \subset K \end{cases} \end{cases}$$

Baillon's map (1978)

Baillon's map is

$$T_B(x) = \left\{ \begin{array}{ll} \cos(\|x\| \frac{\pi}{2}) \cdot e_1 + \frac{\sin(\|x\| \frac{\pi}{2})}{\|x\|} Rx, & x \in B_{\ell_2} \setminus \{0\}, \\ e_1, & x = 0 \end{array} \right\}$$

Known properties of T_B $\left\{ \begin{array}{l} \textit{Fixed point free} \\ UL(T_B, B^+_{\ell_2}) \geqslant \frac{\pi}{2} \end{array} \right.$ **[Enrique** (2001)]

Recent properties $\left\{ \begin{array}{l} \textit{Loose map} \\ \textit{Fixable map} \\ \textit{Determined map} \\ UL(T_B, K) = \frac{\pi}{2}, \quad \textit{for} \left\{ \begin{array}{l} K \equiv 0 - bcc \\ T_B(K) \subset K \end{array} \right. \end{array} \right.$

P. K. Lin's map (1987)

P. K. Lin's map is

$$F_1(x) = \frac{(\eta(x), x_1, x_2, ...)}{\sqrt{\eta(x)^2 + \| x \|^2}} = \frac{\eta(x)}{\sqrt{\eta(x)^2 + \| x \|^2}} \cdot e_1 + \frac{1}{\sqrt{\eta(x)^2 + \| x \|^2}} \cdot Rx, \qquad x \in \mathbf{K},$$

where $\mathbf{K} := \{x = (x_1, x_2, ...) \in B_{\ell_2} : x_1 \geqslant x_2 \geqslant ... \geqslant 0 \}$, $\eta(x) := \max\{x_1, 1 - \|x\|\}$.

Known properties of F_1:
- *Fixed point free*
- *2 – Lipschitz*
- *Not uniformly Lipschitz* [**Enrique** + *JF* (2015)]

Recent properties
- *Loose map*
- *Determined map*
- *Not uniformly Lipschitz in any* 0 – *bcc set in which it is a self – map*

Modifications of P. K. Lin's map (2016)

The modifications of P. K. Lin's map are the maps F_k, $k \geqslant 2$,

$$F_k(x) = \frac{(\eta(x), x_1, \overset{(k)}{\ldots}, x_1, x_2, \overset{(k)}{\ldots}, x_2, x_3, \overset{(k)}{\ldots}, x_3, \ldots)}{\sqrt{\eta(x)^2 + k \parallel x \parallel^2}}, \qquad x \in B_{\ell_2},$$

where η is as before.

Known properties of F_k $\begin{cases} \textit{Fixed point free} \\ \frac{k+1}{\sqrt{k}} - \textit{uniformly Lipschitz} \end{cases}$ [**Enrique** + *JF* (2016)]

Recent properties $\begin{cases} \textit{Loose map} \\ \textit{Determined map} \\ UL(F_k, K) \geqslant \sqrt{\frac{k}{k-1}}, \quad \textit{for} \begin{cases} K \equiv 0 - bcc \\ F_k(K) \subset K \end{cases} \end{cases}$

Partial answer to the Problem for a class of maps defined in the unit ball

Loose maps

$$\begin{cases} B_X \equiv \textit{closed unit ball of the Banach space } X \\ T : B_X \to B_X \end{cases}$$

T is a *loose map* whenever:

$$\sup_{0<\lambda\leqslant 1} \frac{d(T(0), T(S_\lambda))}{\lambda} > 1.$$

where $S_\lambda = \{x \in B_X : \|x\| = \lambda\}$.

Loose maps

Result 1 [Beauzamy's technique]

$X \equiv$ Any Banach space

$$\textit{If} \quad T : B_X \to B_X \text{ is a loose map} \quad \Rightarrow \quad T \text{ is never nonexpansive.}$$

The classical maps are all loose

$\varphi, \psi : [0,1] \to \mathbb{R} \equiv$ Real functions satisfying

$$\begin{cases} 1) & \varphi(0) \neq 0 \\ 2) & |\psi(t)| \geqslant 1,\ t \in]0,1] \\ 3) & \varphi(t)^2 + \psi(t)^2 t^2 \leqslant 1,\ t \in [0,1]. \end{cases}$$

Let

$$T_{\varphi,\psi}(x) := \varphi(\|x\|) \cdot e_1 + \psi(\|x\|) \cdot Rx, \qquad x \in B_{\ell_2} \ [\textit{The}\ (\varphi,\psi) - \textbf{map}]$$

$e_1 \equiv$ First unit vector; $R \equiv$ Right-shift.

Condition 2) implies that the maps $T_{\varphi,\psi}$ have no fixed points.

The classical maps are all loose

Result 2

If $T : B_{\ell_2} \longrightarrow B_{\ell_2}$ is such that

$$\exists \lambda_0 \in]0,1] \textit{ such that } \left\{ \begin{array}{l} T_{|\lambda_0 B_{\ell_2}} \textit{ is a } (\varphi,\psi)-\textit{map} \\ \\ \varphi(t) \textit{ is not constant in } [0,\lambda_0] \end{array} \right\} \Rightarrow T \quad \textit{is a loose map.}$$

Notice that all classical maps, except the modifications of P. K. Lin's, satisfy the requirement of Result 2.
The modifications of P. K. Lin's map, F_k, $k \geqslant 2$, can be seen to also be loose maps in a direct way.

Fixable maps in a Hilbert space

$H \equiv$ Hilbert space
$T : B_H \to B_H$

$$\begin{cases} \Lambda_1(T) := \{\lambda \in]0,1] : \; \forall x \in S_\lambda, \; \|T(x)\| \geqslant \lambda\} \\ \\ \Lambda_2(T) := \{\lambda \in]0,1] : \; \forall x \in S_\lambda, \; \langle T(0), T(x) \rangle \leqslant 0\}. \end{cases}$$

T is a fixable map when at least one of the 3 next conditions is satisfied

$$(a) \quad \inf(]0,1] \setminus \Lambda_1(T)) = 0$$

$$(b) \quad \inf(\Lambda_2(T)) = 0$$

$$(c) \quad \Lambda_1(T)) \cap \Lambda_2(T) \neq \emptyset.$$

Fixable maps in a Hilbert space

Result 3 [Slight extension of a result of Antonio, **Enrique** and Jesús (1997)]

$$\left.\begin{array}{l} T : B_H \to B_H \textit{ is a fixable map} \\ T(0) \neq 0 \end{array}\right\} \Rightarrow T \textit{ is a loose map}.$$

Consequently, such map T is never nonexpansive.

The classical maps, except P. K. Lin's and its modifications, are fixable

Result 4

For maps of the form $T_{\varphi,\psi}$, with φ, ψ as before, *the map $T_{\varphi,\psi}$ is fixable*.

P. K. Lin's map and its modifications F_k, $k \geqslant 1$, are in fact not fixable.

Partial answer to the Problem for a class of maps defined in an arbitrary set

Relevant sets for self-maps

$$K \equiv 0 - bcc\ subset\ of\ the\ separable\ reflexive\ space\ X$$

$$T : K \to K \equiv Any\ map$$

$$For\ t \in]0, 1[, \qquad I_t :=]0, \min\{t, 1 - t\}[$$

$$(u_m)_m \subset K$$

Relevant sets for self-maps

$$\begin{cases} D_{u_m,n,t}(T) := \left\{ \frac{T^n((t+h)u_m)-T^n((t-h)u_m)}{2h} \ : \ h \in I_t \right\} \ [\textit{The } T - \textit{Gateaux quotients at } u_m] \\ D_{(u_m),n,t}(T) := \left\{ (y_m)_m \ : \ y_m \in \overline{D_{u_m,n,t}(T)}^{w}, \ m \geqslant 1 \right\} \\ \qquad [\textit{Sequences of weak} - \textit{limits of } T - \textit{Gateaux quotients}] \\ \widetilde{D}_{(u_m),n,t}(T) \equiv \{\textit{Weak limits of subsequences of } (y_m)_m \in D_{(u_m),n,t}(T)\} \\ D_{(u_m)}(T) := \bigcup_{n\geqslant 1,\ 0<t<1} \widetilde{D}_{(u_m),n,t}(T) \ \ [\textit{The } T - \textit{set of } (u_m)] \end{cases}$$

Relevant sets for self-maps

Result 5 [**Enrique**'s technique]

If

$$T \;:\; K \to K \;\equiv\; M - unif.\ Lipschitz\ resp.\ renorming\ ||| \cdot |||,$$

then, for any sequence $(u_m) \subset K$,

$$M \cdot \limsup_m ||| u_m ||| \geqslant \sup\{||| z ||| : \quad z \in D_{(u_m)}(T) \}$$

Corollary

$\exists\, (u_m) \subset K$ *whose* $T-$*set* $D_{(u_m)}(T)$ *is not bounded* $\Rightarrow$ *T is not uniformly Lipschitz*

Relevant sets for self-maps

Result 6 [Kakutani's, Nirenberg's and P. K. Lin's maps are never unif. Lipschitz]

1) $\underline{T_K} \left\{ \begin{array}{l} K \equiv 0 - bcc \subset B_{\ell_2} \\ T_K(K) \subset K \end{array} \right\} \Rightarrow \sup\{\| z \| : z \in D_{(T_K^m(0))}(T_K)\} = \infty$

2) $\underline{T_N} \left\{ \begin{array}{l} K \equiv 0 - bcc \subset B_{\ell_2} \\ T_N(K) \subset K \end{array} \right\} \Rightarrow \sup\{\| z \| : z \in D_{(T_N^m(0))}(T_N)\} = \infty$

3) $\underline{F_1} \left\{ \begin{array}{l} K \equiv 0 - bcc \subset B_{\ell_2} \\ F_1(K) \subset K \end{array} \right\} \Rightarrow \sup\{\| z \|: z \in D_{(F_1^m(0))}(F_1)\} = \infty.$

Determined maps

$K \equiv 0-bcc \subset X \equiv$ separable reflexive
$T : K \to K$
$u \in K$ is a *determinant point* for T whenever

$$\begin{cases} T^n(u) \neq 0, \quad n \geqslant 1, \\ \Delta(T,u,n) := D_{(T^m(u))}(T) \cap span\{T^n(u)\} \neq \emptyset, \quad n \geqslant 1, \\ \sup\{\frac{\|z\|}{\|T^n(u)\|} : z \in \Delta(T,u,n), \ n \geqslant 1\} > 1. \end{cases}$$

By $\Delta(T)$ we represent the set (possibly empty) of the determinant points for T.
We say that T is a *determined map* whenever $\Delta(T) \neq \emptyset$

Determined maps

Result 7 [**Enrique**'s technique]

$$\left.\begin{array}{l} X \equiv \textit{separable reflexive} \\ \\ K \equiv 0 - bcc \ \subset X \\ \\ T : K \to K \textit{ is a determined map} \\ \\ T(0) \ \neq \ 0 \end{array}\right\} \Rightarrow T \textit{ is never nonexpansive}.$$

All classical maps, except Kakutani's, admit zero as a determinant point

Result 8

$$\left\{\begin{array}{l} T_{\varphi,\psi} \equiv \textit{as before} \\ \varphi(t)^2 + t^2\psi(t)^2 = 1, \quad t \in [0,1] \\ \exists \lambda_0 \in]0,1], \ \varphi \ \textit{differentiable in }]0,\lambda_0[. \\ K \ \equiv 0 - bcc \ \subset B_{\ell_2} \\ T_{\varphi,\psi}(K) \subset \ K \\ T := T_{\varphi,\psi_{|K}} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \textit{If } \sup_{0<t<\lambda_0} \ |\varphi'(t)| \ > \ 1, \\ \textit{then } 0 \textit{ is a determinant point for } T \\ \textit{i.e., } T \textit{ is a determined map}. \end{array}\right.$$

Corollary

All classical maps, except Kakutani's T_K,

(i) Satisfy the above requirements

(ii) Have 0 as a determinant point and so they are determined maps

(iii) Since they are fixed point free, they are never nonexpansive in any such K

Never pseudo-contractiveness of P. K. Lin's map and its modifications

Never pseudo-contractiveness of P. K. Lin's map and its modifications

$H \equiv$ Hilbert
$T : B_H \to H, \quad A \subset B_H,$ we define

$$L(T, A) := \inf\{M > 0 : T \text{ is M -- Lipschitz in A respect to some renorming of } H\}.$$

Notice that, if T is nonexpansive in $A \subset B_H$ for some renorming of H, then $L(T, A) \leqslant 1$.

Never pseudo-contractiveness of P. K. Lin's map and its modifications

$T: \; D \subset X \to X$ is a **pseudo-contractive map** whenever

$$\| x - y \| \leqslant \| (1 + t)(x - y) - t(T(x) - T(y)) \|, \quad t > 0, \;\; x, y \in D \; .$$

Jesús GF (2002) proved the following:

If $T : B_H \to H$ satisfies

$$\left\{ \begin{array}{ll} (1) & T(S_H) \subset S_H \\ (2) & T(0) \textit{ is orthogonal to } T(S_H) \\ (3) & T \textit{ is nonexpansive in } S_H \\ (4) & T(0) \neq 0 \end{array} \right\} \Rightarrow \textit{Then } T \textit{ is never pseudo} - \textit{contractive}.$$

As a consequence, he proved that all classical maps (by that time it seems that P. K. Lin's was not too famous) are never pseudo-contractive in the closed unit ball of ℓ_2.

Never pseudo-contractiveness of P. K. Lin's map and its modifications

It can be easily seen that neither P. K. Lin's map, nor its modifications, satisfy conditions (2) and (3) before stated and so Jesus' result cannot be applied.

The next result proves that these maps are also never pseudo-contractive in B_{ℓ_2}.

Result 9

Let $T : B_H \to H$ be a map and consider the closed subset of S_H

$N(T) \; := \; \{ \, x \in S_H \; : \; \langle T(0), x \rangle \; \leqslant \; 0 \; \}$. [*The obtuse half – sphere respect to* $T(0)$]

Assume that the following conditions are satisfied

(i) $T(N(T)) \subset S_H$
(ii) $T(0) \perp T(N(T))$
(iii) $L(T, N(T)) \leqslant 1$
(iv) $T(0) \neq 0$.

Then T is never pseudo-contractive.

Woops................ SURPRISE !!!

BIRTHDAY PRESENT

Result 10

All fixed point free maps of the form $T_{\varphi,\psi}$ are such that

$$L(T_{\varphi,\psi}, B_{\ell_2}) \geqslant \frac{\pi}{2} \ .$$

Result 10

All fixed point free maps of the form $T_{\varphi,\psi}$ are such that

$$L(T_{\varphi,\psi}, B_{\ell_2}) \geqslant \frac{\pi}{2} \ .$$

Consequently

$$UL(T_{\phi,\psi}, B_{\ell_2}, \|\cdot\|_2) \ \geqslant \ UL(T_{\varphi,\psi}, B_{\ell_2}) \ \geqslant \ L(T_{\varphi,\psi}, B_{\ell_2}) \geqslant \frac{\pi}{2} \ .$$

Thanks everybody

......and.......

Happy Birthday Enrique
