Playing with equivalent norms and the Fixed Point Property

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Introduction

Definition

A Banach space $(X, \|\cdot\|)$ has the Fixed Point Property (FPP) if for every closed convex bounded set C and for every nonexpansive mapping $T : C \to C$, there is a fixed point.

Nonexpansiveness of a mapping depends on the underlying norm, since it means

$$\|Tx - Ty\| \leq \|x - y\| \qquad \forall x, y \in C$$

A mapping $T : C \to C$ may be nonexpansive for a norm $\|\cdot\|$ and it may fail this property for an equivalent norm $|\cdot|$ on X.

From now on, when we refer to the FPP we have to specify which is the norm in action.

$(\ell_1, \|\cdot\|_1)$ fails the FPP

$$C = \overline{co}(e_n) = \left\{ x = \sum_{n=0}^{\infty} t_n e_n : t_n \geq 0, \sum_{n=1}^{\infty} t_n = 1 \right\}; \quad T : C \to C$$

$T\left(\sum_{n=1}^{\infty} t_n e_n\right) = \sum_{n=1}^{\infty} t_n e_{n+1}$

T is fixed point free and $\|Tx - Ty\|_1 = \|x - y\|_1 \ \forall x, y \in C$.

Corollary

Every Banach space $(X, \|\cdot\|)$ containing an **isometric** *copy of $(\ell_1, \|\cdot\|_1)$ fails to have the FPP: $(L_1(\mu), \|\cdot\|_1)$, $(\ell_\infty, \|\cdot\|_\infty)$, $(C[0,1], \|\cdot\|_\infty)$.*

What do we know if $(X, \|\cdot\|)$ contains an isomorphic copy of ℓ_1?

Renormings of ℓ_1 with the FPP. First concepts

Theorem (James)

If $(X, \|\cdot\|)$ contains an isomorphic copy of ℓ_1 then for all $\epsilon > 0$ there exists $(x_n) \subset X$ such that

$$(1-\epsilon)\sum_{n=1}^{\infty}|t_n| \le \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \le \sum_{n=1}^{\infty}|t_n| \qquad \forall (t_n) \in \ell_1$$

Definition (J. Hagler, 1972)

A Banach space $(X, \|\cdot\|)$ is said to contain an **asymptotically isometric copy of** ℓ_1 if there exist $(x_n) \subset X$ and $(\epsilon_n) \downarrow 0$ with

$$\sum_{n}^{\infty}(1-\epsilon_n)|t_n| \le \left\|\sum_{n}^{\infty} t_n x_n\right\| \le \sum_{n}^{\infty}|t_n| \quad \forall (t_n) \in \ell_1$$

Theorem (P. Dowling, C. Lennard, B. Turett, 1996)

If a Banach space $(X, \|\cdot\|)$ contains an a.i.c. of ℓ_1, then $(X, \|\cdot\|)$ fails to have the FPP.

- Every infinity dimensional subspace of $(\ell_1, \|\cdot\|_1)$ fails the FPP.
- Every nonreflexive subspace of $(L_1[0,1], \|\cdot\|_1)$ fails the FPP.
- Let Γ be an uncountable set. Every renorming of $\ell_1(\Gamma)$ contains an asymptotically isometric copy of ℓ_1. $\ell_1(\Gamma)$ cannot be renormed to have the FPP.
- ℓ_∞ contains $\ell_1(\Gamma)$ for some nonseparable Γ. The space ℓ_∞ cannot be renormed to have the FPP.

There are renormings of ℓ_1 without a.i.c. of ℓ_1

Lemma (P. Dowling, W. Johnson, C. Lennard, B. Turett, 1997)

Let $(\gamma_k) \subset (0,1)$ *such that* $\lim_k \gamma_k = 1$. *Then*

$$|||x||| := \sup_k \gamma_k \sum_{n=k}^{\infty} |x_n|, \qquad x = \sum_{n=1}^{\infty} x_n e_n$$

is an equivalent norm in ℓ_1 *and* $(\ell_1, |||\cdot|||)$ *does not have an a.i.c. of* ℓ_1.

$$\gamma_1 \|x\|_1 \leq |||x||| \leq \|x\|_1 \qquad \forall x \in \ell_1$$

P.Dowling, W. Johnson, C.Lennard, B. Turett, 1997

Fix a sequence $p = (p_n)_n \subset (1, +\infty)$ with $(p_n) \downarrow 1$.

Let c_{00} be the space of all real sequences with finitely many non-null coordinates. We define the norm

$$\nu_p(x) = \lim_n \nu_n(p, x)$$

where

$$\nu_1(p, x) := |x_1|, \qquad \nu_{n+1}(p, x) := \left(|x_1|^{p_1} + \nu_n(Sp, Sx)^{p_1}\right)^{1/p_1},$$

with $x = (x_1, x_2, ...)$ and $Sz := (z_2, z_3, ...)$ when $z = (z_1, z_2, ...)$.

Let X be the completion of c_{00} with the $\nu_p(\cdot)$ norm.

$$X := (\overline{c_{00}}, \nu_p(\cdot))$$

Theorem

Let $q = (q_n)$ *be the sequence satisfying* $\frac{1}{p_n} + \frac{1}{q_n} = 1$*. The following are equivalent:*

a) *The norm* $\nu_p(\cdot)$ *is equivalent to the* ℓ_1 *norm* $\|\cdot\|_1$ *and* $X = (\ell_1, \nu_p(\cdot))$,

b) *There exists some* $\delta > 0$ *so that* $q_n \geq \delta \log n$ *for all* $n \in \mathbb{N}$

- If $a)$ fails, $(X, \nu_p(\cdot))$ is not isomorphic to ℓ_1.

Theorem

If $\nu_p(\cdot)$ *is equivalent to* ℓ_1*, then* $(\ell_1, \nu_p(\cdot))$ *fails to contain an a.i.c. of* ℓ_1*.*

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Questions: (P. Dowling, W. Johnson, C.Lennard, B. Turett, 1997)

1. Does $(\ell_1, |||\cdot|||)$ have the FPP?
2. If $\nu_p(\cdot)$ is equivalent to ℓ_1, does $(\ell_1, \nu_p(\cdot))$ have the FPP?

Theorem

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Theorem (P.K. Lin, 2008)

$(\ell_1, |||\cdot|||)$ has the FPP.

Some key facts of $|||\cdot|||$ used in P.K. Lin's proof:

1. For every $\sigma(\ell_1, c_0)$-null convergent sequence (x_n) and for every $x \in \ell_1$,
$\limsup_n \|x_n + x\|_1 = \limsup_n \|x_n\|_1 + \|x\|_1 \qquad (*)$
$(*)$ fails for $|||\cdot|||$. However, since $\gamma_k\|x\|_1 \le |||x||| \le \|x\|_1$ if $x \geq k$, we can still derive

$$\limsup_n |||x_n||| + |||x||| \le \frac{1}{\gamma_k} \limsup_n |||x_n + x|||$$

for every (x_n) a w^*-null sequence and $x \in \ell_1$ with $x \geq k$.

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2. Fix $k \in \mathbb{N}$. If (x_n) is w^*-null, $\liminf_n |||x_n||| \geq 1$, $|||x||| \leq 1$ and $x \leq k$ then

$$\limsup_n |||x_n + \lambda x||| \leq \limsup_n |||x_n|||$$

for all $0 \leq \lambda < 1 - \gamma_k$.

Sequentially separating norms

Let X be a Banach space with a basis (e_n).
$Q_k(x) = \sum_{n=k}^{\infty} x_n e_n$.
(x_n) is a block basic sequence (b.b.s.) if (x_n) is bounded and

$$x_n = \sum_{i=p_n}^{q_n} a_i e_i$$

with $p_1 \leq q_1 < p_2 \leq q_2 < \cdots$.
An equivalent norm $|\cdot|$ is premonotone if $|Q_k(x)| \leq |x|$ for every $x \in X$.

Definition

$$S_k(X, |\cdot|) := \sup \left\{ \frac{\limsup_n |x_n| + |x|}{\limsup_n |x + x_n|} : (x_n) \text{ b.b.s.}, x \geq k \right\}$$

$$\limsup_n |x_n| + |x| \leq S_k(X, |\cdot|) \limsup_n |x + x_n|$$

for every (x_n) b.b.s. and $x \geq k$.

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Definition (A. Barrera-Cuevas, M.A. Japón, 2015)

An equivalent norm $|\cdot|$ is a sequentially separating norm if

$$\lim_k S_k(X, |\cdot|) = 1.$$

Definition (2016-2017)

Let X be a Banach space with a Schauder basis. A norm $|\cdot|$ on X is called **near-infinity concentrated** (n.i.c.) if it has the following properties:

1. $|\cdot|$ is premonotone and **sequentially separating**.
2. For every $k \in \mathbb{N}$, there exists $F_k : (0, +\infty) \to [0, +\infty)$ with

$$\limsup_n |x_n + \lambda x| \leq \limsup_n |x_n| + F_k(\lambda)|x|,$$

for every b.b.s. (x_n), with $\liminf_n |x_n| \geq 1$, for every $x \in X$ with $|x| \leq 1$ and $x \geq k$, for all $\lambda \in (0, +\infty)$, and such that

$$\lim_{\lambda \to 0^+} \frac{F_k(\lambda)}{\lambda} = 0$$

Theorem

Let X be a Banach space with a boundedly complete Schauder basis and let $|\cdot|$ be a **near-infinity concentrated norm**. *Then $(X, |\cdot|)$* **has the FPP**.

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Corollary (P.K. Lin, 2008)

$(\ell_1, |||\cdot|||)$ has the FPP.

P.K. Lin's norm is just the first element of a sequence of equivalent norms in ℓ_1 with the FPP. This sequence can be defined recursively by

$$p_0(x) = \|x\|_1, \quad p_1(x) = |||x|||, \quad p_2(x) = \sup_k \gamma_k |||Q_k(x)|||, \ldots$$

$$p_3(x) = \sup_k \gamma_k \; p_2(Q_k(x)), \ldots \qquad p_{n+1}(x) = \sup_k \gamma_k \; p_n(Q_k(x))$$

Theorem

1. *The norm* $\nu_p(\cdot)$ *is a* **near-infinity concentrated norm** *on* X.
2. *The Banach space* $(X, \nu_p(\cdot))$ **has the FPP** *for every choice* $p = (p_n) \downarrow 1$, *regardless of whether it is isomorphic to* ℓ_1.

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Theorem

$(\ell_1, |||\cdot||| + \nu_p(\cdot))$ *has the FPP.*

There exist equivalent norms on ℓ_1 with the FPP but failing to be near-infinity concentrated norms.

Theorem (C. Hernández-Linares, M. Japón, E. Llorens-Fuster)

$(\ell_1, \|\cdot\|_1 + \lambda|||\cdot|||)$ *has the FPP for every* $\lambda > 0$.

$\|\cdot\|_1 + |||\cdot|||$ does not satisfy condition 2) in the definition of near-infinity concentrated norms. We can not include these examples with the previous techniques.

Theorem (C. Hernández-Linares, C. Lennard, M. Japón)

The set of renormings in ℓ_1 which fail to contain a.i.c. of ℓ_1 is dense in the set of all renormings of ℓ_1.

Question:

- If $(\ell_1, |\cdot|)$ fails to have an a.i.c. of ℓ_1, does $(\ell_1, |\cdot|)$ have the FPP?
- Does there exist an equivalent norm on ℓ_1
  1. without the FPP and
  2. without asymptotically isometric copies of ℓ_1?
- Can the equivalent norms on ℓ_1 with the FPP be characterized by some geometric property?

Extensions to more general family of functions

Definition (E. Llorens-Fuster, E. Moreno-Gálvez, 2011)

A mapping $T : C \to C$ satisfies the (L) condition if

1. every closed convex bounded T-invariant subset $D \subset C$ contains an approximate fixed point sequence (a.f.p.s.), that is, a sequence $(x_n) \subset D$ with $\lim_n \|x_n - Tx_n\| = 0$.
2. For ever a.f.p.s. (x_n) and $x \in C$ $\limsup_n \|x_n - Tx\| \leq \limsup_n \|x_n - x\|$

Nonepansive mappings, mappings satisfying condition (C) of Suzuki, and some others satisfying condition (L).

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Theorem

$(X, \nu_p(\cdot))$ has the FPP for mappings satisfying condition (L).

Some of the main references

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Feliz Cumpleaños

compañero y amigo