Playing with equivalent norms and the Fixed Point Property

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On the occasion of Enrique Llorens-Fuster’s 70th birthday
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A Banach space \((X, \| \cdot \|)\) has the **Fixed Point Property (FPP)** if for every closed convex bounded set \(C\) and for every **nonexpansive** mapping \(T : C \rightarrow C\), there is a fixed point.

Nonexpansiveness of a mapping depends on the underlying norm, since it means

\[
\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in C
\]

A mapping \(T : C \rightarrow C\) may be nonexpansive for a norm \(\| \cdot \|\) and it may fail this property for an equivalent norm \(| \cdot |\) on \(X\).
From now on, when we refer to the FPP we have to specify which is the norm in action.

$$(\ell_1, \| \cdot \|_1)$$ fails the FPP

$$C = \overline{co}(e_n) = \left\{ x = \sum_{n=0}^{\infty} t_ne_n : t_n \geq 0, \sum_{n=1}^{\infty} t_n = 1 \right\}; \quad T : C \to C$$

$$T \left( \sum_{n=1}^{\infty} t_ne_n \right) = \sum_{n=1}^{\infty} t_ne_{n+1}$$

$T$ is fixed point free and $\|Tx - Ty\|_1 = \|x - y\|_1 \forall x, y \in C$.

**Corollary**

*Every Banach space $(X, \| \cdot \|)$ containing an **isometric** copy of $(\ell_1, \| \cdot \|_1)$ fails to have the FPP: $(L_1(\mu), \| \cdot \|_1)$, $(\ell_\infty, \| \cdot \|_\infty)$, $(C[0, 1], \| \cdot \|_\infty)$.***

What do we know if $(X, \| \cdot \|)$ contains an isomorphic copy of $\ell_1$?
Renormings of $\ell_1$ with the FPP. First concepts

**Theorem (James)**

If $(X, \| \cdot \|)$ contains an isomorphic copy of $\ell_1$ then for all $\epsilon > 0$ there exists $(x_n) \subset X$ such that

$$(1 - \epsilon) \sum_{n=1}^{\infty} |t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq \sum_{n=1}^{\infty} |t_n| \quad \forall (t_n) \in \ell_1$$

**Definition (J. Hagler, 1972)**

A Banach space $(X, \| \cdot \|)$ is said to contain an asymptotically isometric copy of $\ell_1$ if there exist $(x_n) \subset X$ and $(\epsilon_n) \downarrow 0$ with

$$\sum_{n}^{\infty} (1 - \epsilon_n)|t_n| \leq \left\| \sum_{n}^{\infty} t_n x_n \right\| \leq \sum_{n}^{\infty} |t_n| \quad \forall (t_n) \in \ell_1$$
Theorem (P. Dowling, C. Lennard, B. Turett, 1996)

If a Banach space \((X, \| \cdot \|)\) contains an a.i.c. of \(\ell_1\), then \((X, \| \cdot \|)\) fails to have the FPP.

- Every infinity dimensional subspace of \((\ell_1, \| \cdot \|_1)\) fails the FPP.
- Every nonreflexive subspace of \((L_1[0, 1], \| \cdot \|_1)\) fails the FPP.
- Let \(\Gamma\) be an uncountable set. Every renorming of \(\ell_1(\Gamma)\) contains an asymptotically isometric copy of \(\ell_1\). \(\ell_1(\Gamma)\) cannot be renormed to have the FPP.
- \(\ell_\infty\) contains \(\ell_1(\Gamma)\) for some nonseparable \(\Gamma\). The space \(\ell_\infty\) cannot be renormed to have the FPP.
There are renormings of $\ell_1$ without a.i.c. of $\ell_1$

**Lemma (P. Dowling, W. Johnson, C. Lennard, B. Turett, 1997)**

Let $(\gamma_k) \subset (0, 1)$ such that $\lim_k \gamma_k = 1$. Then

$$|||x||| := \sup_k \gamma_k \sum_{n=k}^{\infty} |x_n|, \quad x = \sum_{n=1}^{\infty} x_n e_n$$

is an equivalent norm in $\ell_1$ and $(\ell_1, ||| \cdot |||)$ does not have an a.i.c. of $\ell_1$.

$$\gamma_1 \|x\|_1 \leq |||x||| \leq \|x\|_1 \quad \forall x \in \ell_1$$
Fix a sequence $p = (p_n)_n \subset (1, +\infty)$ with $(p_n) \downarrow 1$.

Let $c_{00}$ be the space of all real sequences with finitely many non-null coordinates. We define the norm

$$\nu_p(x) = \lim_n \nu_n(p, x)$$

where

$$\nu_1(p, x) := |x_1|, \quad \nu_{n+1}(p, x) := (|x_1|^{p_1} + \nu_n(Sp, Sx)^{p_1})^{1/p_1},$$

with $x = (x_1, x_2, \ldots)$ and $Sz := (z_2, z_3, \ldots)$ when $z = (z_1, z_2, \ldots)$.

Let $X$ be the completion of $c_{00}$ with the $\nu_p(\cdot)$ norm.

$$X := (\overline{c_{00}}, \nu_p(\cdot))$$
Theorem

Let $q = (q_n)$ be the sequence satisfying $\frac{1}{p_n} + \frac{1}{q_n} = 1$. The following are equivalent:

a) The norm $\nu_p(\cdot)$ is equivalent to the $\ell_1$ norm $\| \cdot \|_1$ and $X = (\ell_1, \nu_p(\cdot))$,

b) There exists some $\delta > 0$ so that $q_n \geq \delta \log n$ for all $n \in \mathbb{N}$

- If a) fails, $(X, \nu_p(\cdot))$ is not isomorphic to $\ell_1$. 
Theorem

If $\nu_p(\cdot)$ is equivalent to $\ell_1$, then $(\ell_1, \nu_p(\cdot))$ fails to contain an a.i.c. of $\ell_1$. 


1. Does $(\ell_1, |||\cdot|||)$ have the FPP?
2. If $\nu_p(\cdot)$ is equivalent to $\ell_1$, does $(\ell_1, \nu_p(\cdot))$ have the FPP?

Theorem (P. K. Lin, 2008) $(\ell_1, |||\cdot|||)$ has the FPP.
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1. Does $(\ell_1, |||\cdot|||)$ have the FPP?
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Theorem (P.K. Lin, 2008)

$(\ell_1, |||\cdot|||)$ has the FPP.
Some key facts of $||| \cdot |||$ used in P.K. Lin’s proof:

1. For every $\sigma(\ell_1, c_0)$-null convergent sequence $(x_n)$ and for every $x \in \ell_1$,
   \[
   \limsup_n \|x_n + x\|_1 = \limsup_n \|x_n\|_1 + \|x\|_1 \quad (\ast)
   \]
   $\ast$ fails for $||| \cdot |||$. However, since $\gamma_k \|x\|_1 \leq |||x||| \leq \|x\|_1$ if $x \geq k$, we can still derive
   \[
   \limsup_n |||x_n||| + |||x||| \leq \frac{1}{\gamma_k} \limsup_n |||x_n + x|||
   \]
   for every $(x_n)$ a $w^*$-null sequence and $x \in \ell_1$ with $x \geq k$. 
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   \]
   for every $(x_n)$ a $w^*$-null sequence and $x \in \ell_1$ with $x \geq k$.

2. Fix $k \in \mathbb{N}$. If $(x_n)$ is $w^*$-null, $\liminf_n |||x_n||| \geq 1$, $|||x||| \leq 1$ and $x \leq k$ then
   \[
   \limsup_n |||x_n + \lambda x||| \leq \limsup_n |||x_n|||
   \]
   for all $0 \leq \lambda < 1 - \gamma_k$. 

Let $X$ be a Banach space with a basis $(e_n)$. 

$Q_k(x) = \sum_{n=k}^{\infty} x_n e_n$.

$(x_n)$ is a block basic sequence (b.b.s.) if $(x_n)$ is bounded and

$$x_n = \sum_{i=p_n}^{q_n} a_i e_i$$

with $p_1 \leq q_1 < p_2 \leq q_2 < \cdots$.

An equivalent norm $|\cdot|$ is premonotone if $|Q_k(x)| \leq |x|$ for every $x \in X$.

**Definition**

$$S_k(X, |\cdot|) := \sup \left\{ \frac{\lim \sup_n |x_n| + |x|}{\lim \sup_n |x + x_n|} : (x_n) \text{ b.b.s., } x \geq k \right\}$$
\[
\limsup_n |x_n| + |x| \leq S_k(X, \cdot) \limsup_n |x + x_n|
\]
for every \((x_n)\) b.b.s. and \(x \geq k\).
\[
\lim_{n} \sup \| x_n \| + \| x \| \leq S_k(X, \cdot ) \lim_{n} \sup \| x + x_n \|
\]
for every \((x_n)\) b.b.s. and \(x \geq k\).

**Definition (A. Barrera-Cuevas, M.A. Japón, 2015)**

An equivalent norm \(\cdot\) is a sequentially separating norm if

\[
\lim_{k} S_k(X, \cdot \cdot ) = 1.
\]
Definition (2016-2017)

Let $X$ be a Banach space with a Schauder basis. A norm $|\cdot|$ on $X$ is called **near-infinity concentrated** (n.i.c.) if it has the following properties:

1. $|\cdot|$ is premonotone and **sequentially separating**.

2. For every $k \in \mathbb{N}$, there exists $F_k : (0, +\infty) \to [0, +\infty)$ with

$$
\limsup_{n} |x_n + \lambda x| \leq \limsup_{n} |x_n| + F_k(\lambda)|x|,
$$

for every b.b.s. $(x_n)$, with $\liminf_{n} |x_n| \geq 1$, for every $x \in X$ with $|x| \leq 1$ and $x \geq k$, for all $\lambda \in (0, +\infty)$, and such that

$$
\lim_{\lambda \to 0^+} \frac{F_k(\lambda)}{\lambda} = 0
$$
**Theorem**

Let $X$ be a Banach space with a boundedly complete Schauder basis and let $|\cdot|$ be a **near-infinity concentrated norm**. Then $(X, |\cdot|)$ has the FPP.
Theorem

Let $X$ be a Banach space with a boundedly complete Schauder basis and let $\| \cdot \|$ be a near-infinity concentrated norm. Then $(X, \| \cdot \|)$ has the FPP.

Corollary (P.K. Lin, 2008)

$(\ell_1, \|\| \cdot \||)$ has the FPP.

P.K. Lin’s norm is just the first element of a sequence of equivalent norms in $\ell_1$ with the FPP. This sequence can be defined recursively by

\[
\begin{align*}
p_0(x) &= \|x\|_1, \quad p_1(x) = \|\|x\||, \quad p_2(x) = \sup_k \gamma_k \|Q_k(x)\|, \\
p_3(x) &= \sup_k \gamma_k p_2(Q_k(x)), \ldots \quad p_{n+1}(x) = \sup_k \gamma_k p_n(Q_k(x))
\end{align*}
\]
Theorem

1. The norm $\nu_p(\cdot)$ is a near-infinity concentrated norm on $X$.

2. The Banach space $(X, \nu_p(\cdot))$ has the FPP for every choice $p = (p_n) \downarrow 1$, regardless of whether it is isomorphic to $\ell_1$. 
Theorem

1. The norm $\nu_p(\cdot)$ is a near-infinity concentrated norm on $X$.

2. The Banach space $(X, \nu_p(\cdot))$ has the FPP for every choice $p = (p_n) \downarrow 1$, regardless of whether it is isomorphic to $\ell_1$.

Theorem

$(\ell_1, ||| \cdot ||| + \nu_p(\cdot))$ has the FPP.
There exist equivalent norms on $\ell_1$ with the FPP but failing to be near-infinity concentrated norms.

**Theorem (C. Hernández-Linares, M. Japón, E. Llorens-Fuster)**

$(\ell_1, \| \cdot \|_1 + \lambda \| \cdot \|)$ has the FPP for every $\lambda > 0$.

$\| \cdot \|_1 + \| \| \cdot \| \| \| \| \| \|$ does not satisfy condition 2) in the definition of near-infinity concentrated norms. We can not include these examples with the previous techniques.
Theorem (C. Hernández-Linares, C. Lennard, M. Japón)

The set of renormings in $\ell_1$ which fail to contain a.i.c. of $\ell_1$ is dense in the set of all renormings of $\ell_1$.

Question:

- If $(\ell_1, |\cdot|)$ fails to have an a.i.c. of $\ell_1$, does $(\ell_1, |\cdot|)$ have the FPP?

- Does there exist an equivalent norm on $\ell_1$ without the FPP and without asymptotically isometric copies of $\ell_1$?

- Can the equivalent norms on $\ell_1$ with the FPP be characterized by some geometric property?
Extensions to more general family of functions

**Definition (E. Llorens-Fuster, E. Moreno-Gálvez, 2011)**

A mapping \( T : C \to C \) satisfies the \((L)\) condition if

1. every closed convex bounded \( T \)-invariant subset \( D \subset C \) contains an approximate fixed point sequence (a.f.p.s.), that is, a sequence \( (x_n) \subset D \) with \( \lim_n \| x_n - Tx_n \| = 0 \).

2. For ever a.f.p.s. \( (x_n) \) and \( x \in C \)

\[
\limsup_n \| x_n - Tx \| \leq \limsup_n \| x_n - x \|
\]

Nonepansive mappings, mappings satisfying condition \((C)\) of Suzuki, and some others satisfying condition \((L)\).
Extensions to more general family of functions

Definition (E. Llorens-Fuster, E. Moreno-Gálvez, 2011)

A mapping $T : C \rightarrow C$ satisfies the $(L)$ condition if

1. every closed convex bounded $T$-invariant subset $D \subset C$ contains an approximate fixed point sequence (a.f.p.s.), that is, a sequence $(x_n) \subset D$ with $\lim_n \|x_n - Tx_n\| = 0$.

2. For every a.f.p.s. $(x_n)$ and $x \in C$
   
   $$\limsup_n \|x_n - Tx\| \leq \limsup_n \|x_n - x\|$$

Nonepansive mappings, mappings satisfying condition $(C)$ of Suzuki, and some others satisfying condition $(L)$.

Theorem

$(X, \nu_p(\cdot))$ has the FPP for mappings satisfying condition $(L)$. 
Some of the main references


- C. A. Hernández-Linares, C. Lennard, M. A. Japón, Renormings failing to have asymptotically isometric copies of \(\ell_1\) or \(c_0\). Nonlinear Anal. 77 (2013), 1127-1117

- P. K. Lin, *There is an equivalent norm on \(\ell_1\) that has the fixed point property.* Nonlinear Anal., 68 (8) (2008), 2303-2308.

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